A GEOMETRICAL THEORY OF NON TOPOLOGICAL MAGNETIC MONOPOLES

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ABSTRACT: A theory of magnetic monopoles without strings has been recently formulated by Rosa, Recami and Rodrigues[2] using the Clifford bundle formalism. Although that formalism seems to be a perfect mathematical design for the electrodynamics with monopoles withouth strings, it is insufficient for the introduction of analogous monopoles in a non abelian gauge theory without sacrificing the geometrization of the theory. Here, we present a geometrical theory of the generalized monopoles without strings as a principal fiber bundle with group GxG (a spliced bundle). We obtain the generalized field equations from the variational principle in the spliced bundle.

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The model of electrodynamics as a principal fiber bundle theory (PFB) is as follows: Let (M,h,D) be a Lorentzian manifold and let $\pi:P\rightarrow M$ be a PFB with group $U(1)=\{e^{i\theta},\theta\in R\}$ and Lie algebra $\hat{U}(1)=\{i\theta,\theta\in R\}$.

Let ω be a connection 1-form over P,i.e., $\omega\in\Lambda^1(P,\hat{U}(1))$, and let $M\supset U,\sigma_U:U\rightarrow P$ be a local section; The pullback of ω is the electromagnetic potential,and we write

$$\omega_U=\sigma_U^*\ \omega=-iA_U \qquad (1)$$

The electromagnetic fied $\Omega^\omega=d\omega$ relative to σ_U is

$$i\Omega_U=F_U=dA_U \qquad (2)$$

If $\sigma_V:V\rightarrow P$ is another local section, then from the fact that U(1) is abelian we have that $F_U=F_V$. It follows that the curvature of the connection is in this case a 2-form which is well defined in the base manifold M; in other words, the electromagnetic field is gauge invariant.

Then, fixing $U\subset M$, if $A\in\Lambda^1(U,R)$ is the potential , the field is F=dA with $A=A_\mu dx^\mu$, so that $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. From Bianchi identity it follows that

$$dF=0 \quad \text{(homogeneous Maxwell equations)} \qquad (3)$$

As there are no reasons for δF to be null ,we put

$$\delta F=-J_e \quad \text{(inhomogeneous Maxwell equations)} \qquad (3')$$

where $J_e\in\Lambda^1\tau^*M$, is the current one-form here introduced in a purely "*phenomenological*" way.

It is well known [1] that eqs.(3) can be used to describe also magnetic monopoles.Indeed, all we need is a situation where it does not exist a global defined potential such that F=dA.This happens if the PFB,$\pi:P\rightarrow M$ with group U(1) is non trivial.The usual model is obtained by choosing as base of the PFB a non contractible space-time,i.e., by deleting the world line of the monopole from M,which then get a "moving hole".

It is quite clear that this theory does not treat electric charges and monopoles on a equal footing.A non trivial theory treating charges and monopoles symmetrically would be described not by eqs.(3) but by:

$$dF=-{*J_m} \qquad (4)$$

$$J_m\neq kJ_e \text{ ,k a real constant}$$

$$\delta F=-J_e \qquad (4')$$

Since eq.(3) is a immediate consequence of Bianchi's identity, the validity of eq.(4) destroyes the simple U(1)-PFB model of classical electrodynamics. In [2] a model given by eqs.(4) is developed using the Clifford bundle over space-time (monopoles without strings).The theory obtained rivalizes in mathematical beauty with the topological monopole theory,and can be easily generalized for non "abelian "monopoles without strings [3].However these formulations lack a geometrical interpretation since as presented they are not PFB models.

In what follows we shall present a theory of electrodynamics with monopoles where eqs.(4) resut from a PFB model.We emphasize once more that in our theory space-time have no holes.

The mathematical structure we shall use in our gauge theory with monopoles without strings is the spliced bundle $\pi_{12}:P \circ P \to M$ with group $G\times G$ obtained from two identical PFB each one describying an ordinary gauge theory.($\pi:P\to M$ with group G and base the space-time). We also observe that in what follows M may be a general Lorentzian manifold with non zero curvature.We will follow the notations of [4], but our coderivative operator is defined as,

$$\delta\omega_p=(-1)^p(^*)^{-1}d\,{}^*\omega_p, \quad \omega_p\in \Lambda^p\tau^*M \tag{5}$$

In our theory we associate the gauge potential with a connection ω in $\pi_{12}:P \circ P\to M$, i.e., given a choice of the gauge in the PFB, $T_U: \pi_{12}^{-1}(U)\to U\times G\times G$ with the associated local section $\sigma_U:U\to P\circ P$,we define $\omega_U = \sigma_U^* \omega$ the gauge potential associated with the chosen gauge.

We observe that there exist [4] two connections ω_1 and ω_2 in $\pi:P\to M$ such that $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$. It is fundamentally different to use, for describing the gauge potential in the theory of generalized monopoles without string,

(i) a connection $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2 \in \Lambda^1(P\circ P, \hat{G}\oplus\hat{G})$ in the spliced bundle, or

(ii) two connections $\omega_1,\omega_2\in \Lambda^1(P,\hat{G})$ in the original PFB of the theory without monopoles.

Let us consider first the case (i): Let then $T_U:\pi_{12}^{-1}(U) \to U\times(G\times G)$ and $T_V:\pi_{12}^{-1}(V) \to V\times(G\times G)$ be two gauges in $\pi_{12}: P\circ P\to M$ and such that $U\cap V\neq\emptyset$, and let be $\sigma_U:U\to P\circ P$ and $\sigma_V:V\to P\circ P$ the associated local sections.

The transference functions $g_{UV}:U\cap V\to G\times G$ are such that $g_{UV}(x)=((g_1)_{UV}(x),(g_2)_{UV}(x))$ with $x\in U\cap V$. Since

$$\omega = \pi^{1*}\omega_1\oplus\pi^{2*}\omega_2 \tag{6}$$

takes its values in $\hat{G}\oplus\hat{G}$, the gauge transformation between the gauge potentials ω_U and ω_V can be written as the two relations

$$(\omega_i)_V = ((g_i)_{UV})^{-1}d(g_i)_{UV} + ((g_i)_{UV})^{-1}(\omega_i)_U(g_i)_{UV}\ , \ i=1,2. \tag{7}$$

In the case of standard electrodynamics, we can write

$$(\omega_1)_V = (\omega_1)_U + id\chi_{UV} \quad ; \quad (\omega_2)_V = (\omega_2)_U + id\psi_{UV} \tag{8}$$

with $(g_1)_{UV}(x) = \exp i\chi_{UV}$ and $(g_2)_{UV}(x)=\exp i\psi_{UV}$, and $\chi_{UV}, \psi_{UV}:U\cap V\to R$.

In that way a gauge transformation of $\omega_U\in \Lambda^1(U,\hat{G}\oplus\hat{G})$ corresponds to *two independent gauge transformations* of $(\omega_1)_U$ and $(\omega_2)_U\in \Lambda^1(U,\hat{G})$.

In the case (ii) we have two connections ω_1 and ω_2 in the PFB $\pi:P\rightarrow M$ with group G. Let $T_U:\pi^{-1}(U)\rightarrow U\times G$ and $T_V:\pi^{-1}(V)\rightarrow V\times G$ be two gauges an let $U\cap V\neq\emptyset$. Let moreover $\sigma_U:U\rightarrow P$ and $\sigma_V:V\rightarrow P$ be the associated local sections. The transition function now is $g_{UV}:U\cap V\rightarrow G$, and we have

$$(\omega_i)_V = (g_{UV})^{-1}dg_{UV} + (g_{UV})^{-1}(\omega_i)_U g_{UV}\ ,\ \ i=1,2.$$

In the case of standard electrodynamics we have

$$(\omega_i)_V = (\omega_i)_U + id\chi_{UV}\ ;\ \chi_{UV}:\ U\cap V\rightarrow R$$

These relations show that both potentials $(\omega_1)_U$ and $(\omega_2)_U$ change by effects of the same gauge transformation in case (ii. This makes clear the difference between (i) and (ii). In what follows we adopt the choice (i).

To show the necessity of such a choice, let us consider first the case of electrodynamics *with* monopoles described by two potentials: one related to the electric charges and the other to the magnetic charges as in [2,3].

Notice that in the case of standard electrodynamics without monopoles, it is $G=U(1)$ and the gauge potential takes its valued in $iR=\hat{G}$,i.e., $(\omega_1)_U = -iA_\mu dx^\mu \in \Lambda^1(U,iR)$.

The gauge field is then

$$i(\Omega_1)_U = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu \in \Lambda^2(U,iR) \tag{9}$$

Observe that the field is invariant under the gauge transformation

$$(\omega_1)_U\rightarrow(\omega_1)_U + id\chi \tag{10}$$

This information is interpreted geometrically as a choice of gauge, or local trivialization, and the associated transition function is $g = \exp i\chi : U\cap V\rightarrow G$.

As is well known we can write

$$i(\Omega_1)_U = \frac{1}{2}(F_{0k}dx^0\wedge dx^k + F_{lm}dx^l\wedge dx^m)$$
$$= (E_k dx^0\wedge dx^k + B_k{*}(dx^0\wedge dx^k))\ ;\ \ k,l,m = 1,2,3. \tag{11}$$

and, since $(*)^2 = -1$ when applied to 2-forms, the Hodge star operator changes $\vec{E}$ into $-\vec{B}$ and $\vec{B}$ into $\vec{E}$. Let us now consider the electrodynamics with charges and monopoles. The field generated only by electric charges can be described by the usual potential

$$(\Omega_1)_U = d(\omega_1)_U \tag{10a}$$

The field generated by the magnetic charges is a dual field:

$$*(\Omega_2)_U = *d(\omega_2)_U \tag{10b}$$

The total field generated by electric and magnetic charges is given by

$$-iF_U = (\Omega_1)_U + *(\Omega_2)_U \tag{11}$$

In this way one of the potentials describes the field generated by electric, and the other the field generated by magnetic charges.

We observe that, if we make two independent gauge transformations

$$(\omega_1)_U\longrightarrow(\omega_1)_U + id\chi\ ;\ (\omega_2)_U\longrightarrow(\omega_2)_U + id\psi$$

the field F_U does not change. If we interpret the above transformations as changes in the local trivialization of a PFB, we must use a spliced bundle (due to the independence of χ and ψ). Once we justified our choice (i), we now go on with the theory.

Let us observe that the spliced bundle of two copies of the PFB $\pi:P\rightarrow M$ has each point $p\in P \circ P$ associated with two points of P *over* the same fiber. This permits us to understand that a gauge transofrmation in $\pi_{12}:P \circ P \rightarrow M$ corresponds to two gauge transformations in $\pi:P\rightarrow M$.Indeed, $\sigma_U:U\rightarrow P\circ P$ corresponds to $\sigma^1{}_U=\pi^1\circ\sigma_U:U\rightarrow P$ and also to $\sigma^2{}_U=\pi^2\circ\sigma_U:U\rightarrow P$. In this way we can associate to a given connection ω in P two gauge potentials $\omega_U=\sigma_U^{1*}\,\omega$ and $\omega_U=\sigma_U^{2*}\,\omega$. Observe that

$$\omega_U = (\pi^1\circ\,\sigma_U)^*\omega = \sigma_U^*\,(\pi^{1*}\omega) = \sigma_U^*\,(\pi^{1*}\omega\oplus 0) \tag{12a}$$

and

$$\tilde{\omega}_U = (\pi^2\circ\,\sigma_U)^*\omega = \sigma_U^*\,(\pi^{2*}\omega) = \sigma_U^*\,(0\oplus\pi^{2*}\omega) \tag{12b}$$

where ω_U and $\tilde{\omega}_U$ correspond to gauge potentials associated with the 1-forms $\pi^{1*}\omega\oplus 0$ and $0\oplus\pi^{2*}\omega$, which are possible extensions of ω to the spliced bundle. This shows that, given two connections ω_1 and ω_2 in $\pi:P\rightarrow M$, we can associate with them two distinct connections $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$ and $\tilde{\omega}=\pi^{1*}\omega_2\oplus\pi^{2*}\omega_1$ in $\pi_{12}:P\circ P\rightarrow M$. It is well known that, given a connection ω on the spliced bundle, we have two connections ω_1 and ω_2 well defined on the original fiber bundles.

We see now that, when both PFB are equal, the connections ω_1 and ω_2 can generate another connection $\tilde{\omega}$ on the spliced bundle. We call $\tilde{\omega}$ the connection dual to ω .

Observe that we have two curvatures $\Omega^{\omega} = D^{\omega}\,\omega$ and $\Omega^{\tilde{\omega}} = D^{\tilde{\omega}}\,\tilde{\omega}$ associated to the connections ω and $\tilde{\omega}$. These curvatures must, by the Bianchi identities satisfy, $D^{\omega}\omega=0$ and $D^{\tilde{\omega}}\,\tilde{\omega}=0$. Before we analyse these identities, we must understand some of the properties of the horizontal forms in a spliced bundle.

If $\tau\in\overline{\Lambda}^k(P_1\circ P_2,\,\hat{G}_1\oplus\hat{G}_2)$ with the adjoint representation $G_1\times G_2\rightarrow\hat{G}_1\oplus\hat{G}_2$, it holds

(a) we can write $\tau=\pi^{1*}\tau_1+\pi^{2*}\tau_2$ with $\tau_1\in\overline{\Lambda}^k(P_1,\hat{G}_1)$ and $\tau_2\in\overline{\Lambda}^k(P_2,\hat{G}_2)$,where we use the adjoint representations $Ad: G_i\rightarrow\hat{G}_i$,$i = 1,2$.

(b) $$D^{\omega}\tau=\pi^{1*}D^{\omega_1}\tau_1+\pi^{2*}D^{\omega_2}\tau_2\ ;\ \text{for } \omega=\pi^{1*}\omega_1+\pi^{2*}\omega_2 \tag{13}$$

(c) Let $\overline{*}_{12}: \overline{\Lambda}^k(P_1\circ P_2,\hat{G}_1\oplus\hat{G}_2)\rightarrow\overline{\Lambda}^{n-k}(P_1\circ P_2,\hat{G}_1\oplus\hat{G}_2)$ be the Hodge operator for horizontal forms in $P_1\circ P_2$. (n=dimension of M) and $\overline{*}_1:\overline{\Lambda}^k(P_1,\hat{G}_1)\rightarrow\overline{\Lambda}^{n-k}(P_1,\hat{G}_1)$; $\overline{*}_2:\overline{\Lambda}^k(P_2,\hat{G}_2)\rightarrow\overline{\Lambda}^{n-k}(P_2,\hat{G}_2)$ be the Hodge operators for horizontal forms in the original PFB. Then $\overline{*}_{12}\tau = \pi^{1*}(\overline{*}_1\tau_1)\oplus\pi^{2*}(\overline{*}_2\tau_2)$.

We employ the relations (a), (b) and (c) in the following way. Returning to the curvature $\Omega^{\omega}\in \overline{\Lambda}^2(P_1 o P_2,\hat{G}_1\oplus\hat{G}_2)$ we can write $\Omega^{\omega}=\pi^{1*}\Omega_1\oplus\pi^{2*}\Omega_2$, where $\Omega_1,\Omega_2\in \overline{\Lambda}^k(P,\hat{G})$ are well defined. We now prove that if $\omega=\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2$, we have $\Omega_1=\Omega^{\omega_1}=D^{\omega_1}\omega_1$ and $\Omega_2=\Omega^{\omega_2}=D^{\omega_2}\omega_2$.

Indeed,

$$\Omega^{\omega}=d\omega+\frac{1}{2}[\omega,\omega]=d(\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2)+\frac{1}{2}[\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2,\pi^{1*}\omega_1\oplus\pi^{2*}\omega_2]$$
$$=\pi^{1*}D^{\omega_1}\omega_1\oplus\pi^{2*}D^{\omega_2}\omega_2=\pi^{1*}\Omega^{\omega_1}\oplus\pi^{2*}\Omega^{\omega_2}$$

Moreover, we have $\Omega^{\tilde{\omega}}=\pi^{1*}\Omega^{\omega_2}\oplus\pi^{2*}\Omega^{\omega_1}$.

In this way, the Bianchi identities $D^{\omega}\Omega^{\omega}=0$ and $D^{\tilde{\omega}}\Omega^{\tilde{\omega}}=0$ according to (b) are equivalent, and correspond to $D^{\omega_1}\omega_1=D^{\omega_2}\omega_2=0$, which are the Bianchi identities associated to ω_1 and ω_2 in $\pi: P\rightarrow M$. In electrodynamics, these equations imply that $d(\Omega_1)_U=0$ and $d(\Omega_2)_U=0$, which *physically* mean that both the magnetic field of electric origin and the electric field of magnetic origin have null divergences.

In what follows we are going to generalize the gauge principle[4] for a gauge theory with our monopoles.

Let be $\pi: PoP\rightarrow M$ a spliced bundle with group $G\times G$ and let be $G\times G\rightarrow GL(V)$ a representation of $G\times G$. We remember that the space of 1-jets of the mappings from P to V is:

$$J(PoP,V)\equiv\{(p,v,\theta)\ |p\in P\ o\ P,\ v\in V \text{ and } \theta: T_pP\ o\ P \text{ is linear}\}$$

We call a Lagrangian the mapping $L: J(P\ o\ P,V)\rightarrow R$ such that for all $(p,v,\theta)\in J(P\ o\ P,V)$ and $g\in G\times G$, we have

$$L(pg,g^{-1}v,g^{-1}\theta\circ R_{g^{-1}*})=L(p,v,\theta)$$

If $L(p,g^{-1}v,g\cdot\theta)=L(p,v,\theta)$ then L is said to be $G\times G$ -invariant, and in what follows we suppose L to have this property.

Now given a Lagrangian $L: J(P\ oP,V)\rightarrow R$, let $\mathcal{C}$ be the space of the connections in PoP. Define the action density by

$$\mathcal{L}: C(P,V)\times\mathcal{C}\rightarrow C^{\infty}(M) \tag{14}$$

where $C^{\infty}(M)$ are the set of the C^{∞} functions on M.

We have,

$$\mathcal{L}(x)=L(p,\Psi(p),D^{\omega}\Psi(p)) \tag{15}$$

where $x\in M$, $p\in\pi^{-1}(x)$ and the generalized wave function (matter field) describing the electric and the magnetic particles is $\Psi\in\Lambda^o(P\ o\ P,V)\equiv C(P\ o\ P,V)$.

Then $\mathcal{L}$ is not only well defined but is also gauge invariant in the sense that for all $f\in GA(P\ o\ P)$, we have, $\mathcal{L}(f^*\Psi,f^*\omega)=\mathcal{L}(\Psi,\omega)$. $GA(P\ o\ P)$ is the gauge algebra of the spliced bundle, more precisely it is the space $C(Po\ P,\hat{G}\oplus\hat{G})$ with the adjoint representation $G\times G\rightarrow GL(\hat{G}\oplus\hat{G})$; $g\rightarrow Ad_g$.

If we impose that $\mathcal{L}^{\omega}(\Psi)$ is stationary with respect to Ψ, we obtain the Euler-Lagrange equations[4]. We show now that, if we add an appropriate term $S(\omega)$, to $\mathcal{L}^{\omega}(\Psi)$, obtaining then the total action $(\mathcal{L}+S)(\Psi,\omega)$, this density will generate not only the Euler-Lagrange equations for Ψ but also the non-homogeneous field equations. More precisely, these results follow once we impose that $(\mathcal{L}+S)(\Psi,\omega)$ is stationary with respect to the pair (Ψ,ω). We will see that the non-homogeneous equations obtained in this way correspond in the case of electrodynamics to Maxwell equations with monopoles without strings. We define the autoaction by

$$S(\omega) = -\frac{1}{4}\,\hbar k_{12}(\mathcal{F}_U,\mathcal{F}_U)\,; \quad \mathcal{F} = \Omega^{\omega} + \overline{*\Omega^{\tilde{\omega}}}\,; \quad \mathcal{F}_U = \sigma_U^{*}\,\mathcal{F} \tag{16}$$

where

$$\hbar k_{12}\colon \overline{\Lambda}^k(P \circ P,\, \hat{G}\oplus\hat{G}) \times \overline{\Lambda}^k(P \circ P,\, \hat{G}\oplus\hat{G}) \to \mathbf{R}$$

is the metric for horizontal forms in $(\hat{G}\oplus\hat{G})$ (with the adjoint representation). We observe that k_{12} is the Killing-Cartan metric in $(\hat{G}\oplus\hat{G})$ and that $k_{12}(A_1\oplus A_2, B_1\oplus B_2) = k(A_1,B_1)+k(A_2,B_2)$, where k is the Killing-Cartan metric in $\hat{G}$ [4].

Let us observe that, as $S(\omega)$ is F-equivariant, it is gauge invariant as required for the autoaction term[4]. Let us observe also that, had we constructed the autoaction term as $\frac{1}{2}\hbar k_{12}(\Omega^{\omega},\Omega^{\omega})$, there would be no interaction between charges and monopoles. Indeed, take the case of electrodynamics where $\Omega^{\omega} = \pi^{1*}\omega_1 \oplus \pi^{2*}\omega_2$, then Ω_1 and Ω_2 correspond to the fields generated by charges and monopoles:

$$\hbar k_{12}(\Omega^{\omega},\Omega^{\omega}) = \hbar k(\Omega_U^{\omega 1},\, \Omega_U^{\omega 1}) + \hbar k(\Omega_U^{\omega 2},\, \Omega_U^{\omega 2}) \tag{17}$$

and there are not, in this expression, interaction terms between the fields Ω_1 and Ω_2. For $S(\omega)$, instead, we have

$$-\frac{1}{4}\hbar k_{12}(\mathcal{F}_U,\mathcal{F}_U) = -\frac{1}{2}\hbar k(\Omega_U^{\omega 1},\, \Omega_U^{\omega 1}) - \frac{1}{2}\hbar k(\Omega_U^{\omega 2},\, \Omega_U^{\omega 2}) - \hbar k(\Omega_U^{\omega 1}, *\Omega_U^{\omega 2}) \tag{18}$$

where the interaction term appears explicitly.

Before we apply the variational principle to the total action, let us remember the definition of the current in terms of the Lagrangian

$$\frac{d}{dt}\mathcal{L}(\Psi,\, \omega + t\,\sigma)\,|_{t=0} = \hbar k_{12}(J^{\omega}(\Psi),\, \sigma)$$

$\forall \sigma \in \overline{\Lambda}^1(P \circ P,\, \hat{G}\oplus\hat{G})$. In this case $J^{\omega}(\Psi) \in \overline{\Lambda}^1(P \circ P, \hat{G}\oplus\hat{G})$ and we can write $J^{\omega}(\Psi) = -\pi^{1*}J_1 \oplus \pi^{2*}J_2$ so that we can asociate J_1 and $J_2 \in \overline{\Lambda}^1(P,\hat{G})$ to the "electric" and "magnetic" currents, respectively.

Effecting the variation at $t=0$ $(\sigma = \pi^{1*}\sigma_1 \oplus \pi^{2*}\sigma_2)$ we get

$$\frac{d}{dt}\int_U (\mathcal{L}+S)(\Psi + t\tau,\, \omega + t\tau)\mu$$

$$= \int_U \mathcal{L}(\Psi + t\tau,\, \omega)\mu + \frac{d}{dt}\int_U S(\Psi,\, \omega + t\tau)\mu$$

$$= \frac{d}{dt}\int_U \frac{1}{2}\,\hbar k(\,\Omega^{\omega_1}_U + t\sigma_1,\ \Omega^{\omega_1}_U + t\sigma_1)\mu$$

$$-\frac{d}{dt}\int_U \frac{1}{2}\,\hbar k(\,\Omega^{\omega_2}_U + t\sigma_2,\ \Omega^{\omega_2}_U + t\sigma_2)\mu$$

$$-\frac{d}{dt}\int_U \frac{1}{2}\,\hbar k(\,\Omega^{\omega_1}_U + t\sigma_1,\ \bar{*}\,\Omega^{\omega_2}_U + t\sigma_2)\mu \qquad (19)$$

We have at t=0 for the four first terms in eq (19)

$$\frac{d}{dt}\int_U \mathcal{L}(\Psi + t\tau, \omega)\mu \;=\; \int_U \hbar\Big(\delta^{\omega}\frac{\partial L}{\partial(D^{\omega}\Psi)} + \frac{\partial L}{\partial \Psi}, \tau\Big)\mu$$

$$\frac{d}{dt}\int_U \mathcal{L}(\Psi, \omega + t\tau)\mu \;=\; \int_U \hbar k_{12}(J^{\omega}(\Psi), \tau)\mu$$

$$\frac{d}{dt}\int_U \frac{1}{2}\hbar k(\,\Omega^{\omega_1}_U + t\sigma_1,\ \Omega^{\omega_1}_U + t\sigma_1)\mu = -\int_U \hbar k(\delta^{\omega_1}\Omega^{\omega_1}, \sigma_1)\mu$$

$$\frac{d}{dt}\int_U \frac{1}{2}\hbar k(\,\Omega^{\omega_2}_U + t\sigma_2,\ \Omega^{\omega_2}_U + t\sigma_2)\mu = -\int_U \hbar k(\delta^{\omega_2}\Omega^{\omega_2}, \sigma_2)\mu$$

Moreover:

$$\frac{d}{dt}\hbar k(\,\Omega^{\omega_1 + t\sigma_1}, \bar{*}\,\Omega^{\omega_2 + t\sigma_2})$$

$$= \frac{d}{dt}\hbar k(\,\Omega^{\omega_1 + t\sigma_1}, \bar{*}\,\Omega^{\omega_2}) + \frac{d}{dt}\hbar k(\,\Omega^{\omega_1}, \bar{*}\,\Omega^{\omega_2 + t\sigma_2})$$

and, at t=0 , we have $\frac{d}{dt}\Omega^{\omega + t\sigma} = D^{\omega}\sigma$[4] .Then

$$\frac{d}{dt}\hbar k(\,\Omega^{\omega_1 + t\sigma_1}, \bar{*}\,\Omega^{\omega_2 + t\sigma_2})$$

$$= \hbar k(\,D^{\omega_1}\sigma_1, \bar{*}\,\Omega^{\omega_2 + t\sigma_2}) + \hbar k(\,\Omega^{\omega_1}, \bar{*}\,D^{\omega_2}\sigma_2)$$

$$= \hbar k(\,\delta^{\omega_1}(\bar{*}\,\Omega^{\omega_2}), \sigma_1) + \hbar k(\,\delta^{\omega_2}(\bar{*}\,\Omega^{\omega_1}), \sigma_2)$$

and we obtain for the last term in eq (19)

$$\frac{d}{dt}\int_U \hbar k(\,\Omega^{\omega_1 + t\sigma_1}, \bar{*}\,\Omega^{\omega_2 + t\sigma_2}) = \int_U \big(\hbar k(\,\delta^{\omega_1}(\bar{*}\,\Omega^{\omega_2}), \sigma_1) + \hbar k(\,\delta^{\omega_2}(\bar{*}\,\Omega^{\omega_1}), \sigma_2)\big)\mu$$

Now, summing all the terms obtained and taking into account that t, σ_1 and σ_2 are all independent, and also that

$$\hbar k_{12}(J^{\omega}(\Psi),\sigma) = -\hbar k(J_1,\sigma_1) + \hbar k(J_2,\sigma_2) \tag{20}$$

we get the equations

$$\delta^{\omega}\frac{\partial L}{\partial(D^{\omega}\Psi)} + \frac{\partial L}{\partial\Psi} \tag{21}$$

$$\delta^{\omega_1}\Omega^{\omega_1} + \delta^{\omega_1}(\bar{*}\,\Omega^{\omega_2}) = -J_1 \tag{22}$$

$$\delta^{\omega_2}\Omega^{\omega_2} + \delta^{\omega_2}(\bar{*}\,\Omega^{\omega_1}) = J_2 \tag{23}$$

Eq (21) corresponds to the Euler-Lagrange equation, which gives the equation of the generalized field describing the motion in PoP of charges and monopoles. We are not going to investigate in this paper the nature of $\mathcal{L}^{\omega}(\Psi)$.

Eqs (22) and (23) can be written, putting $\Omega=\Omega^{\omega_1} + {*}\Omega^{\omega_2}$, as

$$\delta^{\omega_1}\Omega = -J_1 \tag{24}$$

$$\delta^{\omega_2}(\bar{*}\,\Omega) = J_2 \;\Leftrightarrow\; D^{\omega_2}\,\Omega = -{*}J_2 \tag{25}$$

which are the non-homogeneous equations of the theory.

In the case of electrodynamcics these equations reduce to

$$\delta\Omega = -J_1 \tag{26a}$$

$$d\Omega = -{*}J_2 \tag{26b}$$

which we recognize as the Maxwell equations for the electromagnetic field $\Omega=\Omega^{\omega_1} + {*}\Omega^{\omega_2}$ generated by electric and magnetic charges. We have for Ω^{ω_1} and Ω^{ω_2} the equations.

$$\delta\Omega^{\omega_1} = -J_1\,;\;\; \delta\Omega^{\omega_2} = -J_2 \tag{27}$$

since $d\Omega^{\omega_1} = d\Omega^{\omega_2}=0$. Also, since $\Omega^{\omega_1}=d\omega_1$, $\Omega^{\omega_2}=d\omega_2$, we have

$$\Box\omega_1= J_1\,;\;\; \Box\omega_2= J_2 \tag{28}$$

where $\Box = -(d\delta+\delta d)$.

In conclusion we presented in this paper a geometrical theory of magnetic monopoles without strings. It is important to emphazise the crucial difference between our theory and the usual presentations of the subject, namely the string theory by Dirac and the topological monopole theory, where the monopole appears associated with a change in the topology of the world manifold.

We succeed in giving to the theory of magnetic monopoles without strings a principal fiber bundle structure: a spliced bundle with group $G\times G$. We obtain the equation of the generalized field in our theory using in the spliced bundle a generalization of the

Principle of the Stationary Action. We postulate the existence of a Lagrangian density $\mathcal{L}^{\omega}(\Psi)$ for the generalized field, that describes in the spliced bundle the "electric" and "magnetic" matter; but we do not use explicitly any Lagrangian to deduce the equations of the generalized matter field. Indeed, $\mathcal{L}^{\omega}(\Psi)$ is used only to produce the currents. The question of the existence of $\mathcal{L}^{\omega}(\Psi)$ and its form in this formalism will be investigated in another paper.

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