NONNEGATIVELY CURVED SUBMANIFOLDS
IN CODIMENSION TWO

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ABSTRACT. In this paper we study the isometric immersion $f : M^n \longrightarrow \mathbb{R}^{n+2}$ where M is a complete non-compact manifold with nonnegative sectional curvatures. In most of the cases, it is shown that M is a product over its soul and f is a product of immersions. We also study the remaining cases.

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§ 1. Introduction

Let $f: M^n \to \mathbb{R}^{n+2}$ be an isometric immersion of a complete Riemannian manifold M with nonnegative sectional curvatures ($k \geq 0$). If M is compact, we have a complete topological classification of this manifold in [B – M_1] and [B – M_2]. In this paper we want to consider the non–compact case. In [C – G], Cheeger and Gromoll have shown that a complete manifold with $k \geq 0$ is diffeomorphic to the total space of a vector bundle over a compact submanifold, its soul, and classified it in dimensions ≤ 3, up to isometry. Then it is an interesting problem to know under which conditions M turns out to be a product over its soul. In a previous paper, [N_1], we have some topological results along the same lines. In another paper, [N_2], we prove that if there is a point $p \in M$ such that all the sectional curvatures are positive, then the soul of M is a point, implying that M is diffeomorphic to the Euclidean space $\mathbb{R}^n$.

Before stating our results, we recall that f is called a ℓ–cylindrical immersion if there exist the factorizations $M^n = M_1^{n-\ell} \times \mathbb{R}^\ell$ and $f = f_1 \times i$, where $i: \mathbb{R}^\ell \to \mathbb{R}^\ell$ is the identity map. Then, in order to classify M and f, we can suppose that f is non–cylindrical which is equivalent by Hartman [H] to the existence of a point x in M such that the index of relativity nullity $\nu(x) = 0$, where

$$\nu(x) = \dim\{X \in T_x M | \alpha(X,Y) = 0, \quad \forall Y \in T_x M\}$$

and α is the second fundamental form. In what follows, f will always be a non-cylindrical immersion. Our first result states:

Therorem 1: Let $f: M^n \to \mathbb{R}^{n+2}$, $n \geq 3$, be a non–cylindrical isometric immersion of a complete non–compact manifold with nonnegative sectional curvatures and with a m–dimensional soul A^m. Then we have:

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i) If M is simply connected then M is isometric to the product of a manifold homeomorphic to the sphere S^m with a complete manifold $\mathcal{P}^{n-m}$ diffeomorphic to $\mathbb{R}^{n-m}$. If $n-m>1$, f is a product of a hypersurface immersions.

ii) If M is not simply connected, the soul must be homeomorphic to the circle S^1 or to the Real Projective Space $\mathbb{R}P^2$. The latter case occurs only when $n=3$.

It follows by the theorem above that when M is not simply connected, M can be classified topologically. M will be diffeomorphic:

i) either to $S^1 \times \mathbb{R}^{n-1}$ or to the total space of a non–orientable vector bundle over S^1.

ii) either to $\mathbb{R}P^2 \times \mathbb{R}$ or to the total space of a non–orientable line bundle over $\mathbb{R}P^2$.

It also follows by Theorem 1 and the classification of compact manifolds immersed in codimension two that M will be a cylinder over its soul when the soul is homeomorphic to a flat torus or a two–dimensional Klein bottle, or a product of two–spheres, or to the total space of a non–orientable fiber bundle over S^1.

When the codimension of the soul is 1, we cannot expect to have a product of hypersurfaces in the Theorem 1, as we see in the following example: let $g: M_1^{n_1} \to \mathbb{R}^{n_1+1}$ be a codimension one isometric immersion and consider

$$M_1 \times \mathbb{R} \xrightarrow{g \times i} \mathbb{R}^{n_1-2} \xrightarrow{h} \mathbb{R}^{n_1+1}$$

where h is an isometric immersion. $f = h \circ (g \times i)$ is not in general either a product of immersions nor 1–cylindrical. For this case we prove:

Theorem 2: Let $f: M^n \to \mathbb{R}^{n+2}$ be a non–cylindrical isometric immersion, where $M = M_1^{n-1} \times \mathbb{R}$ is a simply connected manifold with nonnegative sectional curvatures and $n \geq 3$. Then:

i) There exists an isometric immersion g from M^{n-1} to $\mathbb{R}^n$.

ii) Suppose that for each $x \in M$, there is a two–plane $\sigma \subset T_x M$ such that $k(\sigma) > 0$; then f is homotopic through isometric immersions to the immersion $g \times i$: $M = M_1 \times \mathbb{R} \to \mathbb{R}^{n-1}$, where i is the identity map.

We have not been able to construct a proof for Theorem 2, (ii) without an extra condition for $k(\sigma)$. But we want to observe that the same statement can be proved if we don't require differentiability of the isometric homotopy.

Finally, we add one extrinsic property to the immersion and for this case we get a complete classification.

Theorem 3: With the same hypothesis of Theorem 1, suppose that the immersion has flat normal bundle ($\mathcal{R}^{\perp} = 0$). Then M is a product over the soul A and f is a product of hypersurface immersions. It follows that the soul is homeomorphic to a sphere S^m, $m \geq 1$.

In the end of the Section 5 in this paper, modifying slightly the Yeaton Cliffon example which appears in [A - M], we see that we can have a complete non-compact manifold isometrically immersed in codimension two, which is a product over the soul S^1 but the immersion f is not a product of hypersurface immersions.

In order to have a complete understanding of isometric immersions of nonnegatively curved manifolds in $\mathbb{R}^{n+2}$, it seems reasonable to ask if a manifold homeomorphic to $\mathbb{R}P^2$ with $K \geq 0$ can be isometrically immersed in $\mathbb{R}^N$ with the dimension of the first normal space at most two, since Gromov and Rokhlin in [G - P] (pg. 42) have proved that if $k > 0$, $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^4$.

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§ 2. Preliminaries and Reducibility along the soul

It is a well known result of Weinstein [W], that if the codimension of an isometric immersion is two then the nonnegativity of the sectional curvatures ($k \geq 0$) implies the nonnegativity of the curvature operator ($\mathcal{R} \geq 0$).

For the case where M^n is complete non-compact manifold with $\mathcal{R} \geq 0$, we now collect some known results of a soul A of M.

(2.1) A soul A of M is a compact, totally geodesic submanifold of M, without boundary and has $\mathcal{R} \geq 0$ (see [C - G]).

(2.2) If M is simply connected, M is isometric to the product $A^m \times \mathcal{P}^{n-m}$ where $\mathcal{P}^{n-m}$ is a complete manifold diffeomorphic to $\mathbb{R}^{n-m}$. (See N_2).

(2.3) If M is not simply connected, M is locally isometric to a product over the soul A (see [N_2]).

In order to prove our theorems, we need the following concept

(2.4) Definition: An isometric immersion $f : M^n \to \mathbb{R}^{n+p}$, $p \geq 1$, is called *reducible along* A if for all $X \in TA$ and $Y \in TA^{\perp}$ we have $\alpha(X, Y) = 0$.

As it is proved in $[B - N]$, this property implies that M is diffeomorphic to $A^m \times \mathbb{R}^{n-m}$. Another consequence of the reducibility along A is given by the following proposition.

(2.5) Proposition: Let $f : M^n \to \mathbb{R}^{n+p}$, $p \geq 1$, be an isometric immersion of a complete non-compact manifold with non negative curvature operator. If f is reducible along a soul A then M is isometric to a product $A^k \times \mathcal{P}^{n-k}$.

Proof: By (2.3) there is a Riemannian submersion $P : M \to A$ where the fibers, F_x are totally geodesic. Hence it follows from [0] that to show that M splits as a product, it is sufficient to prove that for each geodesic loop $\gamma : [0, 1] \to A$, starting from a point $x \in A$, the map

$$L_\gamma : \pi^{-1}(x) = F_x \to F_x, \quad L_\gamma(p) = \tilde{\gamma}(1),$$

(where $\tilde{\gamma}$ denotes the horizontal lift of γ with $\tilde{\gamma}(0) = p$) is the identity on F_x.

This will follow easily from the fact that $\tilde{\gamma}(0) = p$ belongs to the pseudo-soul $\overline{A}$ (see [Y]) obtained from the following way:

Consider $\sigma : [0, \ell] \to M$ the minimal unit speed geodesic connecting p to the soul A. The vector $Y_{\sigma(\ell)} = -\sigma'(\ell)$ is a normal vector of A. Since f is reducible along A, we have:

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) = \nabla_X Y \quad \text{for all } X \in TA$$

(where $\tilde{\nabla}$ and ∇ are the Riemannian connections of $\mathbb{R}^{n+p}$ and M respectively) implying that the parallel translation of the vector $Y_{\sigma(\ell)}$ along A in $\mathbb{R}^{n+p}$ gives a normal parallel section Y along A.

Then the map $\varphi_Y : A \times \mathbb{R} \to M$, given by $\varphi_Y(x, t) = \exp_x tY(x)$ is an isometric immersion (see [Y], Proposition 3.2) and $\overline{A} = \varphi_Y(A, \ell)$.

§ 3. Proof of Theorem 1:

Let x be a point in M such that $\nu(x) = 0$. Let us denote by $r(x)$ the Lie algebra generated by the range of the curvature operator at the point x. Since by (2.3) M is locally a product over the soul,

$$r(x) = \theta(V_1) \oplus \theta(V_2)$$

where $V_1 = T_xA$, $V_2 = T_x^{\perp}A$ and $\theta(V_i)$ is the orthogonal algebra of the subspace V_i. Denoting by m the dimension of the soul we have:

(3.1) Lemma: If $m \geq 2$ and $\sigma \subset T_xA$, the sectional curvature $K(\sigma)$ is positive.

Proof: If dim $V_2 > 1$, since dim $V_1 \geq 2$, we have by $[M_1]$ that the second fundamental form $\alpha(X,Y) = 0$ for $X \in V_1$ and $Y \in V_2$. Then, there is a choice of a tangent and normal frames at x such that the matrices of the second fundamental form operators have the form

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}$$

where each A_i is (dim V_i × dim V_i) non-singular matrix, $i = 1,2$ and this implies the lemma.

Now, let us suppose that dim $V_2 = 1$ and V_2 is generated by the unit vector Z. Since $r(x) = \theta(m) = \theta(n-1)$, following the proof of Theorem 1 in $[Bi]$, there is one normal vector ξ_2 such that rank $A_{\xi_2} = 1$ (A_ξ is the Weingarten operator). If ξ_1 is a normal vector orthogonal to ξ_2 we have

$$\mathcal{R} = A_{\xi_1} \wedge A_{\xi_1} \tag{3.2}$$

We will prove that $A_{\xi_1}(Z) = 0$ and range of A_{ξ_1} is contained in V_1. This together with (3.2) and the fact the that $r(x) = \theta(n-1)$ imply that A_{ξ_1} is non-singular on V_1, proving the lemma.

Since dim $(V_1 \oplus V_2) \geq 3$ there exist $X, Y \in T_xM$ such that $\mathcal{R}(X \wedge Y) \neq 0$. Denoting by X' and X'' the ortogonal projection on V_1 and V_2 respectively we have:

$$\mathcal{R}(X \wedge Y) = (A_{\xi_1}X)' \wedge (A_{\xi_1}Y)' + (A_{\xi_1}X)' \wedge (A_{\xi_1}Y)'' + (A_{\xi_1}X)'' \wedge (A_{\xi_1}Y)' + (A_{\xi_1}X)'' \wedge (A_{\xi_1}Y)''$$

where

$$\omega_1 = (A_{\xi_1}X)'' \wedge (A_{\xi_1}Y)'' = 0 \text{ and}$$

$$\omega_2 = (A_{\xi_1}X)' \wedge (A_{\xi_1}Y)'' + (A_{\xi_1}X)'' \wedge (A_{\xi_1}Y)' = 0,$$

since dim $V_2 = 1$ and $\mathcal{R}$ is zero for 2–forms given by one vector in V_2.

Taking interior product of ω_2 with $(A_{\xi_1}X)'$ we get

$$0 = i(A_{\xi_1}X)'\omega_2 = \|(A_{\xi_1}X)'\|^2(A_{\xi_1}Y)'' - \langle (A_{\xi_1}Y)', (A_{\xi_1}X)'\rangle(A_{\xi_1}X)''$$

and therefore

$$(A_{\xi_1}Y)'' = \langle (A_{\xi_1}Y)', (A_{\xi_1}X)'\rangle \|(A_{\xi_1}X)'\|^{-2}(A_{\xi_1}X)''$$

Taking interior product with $(A_{\xi_1}Y)'$ we get

$$\begin{aligned} 0 &= i(A_{\xi_1}Y)'\omega_2 = < (A_{\xi_1}X)', (A_{\xi_1}Y)' > (A_{\xi_1}Y)'' - \|(A_{\xi_1}Y)'\|^2(A_{\xi_1}X)'' \\ &= \|(A_{\xi_1}X)'\|^{-2}\{(< (A_{\xi_1}X)', (A_{\xi_1}Y)' >)^2 - \|(A_{\xi_1}X)'\|^2\|(A_{\xi_1}Y)'\|^2\}(A_{\xi_1}X)''. \end{aligned}$$

If $(A_{\xi_1}X)'' \neq 0$ the above relation implies $(A_{\xi_1}Y)' = \lambda(A_{\xi_1}X)'$ and then $\mathcal{R}(X\wedge Y) = (A_{\xi_1}X)' \wedge (A_{\xi_1}Y)' = 0$. Hence,

(3.3) if $\mathcal{R}(X\wedge Y) \neq 0$ we have $(A_{\xi_1}X)'' = (A_{\xi_1}Y)'' = 0$

Consider now the orthonormal frame $\{Z_1,\dots,Z_n\}$ which diagonalizes the operator A_{ξ_2} such that $A_{\xi_2}(Z_1) = \lambda Z_1$ and $A_{\xi_2}(Z_i) = 0$, $i \geq 2$. Since $\mathcal{R} \neq 0$ at x, there exist Z_i, Z_j such that $\mathcal{R}(Z_i \wedge Z_j) \neq 0$. By (3.3), we have $\langle\alpha(Z_i, Z), \xi_1\rangle = 0 = \langle\alpha(Z_j, Z), \xi_2\rangle = 0$. This implies $\alpha(Z_i, Z) = 0$ for $i \geq 2$, as we have supposed $A_{\xi_2}(Z_i) = 0$. Since $\mathcal{R}(Z\wedge Z_i) = \mathcal{R}(Z\wedge Z_i') + \mathcal{R}(Z\wedge Z_i'') = 0$ we will have in the Gauss equation $\langle\alpha(Z_i, Z_i), \alpha(Z, Z)\rangle = 0$. Because $\alpha(Z_i, Z_i)$ is orthogonal to ξ_2, we have $\alpha(Z, Z)$ orthogonal to ξ_1. Now, writing the Gauss equation for the sectional curvature of a plane spanned by $X \in V_1$ and $Z \in V_2$ we get:

$$0 = \langle A_{\xi_1}X, X\rangle\langle A_{\xi_1}Z, Z\rangle - (\langle A_{\xi_1}Z, X\rangle)^2 = -(\langle A_{\xi_1}Z, X\rangle)^2$$

This together with (3.3) implies $A_{\xi_1}Z = 0$. Thus, range $A_{\xi_1} \subset V_1$ and $A_{\xi_1}Z = 0$, as we claimed.

Now let $\overline{f} = f|_A : A \to I\!R^{n+2}$ be the isometric immersion f restricted to the soul. Since A is a totally geodesic submanifold of M, the first normal space of $\overline{f}$ is at most two-dimensional. The Lemma (3.1) implies that there is a point $p \in A$ such that all the sectional curvatures along planes tangent to A at p are positive. We can easily to generalize to f, using the same arguments, Theorems (2.2) and (2.3) of [B − M_1], obtaining the same results, since they need only the fact of the first normal space be at most two-dimensional. These slight generalizations imply that if $m \geq 3$, A and consequently M, are simply connected and A is homeomorphic to a sphere S^m. Then if $m = 2$, A has to be homeomorphic either to a sphere S^2 or to a Real Projective space $I\!RP^2$.

If M is simply connected, then by (2.2) we have that M is a product $A^m \times P^{n-m}$. If $n - m > 1$, it follows by [M_1] that either f is a product of hypersurface immersions or f takes a complete geodesic into a straight line. In the latter case, this geodesic is a line (in the sense that each segment realizes the distance between its endpoints), hence it must split off isometrically (cf. [T]). This would imply that f is at least 1–cylindrical which contradicts our assumption. Then f is a product of hypersurface immersions.

The only thing we need to prove now is that if $n \geq 4$, the soul A cannot be homeomorphic to $\mathbb{R}P^2$. Suppose it is and consider the universal covering $\tilde{M}$ of M. Let us consider $\tilde{A}$, the soul of $\tilde{M}$, which is homeomorphic to S^2 and the immersion $\tilde{f} = P \circ f$ where P is the covering map. Since $n - 2 \geq 2$, $\tilde{f}$ will be a product of hypersurface immersions and then if $\tilde{X} \in TS^2$ and $\tilde{Y} \in (TS^2)^\perp$, $\tilde{\alpha}(\tilde{X}\ ,\ \tilde{Y}) = 0$ ($\tilde{\alpha}$ is the second fundamental form of $\tilde{f}$). This implies that f is reducible along A. By proposition (2.5) M is a product over the soul and again because $n - 2 \geq 2$, we would have A, homeomorphic to $\mathbb{R}P^2$, isometrically immersed in codimension 1, which is the required contradiction.

§ 4. Proof of Theorem 2:

To prove this theorem, we will use the Fundamental Existence and Uniqueness Theorem for submanifolds (see $[B - C]$, § 10.8). First, we will prove (ii) of the Theorem and then (i) will follow easily using the same arguments. Following Moore's argument in $[M_2]$, for each $x \in M$, we will take a neighborhood $U \subset M$ of x and on the tube $U \times \mathbb{R}^2$ contained in the normal bundle, consider differentiable sections ξ_1 , ξ_2 such that if we define a second fundamental form $\alpha_t : TU \times TU \to E$ by

$$\alpha_t(X, Y) = \langle\alpha(X, Y),\ \xi_1\rangle\xi_1 + t\langle\alpha(X, Y),\ \xi_2\rangle\xi_2$$

and a normal connection $\nabla_t^\perp : TU \times E \to E$ by $\nabla_t^\perp\xi_\alpha = t\,\nabla^\perp\,\xi_\alpha$ for $\alpha = 1, 2$. $(E,\ \alpha_t,\ \nabla_t^\perp)$ will satisfy the Gauss, Codazzi and Ricci equations for $0 \leq t \leq 1$ in such way that if $\{f_t\}$ denotes the family of isometric immersions, we will have $f_o(U)$ lying in a hyperplane of $\mathbb{R}^{n+2}$. Since M is simply connected, and α_t will globally defined on $TM \times TM$, the family $\{f_t\}$ is globally defined.

In order to construct E and $\{\xi_1,\ \xi_2\}$ as above we will prove the lemmas below. Given $x \in M$, $T_xM = T_xM_1 \oplus \mathbb{R}$, we will always denote by Z the tangent vector belonging to $\mathbb{R}$.

(4.1) Lemma: With the same hypothesis of the Theorem, if at a point $x \in M$, $\alpha(Z, Z)(x) \neq 0$, then there is an orthonormal frame $\{\xi_1,\ \xi_2\}$ in the normal space, such that $A_{\xi_1}(T_xM_1) \subseteq T_xM_1$, $A_{\xi_1}(Z) = 0$ and rank $A_{\xi_2} = 1$.

Proof: Let U be the orthogonal complement of the relative nullity subspace $N(x)$. If there is a 2-plane σ such that the sectional curvature $K(\sigma) > 0$, the same proof as Lemma (3.1) applied to U gives the frame $\{\xi_1, \xi_2\}$, since $r(x)$ will be the nonnull algebra $\theta\ (dim\ U - 1) \oplus \theta(1)$.

Now, let us suppose $K(\sigma) = 0$ for all $\sigma \subset T_x M$. If dim $U = 1$ there exists a normal vector ξ such that A_ξ is identically null and then $\xi_1 = \xi$ and ξ_2 orthogonal to ξ satisfy the lemma.

If dim $U = 2$, as in the proof of the lemma (3.1) we will define ξ_2 by $\frac{\alpha(Z,Z)}{\|\alpha(Z,Z)\|}$. Taking ξ_1 orthogonal to ξ_2, $\langle A_{\xi_1} Z, Z\rangle = 0$. But there exists a vector Z_1 which is the eigenvector of A_{ξ_1} associated to its only nonnull eigenvalue. We claim that $Z_1 \in T_x M_1$. In fact, $Z_1 = aX + bZ$, where X is the normalized orthogonal projection of Z_1 onto $T_x M_1$. If we take $Z_1^\perp = -bX + aZ$, $A_{\xi_1}(Z_1^\perp) = 0$ and then,

$$0 = -bA_{\xi_1} X + aA_{\xi_1} Z$$

Taking inner product with Z (remembering that $\langle A_{\xi_1} Z, Z\rangle = 0$) we have

$$0 = -b\langle A_{\xi_1} X, Z\rangle$$

If $b = 0$ we have $Z_1 = X$ and if $\langle A_{\xi_1} X, Z\rangle = 0$ we have, because $A_{\xi_1}(Y) = 0$ for Y orthogonal to X, $Y \in T_x M_1$, that $T_x M_1$ is invariant by A_{ξ_1} and $A_{\xi_1}(Z) = 0$. Hence the lemma foolows.

By the above lemma, we can always consider a subspace W of $T_x M_1$, which is at least $(n-2)$-dimensional, such that if $Y \in W$, $A_{\xi_1}(Y) = 0$. We want to observe that, since $\alpha(Z, Z)$ is non- null in a neigborhood V of x, $\alpha(Z, Z)$ will define a differentiable normal vector field on V, ξ_2, which is defined parallel to $\alpha(Z, Z)$. Since rank $A_{\xi_2} = 1$, the operator A_{ξ_2} has eigenvalues with constant multiplicity on V and then its eigenvectors are differentiable vector fields on V. Consequently, their orthogonal projections onto $T_x M_1$ for $x \in V$, are differentiable vector fields on V. The normal vector field ξ_1, taken orthogonal to ξ_2, is also a differentiable normal vector field on V.

4.2 Lemma: With the notations above we have:
i) If $Y \in W$, $\nabla_Y^\perp \xi_1 = \nabla_Y^\perp \xi_2 = 0$
ii) If λ is an eigenvalue of A_{ξ_1} at x, λ is constant along the segment of the geodesic $\gamma(t) = exp_x tZ$, contained in V.

Proof: If dim $W = n-1$, $A_{\xi_2}|_{T_x M_1}$ is identically null and then A_{ξ_1} and A_{ξ_2} are diagonalized by the same basis. This implies that the normal curvature $R^\perp$ is zero and the lemma is obvious.

If dim $W = n-2$, from the lemma (4.1) we have $\langle\alpha(Y, Z), \xi_1\rangle = 0$ and $\langle\alpha(Y, Z), \xi_2\rangle = 0$. Denoting by X the vector generating $T_xM_1 \cap W^{\perp}$, we can apply the Codazzi equation to X, Y, Z and ξ_1 to get

$$\langle\nabla_X^{\perp}\alpha(Y, Z), \xi_1\rangle - \langle\alpha(\nabla_X Y, Z)\xi_1\rangle - \langle\alpha(Y, \nabla_X Z), \xi_1\rangle =$$
$$= \langle\nabla_Y^{\perp}\alpha(X, Z), \xi_1\rangle - \langle\alpha(\nabla_Y X, Z), \xi_1\rangle - \langle\alpha(X, \nabla_Y Z), \xi_1\rangle$$

Observe that the left hand side is equal to zero, since $\alpha(Y, Z) = 0$, $\nabla_X Y$ and $A_{\xi_1}(\nabla_X Y)$ belong to T_xM_1 and $\nabla_X Z = 0$. The same reasons together with $\langle\alpha(X, Z), \xi_1\rangle = 0$, imply that the right hand side is equal to

$$\langle\nabla_Y^{\perp}\alpha(X, Z), \xi_1\rangle = -\langle\alpha(X, Z), \nabla_Y^{\perp}\xi_1\rangle$$

As we are supposing that $\langle\alpha(X, Z), \xi_2\rangle \neq 0$ (dim $W = n-2$), we have $\nabla_Y^{\perp}\xi_1 = \nabla_Y^{\perp}\xi_2 = 0$.

To prove the second part of the Lemma, let us consider the basis $\{X_1, \ldots, X_n\}$ of T_xM_1 diagonalizing $A_{\xi_1}|_{T_xM_1}$. From the Codazzi equation applied to Z, X_i and ξ_1 we have:

$$Z\langle\alpha(X_i, X_i), \xi_1\rangle - \langle\alpha(X_i, X_i), \nabla_Z^{\perp}\xi_1\rangle - 2\langle\alpha(\nabla_Z X_i, X_i), \xi_1\rangle =$$
$$X_i\langle\alpha(Z, X_i), \xi_1\rangle - \langle\alpha(Z, X_i), \nabla_{X_i}^{\perp}\xi_1\rangle - \langle\alpha(\nabla_{X_i}Z, X_i)\rangle - \langle\alpha(Z, \nabla_{X_i}X_i), \xi_1\rangle$$

Since $A_{\xi_2}|_W = 0$ and $\nabla_Y^{\perp}\xi_2 = 0$ for $Y \in W$, the above equation is equivalent to

$$(4.3) \qquad Z\langle\alpha(X_i, X_i, \xi_1)\rangle - \textstyle\sum_{i=1}^{n-1}(X_i, X)^2\langle\alpha(X, X), \nabla_Z^{\perp}\xi_1\rangle =$$
$$= -\sum_{i=1}^{n-1}(X_i, X)^2\langle\alpha(Z, X), \nabla_X^{\perp}\xi_1\rangle$$

Therefore, to prove (ii) of the lemma is equivalent to prove that $\langle\alpha(X, X), \nabla_Z^{\perp}\xi_1\rangle = \langle\alpha(Z, X), \nabla_X^{\perp}\xi_1\rangle$ which is equivalent to

$$(4.4) \qquad \langle\alpha(X, X), \xi_2\rangle\langle\nabla_Z^{\perp}\xi_1, \xi_2\rangle = \langle\alpha(Z, X), \xi_2\rangle\langle\nabla_X^{\perp}\xi_1, \xi_2\rangle$$

For that, consider the orthonormal basis $\{Z_1, Z_2, \ldots, Z_n\}$ diagonalizing the operator A_{ξ_2} such that $A_{\xi_2}(Z_i) = 0$, $i \geq 2$. We can consider $Z_i \in W \subset T_xM_1$, for all $i \geq 3$. Applying the Codazzi equation to Z_2, Z_i $(i \geq 3)$ and ξ_2, we have

$$\langle\nabla_{Z_2}^{\perp}\alpha(Z_i, Z_i), \xi_2\rangle - 2\langle\alpha(\nabla_{Z_2}Z_i, Z_i), \xi_2\rangle =$$
$$= \langle\nabla_{Z_i}^{\perp}\alpha(Z_2, Z_i), \xi_2\rangle - \langle\alpha(\nabla_{Z_i}Z_2, Z_i), \xi_2\rangle - \langle\alpha(Z_2, \nabla_{Z_i}Z_i), \xi_2\rangle$$

On the right hand side, because $A_{\xi_1} Z_i = 0$, we will have only

$$(4.5) \qquad \langle \nabla^{\perp}_{Z_2} \alpha(Z_i, Z_i), \xi_2 \rangle = -\langle \alpha(Z_i, Z_i), \nabla^{\perp}_{Z_2} \xi_2 \rangle$$

Since $i \geq 3$, $Z_i \in W$ from (i) of this lemma we have $\nabla^{\perp}_{Z_i} \xi_2 = 0$. Again, because $A_{\xi_2} Z_2 = A_{\xi_2} Z_i = 0$, the left hand side is equal to zero. Then (4.5) becomes

$$\langle \alpha(Z_i, Z_i), \xi_1 \rangle \langle \nabla^{\perp}_{Z_2} \xi_2, \xi_1 \rangle = 0$$

If there is i such that $\langle \alpha(Z_i, Z_i), \xi_1 \rangle \neq 0$, we have $\langle \nabla^{\perp}_{Z_2} \xi_2, \xi_1 \rangle = 0$. If $\langle \alpha(Z_i, Z_i), \xi_1 \rangle = 0$ for all $i \geq 3$, we have $A_{\xi_1}|_W = 0$ and then X is an eigenvector of A_{ξ_1}. Writing the Codazzi equation for Z_2, X, Z and ξ_1 we will have

$$\langle \nabla^{\perp}_{Z_2} \alpha(X, Z), \xi_1 \rangle - \langle \alpha(\nabla_{Z_2} X, Z), \xi_1 \rangle - \langle \alpha(X, \nabla_{Z_2} Z), \xi_1 \rangle =$$
$$= \langle \nabla^{\perp}_{X} \alpha(Z_2, Z), \xi_1 \rangle - \langle \alpha(\nabla_X Z_2, Z), \xi_1 \rangle - \langle \alpha(Z_2, \nabla_X Z), \xi_1 \rangle$$

Since X is the only eigenvector of A_{ξ_1} with nonnull eigenvalue, $Z_2 \in span\{X, Z\}$ and $\langle \nabla_X X, Z \rangle = 0$, the left hand side is equal to zero. On the right hand side, for the same reasons, the only term we will have is

$$\langle \nabla^{\perp}_{Z_2} \alpha(X, Z), \xi_1 \rangle = -\langle \alpha(X, Z), \nabla^{\perp}_{Z_2} \xi_1 \rangle$$

which will be zero too. Since we are considering $\langle \alpha(X, Z), \xi_2 \rangle \neq 0$ (ortherwise $\dim W = n - 1$) we have again $\nabla^{\perp}_{Z_2} \xi_1 = \nabla^{\perp}_{Z_2} \xi_2 = 0$. This fact will imply (4.4). In fact, write $Z_2 = aX + bZ$.

$$A_{\xi_2}(Z_2) = 0 \Rightarrow a \langle \alpha(X, X), \xi_2 \rangle = -b \langle \alpha(X, Z), \xi_2 \rangle$$
$$\nabla^{\perp}_{Z_2} \xi_1 = 0 \Rightarrow a \, \nabla^{\perp}_X \xi_1 = -b \, \nabla^{\perp}_Z \xi_1$$

Therefore

$$(4.6) \qquad b \langle \alpha(X, Z), \xi_2 \rangle \, \nabla^{\perp}_X \xi_1 = b \langle \alpha(X, X), \xi_2 \rangle \, \nabla^{\perp}_Z \xi_1$$

Since we are supposing $b \neq 0$ $[(b = 0 \Rightarrow Z_2 \in T_x M_1 \Rightarrow \dim W = n - 1)]$, (4.6) implies (4.4) taking inner product with ξ_2.

(4.7) Lemma: With the notations above, if $Y \in W$ and X is orthogonal to W, Y and X are parallel along the segment of geodesic $\gamma(t) = exp_x tZ$ contained in V.

Proof: If M_1 is two–dimensional we consider isothermal parameters (x_1, x_2) defined on V such that $\frac{\partial}{\partial x_1} = \lambda_1 X$ and $\frac{\partial}{\partial x_2} = \lambda_2 Y$. If M_1 is at least three dimensional then dim $W \geq 2$. Consider $Y_1, Y_2 \in W$ and the Codazzi equation for Y_1, Y_2, X and ξ_2. We have:

$$\langle \nabla^{\perp}_{Y_1} \alpha(Y_2, X), \xi_2 \rangle - \langle \alpha(\nabla_{Y_1} Y_2, X), \xi_2 \rangle - \langle \alpha(Y_2, \nabla_{Y_1} X), \xi_2 \rangle =$$
$$= \langle \nabla^{\perp}_{Y_2} \alpha(Y_1, X), \xi_2 \rangle - \langle \alpha(\nabla_{Y_2} Y_1, X), \xi_2 \rangle - \langle \alpha(Y_1, \nabla_{Y_2} X), \xi_2 \rangle$$

Since $A_{\xi_2}(Y_i) = 0$ and $\nabla^{\perp}_{Y_i} \xi_2 = 0$ for $i = 1, 2$, we will have

$$\langle \nabla_{Y_1} Y_2, X \rangle \ \langle \alpha(X, X), \xi_2 \rangle = \langle \nabla_{Y_2} Y_1, X \rangle \ \langle \alpha(X, X), \xi_2 \rangle$$

Since $\langle \alpha(X, X), \xi_2 \rangle \neq 0$ (otherwise dim $W = n - 1$ which implies $\mathcal{R}^{\perp} = 0$ and then the lemma is obvious) we have $\langle [Y_1, Y_2], X \rangle = 0$ and then W is integrable. By Frobenius we can consider local coordinates $(x_1, \ldots, x_{n-1}, t)$ defined on V such that $\frac{\partial}{\partial x_1} = \lambda_1 X$, $\frac{\partial}{\partial x_i} = \lambda_i Y_i$ for $Y_i \in W$ and $\frac{\partial}{\partial t} = \lambda_n Z$. Since $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}] = 0$, we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial_t} - \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial_{x_i}} = \nabla_{\lambda_i Y_i} \lambda_n Z - \nabla_{\lambda_n Z} \lambda_i Y_i = 0$$

Therefore

$$\lambda_i Y_i(\lambda_n) Z + \lambda_i \lambda_n \nabla_{Y_i} Z - \lambda_n Z(\lambda_i) Y_i - \lambda_n \lambda_i \nabla_Z Y_i = 0$$

Since $\nabla_{Y_i} Z = 0$ (because M is product) and $\nabla_Z Y_i$ is orthogonal to Z and Y_i, it follows that $\nabla_Z Y_i = 0$. The same proof implies $\nabla_Z X = 0$.

Now, we can start the proof of Theorem 2. Given $x \in M$, we consider the geodesic $\gamma(t) = exp_x tZ$. We claim that there is a point $y \in \gamma$ such that $\alpha(Z, Z)(y) \neq 0$ because if not, f would take a complete geodesic to a straight line in $\mathbb{R}^{n+2}$ contradicting our assumption. As we have already observed, in a neighborhood V of y, ξ_1 and ξ_2 satisfying Lemmas (4.1) and (4.2) are differentiable normal vectors. We want to prove that for each $x \in \gamma$, there is a neighborhood U of x such that we have differentiable normal vector fields ξ_1 and ξ_2, satisfying the lemmas above. Take x on the boundary of V. Let $\overline{V}$ be another neighborhood where x belongs to its boundary too.

If $\alpha(Z, Z)|_{\overline{V}} \neq 0$, we have $\overline{Y} \in \overline{W}$, and $\overline{\xi_1}$ and $\overline{\xi_2}$ satisfying the lemmas above. We claim that $\xi_1 = \overline{\xi_1}$ and $\xi_2 = \overline{\xi_2}$ (defined by limit) at the point x. In fact, if $K(\sigma) > 0$ for some plane $\sigma \subset T_y M_1$, by Lemma (4.2) (ii), $K(\sigma) > 0$ in a neighborhood of x and then $K(\sigma) > 0$ on $\overline{V}$. Therefore, as it can be seen in the proof of Lemma (3.1), there exists $Y \in TV$ such that ξ_1 is $\alpha(Y, Y)$ normalized and there exists $\overline{Y} \in T\overline{V}$ such that $\overline{\xi_1}$ is $\alpha(\overline{Y}, \overline{Y})$ normalized. Now, we define the vector Y on $\overline{V}$ and $\overline{Y}$ or V by parallel translation along geodesics tangent to Z (see Lemma (4.7)). Since $A_{\overline{\xi_2}}(\overline{Y}) = 0$ and $A_{\xi_2}(Y) = 0$, ξ_1 is parallel to $\alpha(Y, \overline{Y})$ in V and ξ_1 is parallel to

$\alpha(Y, \overline{Y})$ in $\overline{V}$. This proves that $\xi_1 = \overline{\xi}_1$ at the point x, because $\xi_1 = \overline{\xi_1} = \frac{\alpha(Y, \overline{Y})}{\|\alpha(Y, \overline{Y})\|}$. This also proves the differentiability of ξ_1 in a neighborhood of x.

If $\alpha(Z, Z)|_{\overline{V}} = 0$, we define ξ_1, ξ_2 on $\overline{V}$ by parallel translation along geodesics tangent to Z. They will satisfy Lemmas (4.1) and (4.2), as it can be seen below. Consider X_i, an eigenvector of A_{ξ_j}, $j = 1, 2$. From the Codazzi equation we have

$$\begin{aligned} &Z\langle\alpha(X_i, X_i), \xi_j\rangle - \langle\alpha(X_i, X_i), \nabla_Z^{\perp}\xi_j\rangle - 2\langle\alpha(\nabla_Z X_i, X_i), \xi_j\rangle = \\ &= X_i\langle\alpha(Z, X_i), \xi_j\rangle - \langle\alpha(Z, X_i), \nabla_{X_i}^{\perp}\xi_j\rangle - \langle\alpha(\nabla_{X_i} Z, X_i), \xi_j\rangle - \\ &- \langle\alpha(Z, \nabla_{X_i} X_i), \xi_j\rangle \end{aligned}$$

Since $A_{\xi_j}(Z) = 0$, $\nabla_Z^{\perp}\xi_j = 0$ and $\nabla_{X_i} Z = 0$ we have

$$Z\langle\alpha(X_i, X_i), \xi_j\rangle = 0$$

implying that ξ_1 and ξ_2 satisfy in $\overline{V}$ the lemmas above.

To finish the proof of (ii) in Theorem 2, we have to prove that α_t defined previously, satisfies Gauss, Codazzi and Ricci equations. The Gauss equation will follow from (3.2). The Codazzi and Ricci equations follow easily if $0 < t \leq 1$. For $t = 0$, $\alpha_t(X, Y) = \langle\alpha(X, Y), \xi_1\rangle\xi_1$, which implies that f_0 will be an isometric immersion in a hyperplane of $\mathbb{R}^{n+2}$. Therefore, we need to prove only the Codazzi equation. This will follow from the Codazzi equation for the given isometric immersion f, remembering that for $Y \in W$,

$$\begin{aligned} &\langle\nabla_Y^{\perp}\alpha(X_1, X_2), \xi_1\rangle = Y\langle\alpha(X_1, X_2), \xi_1\rangle \quad \text{since} \quad \nabla_Y^{\perp}\xi_1 = 0 \\ &\langle\nabla_{X_1}^{\perp}\alpha(Y, X_2), \xi_1\rangle = X_1\langle\alpha(Y, X_2), \xi_1\rangle \quad \text{since} \quad A_{\xi_1}(Y) = 0 \end{aligned}$$

for each X_1, X_2 tangent to M. Hence, we are reduced to prove it for X and Z, X orthogonal to W. We have:

$$\begin{aligned} &X\langle\alpha(Z, Z), \xi_1\rangle - 2\langle\alpha(\nabla_X Z, Z), \xi_1\rangle = 0 \\ &= Z\langle\alpha(X, Z), \xi_1\rangle - \langle\alpha(\nabla_Z X, Z), \xi_1\rangle - \langle\alpha(X, \nabla_Z Z), \xi_1\rangle \end{aligned}$$

because $A_{\xi_1}(Z) = 0$ and $\nabla_Z Z = 0$. The other equation will be

$$\begin{aligned} &Z\langle\alpha(X, X), \xi_1\rangle - 2\langle\alpha(\nabla_Z X, X), \xi_1\rangle = Z\langle\alpha(X, X), \xi_1\rangle \\ &X\langle\alpha(Z, X), \xi_1\rangle - \langle\alpha(\nabla_X Z, X), \xi_1\rangle - \langle\alpha(Z, \nabla_X X), \xi_1\rangle = 0 \end{aligned}$$

because $A_{\xi_1}(Z) = 0$ and $\nabla_X Z = \nabla_Z X = 0$. We can see that the only term left, $Z\langle\alpha(X, X), \xi_1\rangle$ is zero from the Codazzi equation for f which has only one more term on the left hand side, ie, $\langle\alpha(X, X), \nabla_Z^{\perp}\xi_1\rangle$ and only one more term on the right hand side. $\langle \alpha(Z, X), \nabla^{\perp}_{Z}\xi_1 \rangle$ which are equal by (4.4).

Now, part (i) of the theorem. is easily proved. For each $\overline{x} \in M_1$. we consider $x = (\overline{x}, t) \in M_1 \times \mathbb{R}$, such that $\alpha(Z, Z)(x) \neq 0$. For a neighborhood U of x. which is $\overline{U} \times I$ ($\overline{U}$ is a neighborhood of x in M_1 and I an interval of $\mathbb{R}$) we have a differentiable normal vector ξ_1. such that A_{ξ_1} satisfies the Gauss and Codazzi equations for an isometric immersion from $\overline{U}$ to $\mathbb{R}^n$. Since M_1 is simply connected. we have the required immersion g.

We observe that f_*, constructed above, is congruent to $g \times i$ since $A_{\xi_1}(Z) = 0$.

§ 5. Theorem 3 and Immersions which are not products

To prove Theorem 3. by Theorem 1 we have to consider only two cases, namely, when the soul is the circle S^1 and when $n = 3$ and the soul is homeomorphic to $\mathbb{R}P^2$. In both cases, the universal covering $\tilde{M}$ of the manifold M will be the product $M_1 \times \mathbb{R}$.

We will denote by $\tilde{f} = f \circ P$ (P is the covering map), $\tilde{\alpha}$ the second fundamental form of $\tilde{f}$. and $\mathcal{V}_{\tilde{f}}(\tilde{x})$ the index of relative nullity of $\tilde{f}$ at $\tilde{x} \in \tilde{M}$. Let us take $\tilde{x} \in \tilde{M}$ such that $\mathcal{V}_{\tilde{f}}(\tilde{x}) = 0$. Following the proof of Lemma (3.1) we see that there are normal vectors ξ_1 and ξ_2 such that $A_{\xi_1}(T_{\tilde{x}}M_1) = T_{\tilde{x}}M_1$, $A_{\xi_1}(Z) = 0$ and rank $A_{\xi_2} = 1$ (Z denotes the tangent vector belonging to $\mathbb{R}$). Since $\mathcal{R}^{\perp} = 0$, the same orthonormal basis diagonalizes A_{ξ_1} and A_{ξ_2}. Then Z_1, eigenvector of A_{ξ_2} coresponding to nonnull eigenvalue. has to be tangent either to M_1 or to $\mathbb{R}$. If $Z_1 \in T_{\tilde{x}}M_1$, $A_{\xi_2}(Z) = 0$ and this implies that Z is relative nullity vector. which contradicts $\mathcal{V}_{\tilde{f}}(\tilde{x}) = 0$. Then $Z_1 = Z$ and $\tilde{\alpha}(X, Z) = 0$ for all $X \in T_{\tilde{x}}M_1$. The proof of Theorem 2 in [M_1] implies $\tilde{\alpha}(X, Z) = 0$ for all $X \in TM_1$ since no complete geodesic in $\tilde{M}$ can be taken to a straight line in $\mathbb{R}^{n+2}$. Therefore f is reducible along the soul A. By Proposition (2.5) M is a product over the soul and because $\alpha(X, Z) = 0$. f is a product of hypersurface immersions, proving the theorem.

Now. we show an example of a complete manifold M^n with S^1 as a soul. isometrically immersed in $\mathbb{R}^{n+2}$ which is not a product of immersions. The reader is referred to [$A - M$]. section 4, for the details.

Let $f_1 : M_1^{n-1} \to \mathbb{R}^n$ be any isometric immersion of codimension one and $f_2 : S^1 \to \mathbb{R}^2$ an isometric immersion such that f_2 is totally geodesic on the interval $I = (-\varepsilon, \varepsilon)$. Now consider $f = f_1 \times f_2 : M = M_1 \times S^1 \to \mathbb{R}^{n+2}$. The restriction of the immersion f to $M_1 \times I$ is given by

$$(m, t) \mapsto \sum_{i=1}^{n+1} f_1^i(m)e_i + te_{n+2}$$

where $\{e_1, \ldots, e_{n+2}\}$ is the standard basis of $\mathbb{R}^{n+2}$.

If M_1 is compact in section 4 of $[A - M]$, the authors show how to construct an isometric immersion $\bar{f}$ of $M_1 \times I$ which is not a product of hypersurface immersions and agrees with f whenever $\frac{\varepsilon}{2} < |t| < \varepsilon$.

We want to point out that the compactness of M_1 is necessary to guarantee that equation (7) in $[A - M]$ never vanishes and then h, defined by them is an immersion. We are interested when M_1 is non-compact. So, we will fix one copy S^1 of M, denote it by $\overline{S}$, and for a real number σ, take the compact subset $B(\overline{S}, \sigma) = \{m \in M/d(m, \overline{S}) < \sigma\}$ ($d(m, \overline{S}$ is the distance from m to $\overline{S}$). We redefine the immersion $h : M_1 \times I \rightarrow \mathbb{R}^{n+2}$ by

$$h(m, t) = \sum_{i=1}^{n-1} f_1^i(m)e_i + f_1^n(m)[X_n + \alpha X_{n+1}](t) + \sigma(t)$$

where now α is a smooth function depending on (m, t). For $\bar{m} \in B(\overline{S}, \alpha)$, $\alpha(\overline{m}, t)$, satisfies the same conditions as α in $[A - M]$ as a function of t. and we require that $\alpha(m, t)$ goes to zero when m is such that $d(m, \overline{S}) = \sigma$.

Following the same procedure as in $[A - M]$ we can produce an isometric immersion $\overline{f}$ of $M_1 \times I$ which agrees with f whenever $\frac{\varepsilon}{2} < |t| < \varepsilon$ and $m \notin B(\overline{S}, \sigma)$.

Piecing f and $\overline{f}$ together we obtain 2– codimensional isometric immersion of $M_1 \times S^1$ which is not a product of immersions.

Before finishing this paper we would like to say that it would be of interest to construct examples of manifolds having S^1 as soul in $\mathbb{R}^{n+2}$, which are not product over the soul.

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