ROTATION NUMBERS OF DIFFERENTIAL EQUATIONS. A FRAMEWORK IN THE LINEAR CASE

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1. INTRODUCTION

The rotation number $rot_{A(t)}$ of a two-dimensional linear system

$$\dot{x} = A(t)x \quad x \in \mathbb{R}^2 \quad t \in \mathbb{R} \tag{1}$$

is defined as the assymptotic mean of the values of the angles assumed by its solutions, that is,

$$rot_{A(t)} = \lim_{t\to\infty} \frac{\theta(t)}{t}$$

where $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function defining the angular part of a nonzero solution of (1). This limit, its existence or not, is shown to be independent of the solution taken and when $A(t) \equiv A$ is time independent, rot_A is just the imaginary part of the eigenvalues of A.

The aim of this paper is to propose an extension of this notion to higher dimensions and more general systems.

Our procedure is patterned after the theory of Lyapunov characteristic numbers whose main effort is to describe the assymptotic stability of solutions or trajectories of dynamical systems. Thus if T_t is a differentiable dynamical system evolving on a manifold M, its Lyapunov exponent in the direction of a vector v tangent to $x_0 \in M$ is defined as $\lim_{t\to\infty} 1/t \quad log||T_{t^*}v||$, T_{t^*} being the differential of T_t. This exponent gives information about the relative distance between the trajectory starting at x_0 and the trajectories starting at points which are near x_0 and which are in the direction of v from x_0 (c.f. [1], [4]). On the algebraic side, the Lyapunov exponents of a time independent linear system $\dot{x} = Ax$ at the trivial solution are the real parts of the eigenvalues of A, thus giving a meaning for these real parts when the issue is a more complicated system like nonlinear systems, stochastic systems, etc... . Note that if we look inside the one-dimensional subspace spanned by $T_t v$, $||T_t v||$ gives the position of the point $T_t v$ when this is measured with respect to the set of vectors u with $||u|| = 1$. In other words, a Lyapunov characteristic number is obtained by performing successive measurements of length inside a moving one-dimensional subspace.

Rotation numbers will be given in a similar way, by measuring movements inside subspaces which are themselves moving. The point now is that we consider rotation as a bidimensional phenomenon. Therefore we shall talk of the rotation number of the trajectory starting at x_0 in the direction of a two-dimensional subspace tangent to x_0 and not any more in the direction of a one-dimensional subspace like the Lyapunov characteristc numbers. We thus look at a point moving inside a two-dimensional subspace and measure its angle with a reference point. However, unlike the measuring of length – which can always be made by referring to a fixed set – there is no global reference set to measure angles. Because of this we shall instead measure "infinitesimal angles" with the aid of a connection on a fibre bundle and add up these infinitesimal angles along the trajectories. The reference point inside the two-dimensional subspaces will then be given by parallel transporting the starting point. This approach will force us to deal only with dynamical systems which are infinitesimally defined, that is, with differential equations and thus avoiding discrete time systems.

The purpose of this paper is to detail this framework for linear systems. We leave out for the future further advances like the details for nonlinear systems and some convergence theorems (in the spirit of the Multiplicative Ergodic Theorem of Oseledec). We try to motivate the whole discussion and we hope to justify our definition with a study of the time independent linear systems in §4, showing the way in which the imaginary parts of the eigenvalues can be recovered.

2. LYAPUNOV EXPONENTS

We shall start by drawing some comments about a geometrical framework underlying the theory of Lyapunov exponents which inspired our notion of rotation numbers.

Consider the linear differential equation

$$\dot{v} = A(t)v \quad v \in \mathbb{R}^n \tag{2}$$

and put $\dot{g} = A(t)g_t$ with $g_t \in G\ell^+(n, \mathbb{R})$, $g_0 = 1$. Its Lyapunov exponent in the direction of $v_0 \in \mathbb{R}^n - 0$ is $\lambda(v_0) = \lim_{t\to\infty} \frac{\Lambda(t)}{t}$ where $\Lambda(t) = log||v_t||$ and $v_t = g_t v_0$. This exponent gives information about the assymptotic behaviour of $||v_t||$, which is, of course, the $\mathbb{R}^+$-component of v_t with respect to the polar decomposition $\mathbb{R}^n - 0 = S^{n-1} \times \mathbb{R}_+$.

Thus if we speak the language of fibre bundles, the issue is to look at the fibre component of the principal bundle $S^{n-1} \times \mathbb{R}_+ \longrightarrow S^{n-1}$. The group of this bundle is $\mathbb{R}_-$ and since it is a trivial bundle it makes sense to talk about the projection onto fibres and hence of the fibre component of its elements.

In the sequel we shall be dealing with non trivial bundles for which this global projection onto fibres is meaningless. For these bundles we will make instead infinitesimal projections, that is, projections with respect to decompositions of the tangent space furnished by connections (see [1] for the theory of connection on principal bundles). So let us forget for a moment the possibility of globally trivializing $\mathbb{R}^n - 0 \longrightarrow S^{n-1}$ and consider the decomposition $T_v(\mathbb{R}^n - 0) = V_v \oplus H_v$ of the tangent space at v into its radial direction V_v and H_v, the tangent space to the sphere centered at the origin passing through v.

The assignment $v \longrightarrow H_v$ is a connection on $R^n - 0 \longrightarrow S^{n-1}$ whose connection form δ is at $v \in \mathbb{R}^n - 0$ given by

$$\delta_v(w) = \frac{\langle w, v\rangle}{\langle v, v\rangle}, \quad w \in T_v(\mathbb{R}^n - 0). \tag{3}$$

Returning to the above expression for $\lambda(v_0)$, we can rewrite it as

$$\lambda(v_0) = \lim_{T\to\infty} 1/T \int_0^T \Lambda'(t)dt. \tag{4}$$

But $\Lambda'(t) = \langle A(t)v_t, v_t/||v_t||^2 = \delta_{v_t}(A(t)v_t)$, so we get

$$\lambda(v_0) = \lim_{T\to\infty} 1/T \int_0^T \delta_{v_t}(\dot{v}_t)dt. \tag{5}$$

From this we see that $\lambda(v_0)$ is the same as the Lyapunov exponent of the 1-dimensional linear system $\dot{x} = B(t)x$, where $B(t) = \delta_{v_t}(\dot{v}_t)$. In other words, Lyapunov exponents (which are given by movement along 1–dimensional rays) are obtained from 1-dimensional linear systems constructed from the connection δ and the trajectory v_t. The same kind of idea will be used to introduce rotation numbers. Let us remark however, the following facts

i) If we put $u_t = ||v_0||\frac{v_t}{||v_t||}$ then u_t is the horizontal lifting – starting at v_0 – of the projection of v_t onto S^{n-1}. $B(t)$ measures the relative movement of $u(t)$ and $v(t)$ and we have $\Lambda'(t) = \langle A(t)u_t, u_t\rangle / \langle u_t, u_t\rangle$. This last equality is essentially due to the commutativeness of the structure group $\mathbb{R}_+$ of $\mathbb{R}^n - 0 \longrightarrow S^{n-1}$.

ii) The approach above – via the connection δ – is also possible in the bundle $\mathbb{R}^n - 0 \longrightarrow \mathbb{R}P^{n-1}$ which is not trivial and hence there is no global projection onto fibres. Note that this bundle is identified with the bundle over $\mathbb{R}P^{n-1}$ whose fibre over $\varsigma \in \mathbb{R}P^{n-1}$ is the set of frames $p : \mathbb{R} \longrightarrow \varsigma$ of ς;

iii) The connection δ is integrable in the sense that the distribution $v \longrightarrow H_v$ admit integral manifolds (the spheres). Its curvature anihilates and it is locally flat. This property will not be shared by the connection defining rotation numbers. This will force us to deal with differential equations only. In §6 below we make further comments about flat connections, relating them to the cocycles.

iv) Another feature of the above connection is that it is invariant by the action of the orthogonal group, i.e., $g_* H_v = H_{gv}$ if g an orthogonal matrix.

3. THE DEFINITON

The Lyapunov exponents measures movements inside 1–dimensional subspaces. This is the reason why the projective space appears in the framework depicted in §2.

Now, we wish to deal with rotation as bidimensional rotation so we consider the same kind of movement but inside two-dimensional subspaces. We shall thus establish a framework similar to the one for Lyapunov exponents but in a bundle over the Grassmannian $GR_2(n)$ of 2-spaces in $\mathbb{R}^n$.

The bundle over $Gr_2(n)$ analogous to $\mathbb{R}^n - 0 \longrightarrow \mathbb{R}P^{n-1}$ is the bundle $B_2(n)$ in which the fibre over $\varsigma \in Gr_2(n)$ is the set of frames of the subspace ς. More precisely, set

$$B_2(n) = \{p : \mathbb{R}^2 \longrightarrow \mathbb{R}^n : p \text{ is linear and one} - \text{to} - \text{one}\}$$

and define $\pi : B_2^n \longrightarrow Gr_2(n)$ by $\pi(p) = image\ of\ p$. This defines a principal bundle over $Gr_2(n)$ whose group is $G\ell(2, \mathbb{R})$. The right action of $G\ell(2, \mathbb{R})$ on $B_2(n)$ is $R_a(p) = pa = p \circ a$, $p \in B_2(n)$, $a \in G\ell(2, \mathbb{R})$. Fixing basis of $\mathbb{R}^2$ and $\mathbb{R}^n$, $B_2(n)$ becomes the set of rank 2, $n \times 2$ matrices and $R_a(p)$ is just multiplication on the right.

$G\ell(n, \mathbb{R})$ acts transitively on $B_2(n)$ by left multiplication: $g(p) = g \circ p = gp$, $g \in G\ell(n, \mathbb{R})$, $p \in B_2(n)$. This action commutes with the right action of $G\ell(2, \mathbb{R})$ and is compatible with the projection $\pi : B_2(n) \longrightarrow Gr_2(n)$, i.e., $\pi \circ g = g \circ \pi$, where g on the right hand side represents the action of g on $Gr_2(n)$.

Any $n \times n$ matrix A induces a vector field $\hat{A}(p) = d/dt\ e^{tA_t}|_{t=0} = Ap$ on $B_2(n)$. Since $g \in G\ell(n, \mathbb{R})$ commutes with R_a, $a \in G\ell(2, \mathbb{R})$, we have that $R_a^*(\hat{A}(R_a p))$.

Let $g_t, t \geq 0, g_0 = 1$ be a differentiable path in $G\ell(n, \mathbb{R})$ and form from it a system like (2) by putting $A(t) = \dot{g}_t g_t^{-1}$. Clearly, if $p \in B_2(n), p_t = g_t p$ is differentiable and $\dot{p}_t = \hat{A}(t)p$.

Now, the idea to define the rotation numbers of $A(t)$ (or g_t) is to introduce a connectiono on $B_2(n) \longrightarrow Gr_2(n)$ (which somehow measures "infinitesimal rotation") and use it together with $\hat{A}(t)$ to induce via $\alpha(\hat{A}(t))$ a two-dimensional linear system whose rotation number gives what is desired.

The connection α will be specified later and is chosen in such a way that $\alpha_p(u), u \in T_p B_2(n)$ measures the infinitesimal sliding of the two plane $\varsigma = \pi(p)$ against itself in the direction of u.

Let us see first how to define the linear system on $\mathbb{R}^2$ by means of $\alpha(\hat{A}(t))$. Like in the case of the Lyapunov exponents, this system is required to measure the rising of $p_t = g_t p$ along the vertical part.

Put $\varsigma_t = \phi(p_t) \in Gr_2(n)$ and denote by q_t its horizontal lifting (w.r.t. α) which starts at p_0. Let $a_p \in G\ell(2, \mathbb{R})$ defined by means of

$$p_t = q_t a_t. \tag{6}$$

Then, following the pattern of §2, we should define the rotation number of $A(t)$ starting in or in the direction of the two-frame p_0 as the rotation number of the path $a_t \in G\ell(2, \mathbb{R})$ viewed as maps in $\mathbb{R}^2$.

Evaluating α on both sides of the derivative of (6), we have that $a_t^{-1}\dot{a}_t = \alpha_{p_t}(\dot{p}_t)$ and hence that $\dot{a}_t a_t^{-1} = a_t \alpha_{p_t}(\dot{p}_t) a_t^{-1}$, and since $\dot{p}_t = \hat{A}(t)p_t$ and $\hat{A}(t)$ is right invariant, we finally get that

$$\dot{a}_t a_t^{-1} = \alpha_{q_t}(\hat{A}(t)(q_t)) \tag{7}$$

Therefore the rotation number of $\dot{g} = A(t)g$ in the direction of the two-frame p_0 is defined as the rotation number of the 2-dimensional linear system

$$\dot{x} = \alpha_{q_t}(\hat{A}(t)q_t)x. \tag{8}$$

This way of defining rotation number depends actually only on the orientation of the two-plane $\varsigma = \pi(p_0)$.

In fact, if p_0^1 is another frame of ς_0, there exists $b \in G\ell(2, \mathbb{R})$ such that $p_0^1 = p_0 b$. So if we put $p_t^1 = g_t p_0^1$ then $\varsigma_t = \pi(p^t) = \pi(p_t^1)$. Hence if q_t^1 denotes the horizontal lifting of ς_t starting at p_0^1, we have $q_t^1 = q_t b$. This together with the invariance of the vector field $\hat{A}(t)$ by R_{b^*}, entails that $\alpha_{q_t^1} = b^{-1}\alpha_{q_t}(\hat{A}(t)q_t)b$. So the above affirmation follows from the next proposition.

Proposition 3.1. Suppose that the rotation numbers of the 2-dimensional linear system

(B) $\dot{x} = B(t)x$ exists. Take $b \in G\ell(2, \mathbb{R})$ and form the system

(C) $\dot{y} = C(t)y$; $C(t) = b^{-1}B(t)b$.

Then $rot_{C(t)} = \pm\, rot_{B(t)}$, the signal taken according to the signal of $det b$.

Proof Denote by ς_1 and ς_2 the angular parts of the solutions of (B) and (C) respectively. Since $y = bx$, $\varsigma_2 = b(\varsigma_1)$, where $b(\varsigma_1)$ denotes the action of b on S^1.

Since b is continuous and invertible as a map $S^1 \longrightarrow S^1$, there is a continuous $f_b : \mathbb{R} \longrightarrow \mathbb{R}$ with $b(e^{i\theta}) = e^{if_b(\theta)}, \theta \in \mathbb{R}$ and satisfying

$$f_b(2\pi) = f_b(0) \pm 2\pi.$$

Thus we can write $f_b(\theta) = \pm\,\theta + \varpi(\theta)$ with ϖ bounded. Hence $rot_{C(t)} = \lim_{T\to\infty} \cdot \frac{\theta_2(T)}{T} = \lim_{T\to\infty} \cdot \frac{f_b(\theta_1(t))}{T} = \pm\, rot_{B(t)}$.

To check that the signal is according so that of $det\ b$, note that if $b_t \in G\ell(2, \mathbb{R})$ is continuous then $f_{bt}(2\pi) - f_{b_t}(0)$ is continuous and assume only the values $\pm\, 2\pi$ and hence is constant. Therefore, the fact that $f_1(2\pi) - f_1(0) = 2\pi$ entails that $f_b(2\pi) - f_b(0) = 2\pi$ for every b with $det\ b > 0$. Also, since $f_{b_0}(2\pi) - f_{b_0}(0) = -2\pi$ for

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have the opposite for b with $det\ b < 0$.

Because of this proposition we shall view from now on $B_2(n)$ as a principal bundle over $Gr_2^+(n)$, the Grassmannian of oriented two-planes. This is a principal bundle with group $G\ell^+(2, \mathbb{R})$ and with the projection $\pi : B_2(n) \longrightarrow Gr_2^-(n)$ the same as before but with $\pi(p)$ being oriented by p. The rotation numbers will be dependent only on the elements of Gr_2^-.

Before proceeding, let us note that in case $n = 2$, $B_2(n) = G\ell(2, \mathbb{R})$ and $Gr_2(n)$ is just one point, so the above framework reduces to the usual rotation number of 2-dimensional systems. For these systems the connection plays no role.

For the n-dimensional case we shall use a connection on $B_2(n) \longrightarrow Gr_2^-(n)$ extended from a connection on the Stiefel manifold $St_2(n) \subset B_2(n)$ of two-orthonormal frames in $\mathbb{R}^n$. This is a principal subbundle of $B_2(n)$ with group $SO(2)$. By fixing the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^n$ we get a one-to-one correspondence between $St_2(n)$ and the set of $n \times 2$ matrices p with $p^* p = I_{2\times 2}$, (p^* meaning transposition).

If $n \geq 3$, $SO(n)$ acts transitively on $St_2(n)$ by left multiplication. This action commutes with the right action of $SO(2)$. $G\ell(n, \mathbb{R})$ also acts on $St_2(n)$ by performing the orthonormalization procedure on the columns of the $n \times 2$ matrix $gp, p \in St_2(n), g \in G\ell(n, R)$. In order to distinguish the actions of $G\ell(n, \mathbb{R})$ on $B_2(n)$ and $St_2(n)$, we shall write for $p \in St_2(n)$, gp when it is to be in $B_2(n)$ and $\tilde{g}p$ when it is in $St_2(n)$. $St_2(n)$ identifies with the homogeneous space $G\ell(n, \mathbb{R})/M$ where M is the closed subgroup of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with A and C upper triangular and A, 2×2. The action of $G\ell(n, \mathbb{R})$ on $St_2(n)$ does not commute with the right action of $SO(2)$.

If $p_t \in St_2(n)$ is a differentiable path then $p_t^* p_t = I$ so that $0 = p_t^* \dot{p}_t + \dot{p}_t^* p_t = p_t^* \dot{p}_t + (p_t^* \dot{p}_t)^*$. Thus the tangent space $T_p St_2(n)$ at p identifies with the set of $n \times 2$ matrices v such that $p^* v$ is skew-symmetric. Therefore the expression $\theta_p = p^* dp$ defines so(2)-valued 1-form in $St_2(n)$. This differential form is the connection form of a (canonical) connection on $St_2(n) \longrightarrow Gr_2^-(n)$ (c.f. [3]). This connection is invariant by the action of $SO(n)$ on $St_2(n)$.

Any $n \times n$ matrix A gives rise to a vector field $\tilde{A}$ on $St_2(n)$ by $\tilde{A}(p) = d/dt((exp\ tA)_p^\sim)|_{t=0}$. In case A is skew-symmetric (but not in general) we have in matrix form that $\tilde{A}(p) = Ap = \hat{A}(p)$. So that

$$\theta_p(\tilde{A}(p)) = p^* Ap \quad \text{(A skew)}. \tag{9}$$

Notice that since $SO(n)$ acts transitively on $St_2(n)$, every tangent vector at q is of the form Aq, A *skew*. (9) is thus enough for defining θ. With the aid of (9) we can give the following rotational interpretation of θ_p: Let p_c denote the two-frame formed by the first two vectors of the standard basis in $\mathbb{R}^n$. Then $\theta_{p_c}(\hat{A}(p_c)) = p_c^* A p_c = 2 \times 2$ upper left corner of A. Since this piece of A is the one reponsible for the action of A within the plane $\varsigma_c = \pi(p_c)$, we see that $\theta_{p_c}(\hat{A}(p_c))$ gives a measure of the infinitesimal rotation inside ς_c in the direction of the tangent vector $\hat{A}(p_c)$. Since θ is $SO(n)$-invariant, at the other $p's$ the situation is similar modulo a change of orthonormal basis, so $\theta_p(\hat{A}(p))$ is actually infinitesimal rotation inside $\pi(p)$.

At this point we can introduce rotation numbers as the assymptotic mean of the sum along the trajectories of the infinitesimal rotations.

Definition 3.2. Let $g_t \in G\ell(n, R), t \geq 0$ be a differentiable path and put $A(t) = \dot{g}_t g_t^{-1}$. Take $p_0 \in St_2(n)$ and set $r_t = \tilde{g}_t p_0$. The rotation number of $A(t)$ in the direction of p_0 is defined as

$$rot_{A(t)}(p_0) = \lim_{T \longrightarrow \infty} 1/T \int_0^T \theta_{r_t}(\hat{A}(t) r_t) dt \tag{10}$$

provided the limit exists.

Remarks.

(1) Differentiability above can be weakened to absolute continuity. In this case $A(t)$ does not exist everywhere but is almost everywhere (w.r.t. Lebesgue measure) definable.

(2) Strictly speaking $rot_{A(t)}(p_0)$ is in (10) an element of $so(2)$, which are however identified, in a canonical way with the real numbers.

Let us now relate definition 3.2 with the approach in $B_2(n)$ suggested before.

Since $St_2(n) \longrightarrow Gr_2^+(n)$ is a subbundle of $B_2(n) \longrightarrow Gr_2^-(n)$, any connection on $St_2(n)$ extends uniquely to a connection on $B_2(n)$. In particular θ extends to a connection $\hat{\theta}$. The differential form $\hat{\theta}$ is a 1-form in $B_2(n)$ with values in the space of 2×2 matrices (= the Lie algebra of $G\ell^+(2, \mathbb{R})$). Because of the invariance of θ w.r.t. the action of $SO(n)$, $\hat{\theta}$ is also $SO(n)$-invariant. Let us compute the value of $\hat{\theta}$ of the vector fields $\hat{A}$, A $n \times n$ matrix. Note that once this is done, $\hat{\theta}$ is known because every vector in $T_p B_2(n)$ is of the form $\hat{A}(p)$.

Proposition 3.3. With p_c the canonical two frame, as before, and A, $n \times n$ matrix, we have,

i) $\hat{\theta}_{p_c}(\hat{A}(p_c)) = 2 \times 2$ upper left corner of $A = p_c^* A p_c$.

ii) If $p \in St_2(n)$ then $p = up_c$, some $u \in SO(n)$ and $\hat{\theta}_p(\hat{A}(p)) = p^* Ap$.

iii) If $p \in B_2(n)$ then $pa \in St_2(n)$, some $a \in G\ell^+(2, \mathbb{R})$ and

$$\hat{\theta}_p(\hat{A}(p)) = a\hat{\theta}_{pa}(\hat{A}(pa))a^{-1} = (p^* p)^{-1} p^* Ap$$

iv) $\widehat{\theta}_p(\widehat{A}(p)) = p^{-1}(Q_\varsigma A|\varsigma)p$, where $\varsigma = \pi(p)$ and Q_ς is orthogonal projection onto ς.

Proof.

i) Write $A = \left(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\right)$ with α, 2×2 and put

$$A = A_1 + A_2 + A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma^{\cdot} \\ \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta + \gamma^{\cdot} \\ 0 & \delta \end{pmatrix}.$$

Then $\widehat{A}_1(p_c)$ is vertical, A_2 is skew and satisfies $\widehat{\theta}_{p_c}(\widehat{A}_2)p_c)) = 0$. Also $\widehat{A}_3(p_c) = 0$ so that

$$\widehat{\theta}_{p_c}(\widehat{A}(p_c)) = \alpha.$$

ii) Follows from the $SO(n)$-invariance of $\widehat{\theta}$. The first equality in iii) is valid because $\widehat{\theta}$ is a connection and $\widehat{A}$ is a R_a-invariant. As to the second equality, put $q = pa$. Then $\widehat{\theta}_q(\widehat{A}(q)) = q^{\cdot} Aq$ so

$$\widehat{\theta}_p(\widehat{A}(p)) = aa^{\cdot} p^{\cdot} Ap.$$

but $1 = q^{\cdot} q = a^{\cdot} p^{\cdot} pa$, hence $aa^{\cdot} = (p^{\cdot} p)^{-1}$.

Finally iv) follows from the first equality of iii) and ii).

Going back now to (3.1), write the Iwasava decomposition of a_t as $a_t = u_t s_t$ with u_t ortogonal and s_t upper triangular with positive entries on the diagonal. If we put

$$u_t = \begin{pmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{pmatrix}$$

we get

$$u_t^{-1} \dot{u}_t = \begin{pmatrix} 0 & -\dot{\theta}_t \\ \dot{\theta}_t & 0 \end{pmatrix}$$

so the rotation number of (3.3) is given by $\lim 1/T \int_0^T u_t^{-1} \dot{u}_t dt$.

Now, taking the derivative of a_t we see from(3.2) with $\widehat{\theta}$ instead of α that

$$u_t^{-1} \dot{u}_t + \dot{s}_t s_t^{-1} = u_t^{-1} \widehat{\theta}_{q_t}(\widehat{A}(q_t)) u_t = \widehat{\theta}_{q_t u_t}(\widehat{A}(q_t u_t))$$

because of the invariance of $\widehat{A}$ by the right action.

Since $\dot{s}_t s_t^{-1}$ is upper triangular we conclude that $u_t^{-1} \dot{u}_t$ is the skew–symmetric component of $\widehat{\theta}_{q_t u_t}(\widehat{A}(q_t u_t))$ in the Iwasava decomposition (skew $\oplus$ upper triangular).

Note that $q_t u_t \in St_2(n)$, so to check that the two definition of rotattion numbers coincide (with $\widehat{\theta}$ in place of α in (3.3)) it is enough to verify that

a) $q_t u_t = r_t$ with r_t as in definition 3.2.

and

b) If $p \in St_2(n)$ then $\theta_p(\hat{A}(p))$ is the same as the skew component in the Iwasava decomposition of $\hat{\theta}_p(\hat{A}(p))$.

As for a), note that $r_t = \bar{g}_t p$ is obtained from $p_t = g_t p$ by orthonormalization, that is, by multiplying on the right by a 2×2 upper triangular matrix with positive entries on the diagonal. This produces $p_t s_t^{-1} = q_t u_t$ because of the uniqueness on the Iwasava decomposition. We get b) as a consequence of

Lemma 3.4. If $p \in St_2(n)$ then

$$\hat{\theta}_p(\hat{A}(p)) = \theta(\hat{A}(p)) + C$$

with C upper triangular.

Proof. Write $e^{tA} = u_t s_t$ with $u_t \in SO(n)$ and s_t upper triangular with positive entries an the diagonal.

Then if p_c is a before

$$\hat{A}(p_c) = d/dt(k_t \bar{s}_t(p_c))_{t=0} = d/dt(\hat{k}_t p_c)_{t=0} = k_0 p_c.$$

Since $A = k_0 + s_0$, this together with proposition 3.3 implies the lemma for $p = p_c$. If $p = up_c$ with $u \in SO(n)$, then by the $SO(n)$-invariance of θ we have

$$\begin{aligned} \theta_{up_c}(\hat{A}(up_c)) &= \theta_{up_c}(u_*(u^{-1}Au)^{\sim}(p_c)) \\ &= \theta_{p_c}((u^{-1}Au)^{\sim}(p_c)). \end{aligned}$$

Doing the same for $\hat{\theta}$ and $\hat{A}$, the lemma follows.

These comments show that the limit in definition 3.2 coincides with the rotation number of equation (8) with $\hat{\theta}$ in place of α. We have thus that $rot_{A(t)}(p_0)$ depends only on the oriented two-plane $\varsigma_0 = \overline{u}(p_0)$ and we so we can talk about $rot_{A(t)}(\varsigma)$ with $\varsigma \in Gr_2^{\sim}(n)$.

Now, let us note that lemma 3.4 shows that if

$$\hat{\theta}(\hat{A}(p)) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{then} \quad \theta_p(\hat{A}(p)) = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}$$

so that if $p = up_c$, then by proposition 3.3 ii), $\theta(\hat{A}(p))$ is the skew-symmetric 2×2 matrix whose 2,1-entry coincides with the 2,1-entry of $u^{-1}Au$. Of course, this 2,1-entry is the same as $\langle Av, w \rangle$ where v, w are - in this order - the vectors forming the 2-frame $p(v = pe_1, w = pe_2, \{e_1, e_2\}$ standard bais in $\mathbb{R}^2)$ and $\langle \, , \, \rangle$ is the canonical inner product in $\mathbb{R}^n$.

We can thus rewrite (10) as

$$rot_{A(t)}(p_0) = \lim_{T \longrightarrow \infty} 1/T \int_0^T \langle A(t) v_t, w_t \rangle dt. \tag{11}$$

where v_t, w_t is the two-frame obtained by orthonormalizing the frame $g_t v_0, g_t w_0$ with $p_0 = \{v_0, w_0\}$.

Formula (11) evidentiates the dependence of $rot_{A(t)}(\varsigma)$ upon the inner product $\langle\ ,\ \rangle$, which was implicit in the choice of θ. If the inner product is changed a different subbundle of $B_2(n)$ plays the role of the Stiefel manifold and a corresponding connection is definable. In fact, any inner product $(\ ,\ )$ can be written as $(u, v) = \langle h_u, h_v \rangle$ with h a positive definite symmetric $n \times n$ matrix. So if $St_2^1(n)$ denotes the set of orthonormal 2-frames for $(\ ,\cdot)$, $St_2(n) = hSt_2^1(n)$ and the 1-form $\theta^1 = h^* \theta$ defines a a connection on $St_2^1(n)$ which plays the role of θ for $(\ ,\ )$ (here $h^* \theta$ means the action of the map $h : B_2(n) \longrightarrow B_2(n)$ on forms). Note that the metric in $\mathbb{R}^2$ is mantained, that is, θ^1 is as θ a form with values in $so(2)$.

This connection provides another notion of infinitesimal rotation which also leads to rotation numbers. These should be – as are the Lyapunov exponents – the same for both inner products. We do not know however, how is, in general the dependence of the rotation numbers upon the metrics in $\mathbb{R}^n$. Unless otherwise stated, $rot_{A(t)}$ means rotation numbers w.r.t. the standard inner product in $\mathbb{R}^n$.

Note that in the 2-dimensional case, rotation numbers are independent of the metric. In fact, to take $rot_{A(t)}$ with respect to a different metric amounts to take $rot_{hA(t)h^{-1}}$, some $2\times 2h$, which by proposition 3.1 coincides with $rot_{A(t)}$. Because of this in the n–dimensional case, if $\varsigma \in Gr_2(n)$ is invariant by $A(t), \forall t \geq 0$ then $rot_{A(t)}(\varsigma)$ being the rotation number of $A(t)$ restricted to ς is the same regardless the choice of the inner product.

Finally, let us remark that there is some flexibility in the choice of the dimension of the state space $\mathbb{R}^n$. In fact, if V is a subspace with $A(t)V \subset V, \forall t \geq 0$ then for $\varsigma \subset V, rot_{A(t)}(\varsigma)$ can be seen as the rotation number of ς inside V, as becomes clear from (3.6). Similarly, extensions of $A(t)$ will not affect $rot_{A(t)}$ inside the original space.

4. TIME INDEPENDENT LINEAR SYSTEMS

For a time independent linear system $\dot{x} = Ax$ the existence of its rotation numbers can be assured. In the direction of some two-planes they are given by the imaginary parts of the eigenvalues of A. In contraposition to the Lyapunov exponents of $\dot{x} = Ax$ – which in any direction is the real part of some eigenvalue of A – the rotations within the different eigenspaces can be superposed providing as rotation numbers combinations of the imaginary parts of the eigenvalues of A.

We divide in cases according to the Jordan decomposition of **A**.

I) The eigenvalues of A are purely imaginary

Suppose first that A is skew with respect to the standard inner product in $\mathbb{R}^n$. Take $p \in St_2(n)$.

As before let $\tilde{A}$ be the vector field in $St_2(n)$ induced by A and put $p_t = \tilde{A}_t(p)$, with $\tilde{A}_t$ denoting the flow generated by $\tilde{A}$.

By the $SO(n)$-invariance of θ, we have that $\theta_{pt}(\tilde{A}(p_t)) = \theta_{p_t}(\tilde{A}_{t*}(\tilde{A}(p)) = \theta_p(\tilde{A}(p))$. Thus the integrand appearing in (10) is constant and hence

$$rot_A(p) = \theta_p(\tilde{A}(p)) = p^* Ap \tag{12}$$

and $rot_A(p)$ is an eigenvalue of A if $\phi(p)$ is invariant by A and a combination of the eigenvalues for arbitrary p.

Note that the map $p \longrightarrow rot_A(p)$ is the restriction to $St_2(n)$ of a skew-symmetric bilinear map and as such it can be viewed as the restriction of a linear functional in $\Lambda^2 \mathbb{R}^n$ - the two-fold exterior product of $\mathbb{R}^n$ - to the set of oriented two planes. This set being identified to the set of decomposable vectors of modulus one in $\Lambda^2 \mathbb{R}^n$.

Now, let $(\, , \,)$ be another inner product in $\mathbb{R}^n$ and denote by rot_A^1 the rotation number map of A w.r.t. it. Since the A-trajectories are quasi-periodic, it can be seen at once that $rot_A^1(\varsigma)$ is well defined for any two plane ς.

We wish to compare rot_A^1 with rot_A. Since any B with imaginary eigenvalues is skew for some inner product, this will provide us with some information about rot_B for such B's.

Proposition 4.1. Suppose that A is also skew with respect to $(\, , \,)$. Then $rot_A^1 = rot_A$.

Proof. By choosing a basis $\{e_i\}$ of $\mathbb{R}^n$ we can write A as $diag(A_1, \ldots, A_k)$ with $A_j = \binom{0 \;\; -a_j}{a_j \;\; 0}$ or $(0)_{1\times 1}$.

Then for any $i \neq j$, $rot_A^1(e_i, e_j) = rot_A(e_i, e_j)$. In fact, if $span\{e_i, e_j\}$ is an eigenspace of A, the equality follows by the remarks at the end of the previous section. Otherwise, $(A\, e_i, e_j) = \langle A\, e_i, e_j \rangle = 0$ because the eigenspaces are orthogonal in both metrics. This means that rot_A^1 coincides with rot_A in a basis of $\Lambda^2 \mathbb{R}^n$ and since they are linear, they coincide everywhere.

Now let h be a posotive definite matrix such that $(u, v) = \langle hu, hv \rangle$ $u, v \in \mathbb{R}^n$, denote by $St_2^1(n)$ the bundle of $(\, , \,)$-orthonormal frames and θ^1 the corresponding connection on $St_2^1(n)$. We have $hSt_2^1(n) = St_2(n)$ and $\theta^1 = h^* \theta$, so that $d\theta^1 = h^* d\theta$.

Let Ω and Ω_1 be the curvatures of θ and θ^1 respectively. They are tensorial forms on $St_2(n)$ and $St_2^1(n)$ and are related by $\Omega_1 = h^* \Omega$. Because of the commutativeness of $SO(2)$ there are ω_1 and ω on $Gr_2^+(n)$ such that $\pi_1^* \omega_1 = \Omega_1$ and $\pi^* \omega = \Omega$. Viewing h as a map on $Gr_2^-(n)$ we have $\omega_1 = h^* \omega$. Note that ω and Ω are A-invariant.

We shall express $rot_A^1 - rot_A$ in terms of ω and h.

Let $\varphi : St_2(n) \longrightarrow St_2^1(n)$ be the map given by orthonormalization with respect to $(\, , \,)$. Clearly, φ is a diffeomorphism between $St_2(n)$ and $St_2^1(n)$ and contrary to h, φ takes the fibres into themselves, i.e., $\pi \circ \varphi = \bar{\pi}$. Also, $\varphi_* \tilde{A} = \tilde{A}^1$ where $\tilde{A}^1$ stands for the vector field induced by A on $St_2^1(n)$. So if we put $\beta = \varphi^* \theta^1$ then $\beta(\tilde{A}(p)) = \theta^1(\tilde{A}^1(\varphi(p)))$ hence

$$rot_A^1 \;\; = \;\; lim_{t\to\infty} 1/t \int_0^t \beta(\tilde{A}(\tilde{A}_s(p)))ds$$

But $\tilde{A}(\tilde{A}_s(p)) = \tilde{A}_{s*}(\tilde{A}(p))$ so

$$rot^1_A(p) = 1/T\int_0^T (A_s^{\cdot}\beta)_p(\tilde{A}(p))ds.$$

Consequently, if we put $\alpha = lim_{T\to\infty}1/T\int_0^T \tilde{A}_s^{\cdot}\beta$ then α is an $\tilde{A}$-invariant differential form on $St_2(n)$ and rot^1_A is given by evaluation of α on the vector field $\tilde{A}$.

We have also

$$d\alpha = lim_{T\to\infty}1/T\int_0^T \tilde{A}_s^{\cdot}(d\beta)ds.$$

But $d\beta = d\varphi^{\cdot}\theta^1 = \varphi^{\cdot}d\theta^1 = \varphi^{\cdot}\pi^{\cdot}\omega_1 = \pi^{\cdot}\omega_1$, so

$$d\alpha = \pi^{\cdot}lim_{T\to\infty}(1/T\int_0^T (e^{sA})^{\cdot}\omega_1 ds)$$

(here $(e^{sA})^{\cdot}\omega_1$ means the pull-back of ω_1 by the map e^{sA} on $Gr_2^-(N)$). Using the A-invariance of ω and the fact that $\omega_1 = h^{\cdot}\omega$, we finally have

$$d\alpha \;=\; \pi^{\cdot}\lim_{T\to\infty}(1/T\int_0^T (e^{-sA}\,h\,e^{sA})^{\cdot}\,\omega\,ds.$$

Since $\alpha - \theta$ is $\tilde{A}$-invariant, we have

$$0 = L_{\tilde{A}}(\alpha-\theta) = i_{\tilde{A}}d(\alpha-\theta) + di_{\tilde{A}}(\alpha-\theta)$$

so $d(rot'_A - rot_A) = -i_{\tilde{A}}d(\alpha-\theta) == i_{\tilde{A}}\Omega - i_{\tilde{A}}d\alpha$.

Note that if one can show that $d\alpha = \Omega$ then $rot^1_A - rot_A$ is a constant which must be zero because of the coincidence of the rotation numbers on the eigenspace of A. We do not know if this happens to be the case.

II) A is nilpotent.

Then rot_A is identically zero w.r.t. any inner product.

In fact, the entries of $exp\ tA$ are polinomial functions of t. After orthonormalizing, one sees that any trajectory p_t of $\tilde{A}$ as either a fixed point or converges to a fixed point p_∞ with $d(p_t,p_\infty) \approx t^{-k}, k \geq 1$, when $t \longrightarrow \infty$. Hence

$$\begin{aligned} rot_A(p_0) &= \lim_{T\to\infty} 1/T\int_0^T \theta_{p_t}(\tilde{A}(p_t))dt \\ &\approx \lim_{T\to\infty} 1/T\int_0^T t^{-k}\,dt + rot_A(p_\infty) \end{aligned}$$

so that $rot_A(p_0) = rot_A(p_\infty)$. But p_∞ being a fixed point, $rot_A(p_\infty)$ is the rotation number of the 2×2 nilpotent matrix obtained by restricting A to $\varsigma_\infty = \pi(p_\infty)$, which is zero with respect to any metric.

III) A is diagonal (real eigenvalues)

Then rot_A is also identically zero. The situation is similar to the case of nilpotent A. Instead of having $d(p_t, p_\infty) \approx t^{-k}, p_t \longrightarrow p_\infty$ exponentially fast so that $rot_A(p_0) = rot_A(p_\infty)$ which is zero.

IV) The General Case

Put $A = A_1 + A_2$ with $A_1 A_2 = A_2 A_1$, as the Jordan decomposition of A, with A_1 corresponding to the imaginary parts of the eigenvalues and A_2 englobing the real parts of the eigenvalues and the nilpotent part.

Take $p \in St_2(n)$ and put $q_s = \dot{A}_{2,s}(p)$. Then $p_s = \dot{A}_s(p)$ is the same as $u_s q_s$ if $u_s = e^{sA_1}$. Hence

$$\begin{aligned} \theta_{p_s}(\dot{A}(p_s)) &= \theta_{u_s q_s}(u_s \cdot (Ad(u_s^{-1})A)^{\sim}(q_s)) \\ &= (u_s^{*}\theta)_{q_s}(\dot{A}(q_s)) \quad \text{(because } \ u_s A = A u_s) \\ &= \theta_{q_s}(\dot{A}_1(q_s)) + \theta_{q_s}(\dot{A}_2(q_s)) \end{aligned} \tag{13}$$

The second term on the r.h.s. in this expression gives the rotation number of A_2 which is zero as can be seen by combining cases II) and III) with a decomposition similar to (4.2). As to the first term, note that q_s converges exponentially or t^{-k}–fast to the trajectory of some p_∞ which is fixed by A_2. Hence $rot_A(p) = rot_{A_1}(p_\infty)$, and we are led to case I).

5. Stochastic Equations

We show here how the framework so far developed for ordinary equations. can be adapted linear stochastic differential equations of the form

$$\sum \quad dx = Ax\, dt + \sum_{j=1}^{m} B_j x \circ dW_t^j \tag{14}$$

Here $A, B_1, \ldots, B_m$ are $n \times n$ matrices, $(W^1, \ldots, W^m)$ are m mutually independent standard Wiener processes defined on a probability space with filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and $\circ$ denotes Stratonovich stochastic differential.

System (14) induces the differential equation

$$dp = \dot{A}(p)dt - \sum_{j=1}^{m} \dot{B}_j(p) \circ dW_t^j \tag{15}$$

on $St_2(n)$. Following the pattern of the definition 3.2, we should be able to define for each $p_0 \in St_2(n)$ and each randon path of the solution of (15) starting at p_0 a rotation number which is to be dependent only upon the oriented two plane ς_0 projection of p_0. This rotation number is to be the difference between $p_t(\omega)$ and $q_t(\omega)$, the corresponding horizontal path. Note however that since $p_t(\omega)$ is not differentiable, $q_t(\omega)$ is not directly constructed from $p_t(\omega)$ as it is in the differentiable case. We can however avoid this problem by taking a stochsatic system whose solutions play the role of horizontal curves corresponding to the solutions of (15). Let $A_H, B_{j,H}$ denote the vector fields on $St_2(n)$ obtained by horizontally projecting the vector field $\tilde{A}$ and $\tilde{B}_j$.

Consider the equation

$$\sum_H : \quad dq = A^q_H dt + \sum_{j=1}^m B_{j,H}(q) \circ dW^j_t \tag{16}$$

Its solutions are 'horizontal' curves and follows the same fibres as the solutions of (15) because both system projects onto the system on $Gr_2^+(n)$.

Now, let $p_t(\omega)$ be a solution of (15) with $p_0(\circ) = p_0$ a.s. and consider the equation

$$da = a\theta_{p_t}(\tilde{A}(p_t))dt + \sum_{j=1}^m a\theta_{p_t}(\tilde{B}_j(p_t)) \circ dW^j_t, \quad a_0 = 1. \tag{17}$$

By taking stochastic differentials of $q_t = p_t a_t^{-1}$, one sees that q_t is a solution of (16) wiith $q_0 \approx p_0$, thus (17) is the equation on $SO(2)$ whose soolutions give the rotation numbers of the linear system $\sum$.

So if we put - for a $n \times n$ matrix A - $\gamma(A,p) = \theta_p(\tilde{A}(p))$ we can define the rotation number of $\sum$ as the random variable

$$rot_{\sum}(p_0) = lim_{T\to\infty} 1/T\{\int_0^T \gamma(A,p_s)ds + \sum_{j=1}^m \gamma(B_j,p_s) \circ dW^j_s\} \tag{18}$$

where p_s is the solution of (15) with $p_0(\omega) = p_0$ a.s.

Rewritting (18) in Itô's integral and taking into account that $\int_0^T \gamma(B_j,p_s)dW^j_s$ is a martingale (18) becomes

$$rot_{\sum}(p_0) = \lim_{T\to\infty} 1/T \int_0^T (\gamma(A,p_s) - 1/2\sum_{j=1}^m \rho(B_j,p_s))ds \tag{19}$$

where $\rho(B_j,\cdot) = \tilde{B}_j\gamma(B_j,\cdot)$ (= *derivative of* $\gamma(B_j,\cdot)$ in the direction of $\tilde{B}_j$.

The same kind of reasoning as in §3, shows that the limit in (19) is the rotation number of a 2-dimensional system and as there the rotation numbers turn out to be independent of the choice of a frame and depends only upon the oriented two-plane it spans.

Let us explicitate a bit more the function $\rho(Y,\cdot), Y \quad n \times n$ matrix.

If X is skew-symmetric with $\tilde{X}(p) = \tilde{Y}(p)$ then $\tilde{X}(Y)\cdot\gamma(Y,\cdot) = \tilde{Y}(p)\cdot\gamma(Y,\cdot)$. But if $p = \{u, v\}$ then

$$\begin{aligned} \tilde{X}(p)\gamma(Y,\cdot) &= d/dt\langle Ye^{tX}u, e^{tX}v\rangle_{t=0} \\ &= \langle (YX - XY)u, v\rangle \end{aligned} \tag{20}$$

so $\rho(Y,p) = \langle [X,Y]u, v\rangle$. In case $p = p_e$ is the canonical 2-frame, if we write

$$Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ (with) } A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{2\times 2}$$

then if

$$X = \begin{pmatrix} 0 & -\gamma & \\ \gamma & 0 & -C^{*} \\ & C & 0 \end{pmatrix}$$

we have $\hat{X}(p_c) = \hat{Y}(p_c)$.

Hence

$$\rho(Y, p_c = (\delta - \alpha)\gamma + \{2, 1 - \text{entry of}(B + C^{*})C\}$$

or written another way,

$$\rho(Y, p_c) = \{2, 1 - \text{entry}(Y + Y^{*})Y\} - (A)$$

where

$$\Lambda(A) = \{2, 1 - \text{entry}(A + \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))A\}$$

so

$$\rho(Y, p_c) = \langle (Y + Y^{*})Y\, e_1, e_2\rangle - \Lambda(p_c^{*}\, Y\, p_c)$$

with $p = \{e_2, e_2\}$.

If $p = up_c, u \in SO(n)$ is another 2–frame, $\rho(Y, p) = \rho(u^{-1}Y\, u, p_c)$ because for any $q \in St_2(n)$, $\gamma(u^{1}Y\, u, u^{-1}q) = \gamma(Y, q)$. Hence

$$\rho(y, p) = \langle (Y + Y^{*})Y\, u, v\rangle - \Lambda(p^{*}Y_p)$$

where $p = \{u, v\}$.

6. Cocycles and Connections

We shall see here how the concept of connection extends the notion of multiplicative cocycle which appears more often in questions involving the assymptotics of dynamical systems. As we will see, any multiplicative cocycle turns out to be essentially the global counter part of a flat connection and vice–versa any flat connection on a principal bundle acted by a Lie group gives rise to a multiplicative cocycle.

We take here cocycles with values in arbitrary Lie groups, i.e., maps $\varphi : G \times M \longrightarrow H$ satisfying

$$\varphi(gh, x) = \varphi(g, hx)\varphi(h, x) \quad g, h \in G, \quad x \in M$$

Where G and H are Lie groups and M is a manifold in which G acts.

Proposition 6.1. Let $P \longrightarrow M$ be a principal bundle with group H.

Let G be a simply connected Lie group acting on P and suppose that this action commutes with the right action of H (so G also acts in M).

Let $p \in P \longrightarrow H_p \subset T_pP$ be a flat connection on P.

Fix $p_0 \in P$ and let $\tilde{M}$ be the maximal integral manifold of the connection passing through p_0. $\tilde{M}$ is a covering of M and the projection $\tilde{M} \longrightarrow M$ defines a principal bundle whose group is a discrete subgroup of H and a quotient of the fundamental group of M (c.f. [2]).

G acts on $\tilde{M}$ by lifting its action on M. Denote this action by $\tilde{g}y$, $g \in G$, $y \in \tilde{M}$.

The "diference" between gy and $\tilde{g}y$, $y \in \tilde{M}$ defines a cocycle $\varphi : G \times \tilde{M} \longrightarrow H$.

If M or the fundamental group of M is abelian φ can be defined over M itself.

Proof. The difference above is expressed by $a \in M$ a follows: If $y \in \tilde{M}$, gy and $\tilde{g}y$ are in the same fibre, thus there is a unique $a \in M$ such that $gy = (\tilde{g}y)a$.

Define $\varphi : G \times \tilde{M} \longrightarrow H$ the formula

$$gy = (\tilde{g}y)\varphi(g, y).$$

This is a cocycle:

If $g, h \in G$, $y \in \tilde{M}$ then

$$(gh)y = (gh)^{\sim}(y)\varphi(gh, y)$$

Also.

$$\begin{aligned}(gh)y &= g(hy) = g(\tilde{h}y\varphi(h, y)) \\ &= \tilde{g}(\tilde{h}y)\varphi(g, \tilde{h}y)\varphi(h, y)\end{aligned} \tag{21}$$

because g commutes with $R_{\varphi(h,y)}$. Hence

$$\varphi(gh, y) = \varphi(g, hy)\varphi(h, y).$$

If $y' = yb$ is in the same fibre as y, then

$$gy' = g(yb) = \tilde{g}(y)(\varphi(g, y)b)$$

but also

$$\begin{aligned}gy' &= \tilde{g}(y')\varphi(g, y') = \tilde{g}(yb)\varphi(g, y') \\ &= \tilde{g}(y)(b\varphi(g, y'))\end{aligned} \tag{22}$$

Therefore $\varphi(g, y') = b^{-1}\varphi(g, y)b$ so if the group of $\tilde{M} \longrightarrow M$ is abelian φ can be defined over M it self.

Remarks.

(1) For the cocycle $\varphi(g, v) = ||gv||/_{||v||}$ $g \in G\ell(n, \mathbb{R})$, $v \in \mathbb{R}^n - \{0\}$, defining the Lyapunov exponents, $\tilde{M}$ above is just the unit sphere while $\tilde{g}v = v/_{||v||}$.

(2) Since paths in M can be horizontally lifted to P, the restriction of any connection in $P \longrightarrow M$ to a differentiable curve in M is a flat connection and thus define a coclycle over some covering of M. A connection becomes then the join of a family of cocycles.

The reciproque of proposition 6.1 is as follows:

Let $\varphi : G \times M \longrightarrow H$ be a cocycle. In $M \times H$ define the action

$$g(x, h) = (gx, \varphi(g, x)h); \quad g \in G, \ h \in M, \ x \in M.$$

This is an action because φ is a cocycle. Taking the canonical flat connection on $M \times H \longrightarrow M$, φ turns out to be the cocycle defined by this connection as in proposition 6.1.

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