A SPLITTING THEOREM FOR COMPLETE MANIFOLDS WITH NON–NEGATIVE CURVATURE OPERATOR

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ABSTRACT. In this paper we consider non-compact Riemannian manifolds with non-negative curvature operator. The main result is the following theorem:

"If M^n is a complete non-compact Riemannian manifold, simply-connected with non-negative curvature operator, then M is isometric to the product $\Delta^k \times P^{n-k}$, where Δ^k is a k-dimensional soul of M and P^{n-k} is a complete manifold diffeomorphic to $\mathbb{R}^{n-k}$."

This result provides a complete topological description of such manifolds, since there is a topological classification for compact, simply-connected manifolds with non-negative curvature operator. This result implies, for the non simply-connected case, the following corollary:

"Let M^n be a non-compact Riemannian manifold with non-negative curvature operator. Then M is locally isometric to a product over S. In particular if the curvature operator is positive at one point then M^n is diffeomorphic to $\mathbb{R}^n$."

This corollary gives a positive answer to a Conjecture of Cheeger and Gromoll in the case of non-negative curvature operator.

In the case of codimension two submanifolds of the Euclidean space non-negativity of sectional curvatures is well known to be equivalent to the non-negativity of the curvature operator. In that case, the corollary above generalizes the Theorem of Sackesteder for submanifolds with codimension 1 in the following sense:

"Let M^n be a complete non-compact Riemannian manifold with non-negative sectional curvatures isometrically immersed in $\mathbb{R}^{n+2}$. If there is a point P in M such that all sectional curvatures are positive then M^n is diffeomorphic to $\mathbb{R}^n$".

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I. INTRODUCTION.

It has been an important problem in Riemannian geometry to determine the structure of a complete, non-compact manifold M whose sectional curvatures are non-negative. J.Cheeger and D.Gromoll in [CG] have shown that M is diffeomorphic to the total space of a vector bundle over a compact, totally geodesic submanifold, called the soul, and classified it in dimensions ≤ 3 up to isometry. These are the most significant results in this direction. In the same paper the authors left an interesting problem: "Suppose there is a point $x \in M$ such that all the sectional curvatures are positive. Is the soul of M a point, or equivalently, is M diffeomorphic to the Euclidean space $\mathbb{R}^n$?" This is known to be true for immersed hypersurfaces in euclidean space.

In this paper we want to consider a stronger condition on such manifolds, namely the non-negativity of the curvature operator (see definition below) and answer the Cheeger-Gromoll conjecture affirmatively in this case. This in turn implies a positive answer to the same conjecture for manifolds isometrically immersed in Euclidean space with codimension two, since it is a well known result that in codimension two, the non-negativity of the sectional curvatures is equivalent to the non-negativity of the curvature operator (see [We]). Our result states:

THEOREM: Let M^n be a complete non-compact, simply connected manifold with non-negative curvature operator. Then M is isometric to the product $S^k \times \mathcal{O}^{n-k}$ where S is the k-dimensional soul of M and $\mathcal{O}^{n-k}$ is a complete manifold diffeomorphic to $\mathbb{R}^{n-k}$.

REMARK: This result gives a complete topological description of this manifold since we know the possibilities for the soul S from the classification of simply connected, compact manifolds with non-negative curvature operator which appears in [GM] and [CY]. Namely, S is a riemannian product of manifolds of the following types: compact symmetric spaces, Kähler manifolds biholomorphic to complex projective spaces and manifolds homeomorphic to spheres.

COROLLARY: Let M^n be a complete non-compact manifold with non-negative curvature operator. Then M is locally isometric to a product over S. In particular, if the curvature operator is positive at some point, then M^n is diffeomorphic to $\mathbb{R}^n$.

We want to observe that the non-negativity of the curvature operator is equivalent to the non-negativity of the sectional curvatures in two more cases:

i) Manifolds which can be immersed isometrically into space forms with flat normal connection

ii) Submanifolds in which the second fundamental form satisfies the condition (4.13) in [KW].

For these cases, our theorem also gives an answer to the Cheeger-Gromoll conjecture.

Some of the arguments in this paper can also be found in G.Walschap [Wa].

2.BASIC RESULTS:

For a Riemannian manifold M the curvature operator at x∈M is the linear symmetric map

$$\rho: \Lambda^2(T_xM) \to \Lambda^2(T_xM)$$

characterized by

$$\langle \rho(X\wedge Y), (W\wedge Z) \rangle = \langle R(X,Y)Z,W \rangle$$

where the scalar product on the left hand side is the induced one at the level of two-forms and R is the Riemannian tensor. Since ρ is symmetric, it makes sense to talk about the positivity and the non-negativity of ρ.

Now suppose that M is a complete manifold with a soul denoted by S.

(2.1)PROPOSITION: If the curvature operator is non-negative and dimS ≥2 , then the inclusion i: S→ M has flat normal bundle.

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<u>Proof.</u> For every $x \in M$, let us consider the normal set $\{w_i\}$ in $\Lambda^2(T_xM)$ which diagonalizes ρ with eigenvalues λ_i. Then for $X,Y \in T_xM$ we write $X \wedge Y = \sum a_i w_i$ and therefore

$$\rho(X \wedge Y) = \sum a_i \rho(w_i) = \sum a_i \lambda_i w_i$$

with $\lambda_i \geq 0$. Notice that

(2.2) If the sectional curvature $K(X,Y)=0$ we have $\rho(X \wedge Y)=0$

this following from

$$0 = \langle \rho(X \wedge Y), X \wedge Y \rangle = \sum a_i^2 \lambda_i \text{ and } \lambda_i \geq 0 \text{ for all } i.$$

Now, take $x \in S$, $X,Y \in T_xS$ and $Z \in T_xS^{\perp}$. By Theorem 3.1 in [CG], $K(X,Z)=0$ and $K(Y,Z)=0$ which implies $\rho(X \wedge Z)=0$ and $\rho(Y \wedge Z)=0$. Using the first Bianchi identity, it is easy to see that $R(X,Y)Z=0$. Applying this fact to the Ricci equation for the totally geodesic immersion $i: S \to M$, we have for all $X,Y \in TS$ and $Z,W \in TS^{\perp}$, $\langle R(X,Y)Z,W \rangle = \langle R_i^{\perp}(X,Y)Z,W \rangle$. But the first term is $\langle \rho(X \wedge Y), W \wedge Z \rangle$ which is zero since $R(X,Y)Z=0$ and the conclusion follows.

Now suppose M simply connected so that the soul is simply connected. Proposition (2.1) implies that for each unit normal vector Z at x we can get, by parallel transportation, a parallel section of the flat normal bundle $\nu(S)$. This parallel section together with the proposition below will take us to the concept of the pseudo-soul.

(2.3) <u>PROPOSITION</u> (Proposition 3.2, [Y]): Let S be a soul of M. Then S has minimal volume in its homology class.

This result was used by Yim in [Y] to show that if Z is any parallel section of $\nu(S)$ then the map $\varphi_Z : S \times \mathbb{R} \to M$, given by $\varphi_Z(x,t) = \exp_x tZ(x)$ is an isometric immersion. In fact, by the Rauch Comparison Theorem [CE], $\varphi_Z(.,t)$ is distance non-increasing for small t which implies that φ_Z is an isometry since for each t, $S_t = \varphi_Z(S,t)$ is in the same homology class as S and its volume is not less than that of S. By the connectedness of $\mathbb{R}$, φ_Z is an isometric immersion for all $t \in \mathbb{R}$ and its image is isometric to a product manifold $S \times \mathbb{R}$. Actually this immersion is totally geodesic (see [2.7] below), and then for each t, $S_t = \varphi_Z(S,t)$ is a totally geodesic manifold isometric to S. Yim has called it a pseudo-soul.

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(2.4)<u>PROPOSITION</u>: If the curvature operator is non-negative, S is simply connected and $\dim S \geq 2$, then the pseudo-soul $\bar{S}$ also has flat normal bundle.

<u>Proof</u>: Let us consider the pseudo-soul $\bar{S}=\exp_S \bar{t}Z_1$, where Z_1 is a parallel section of $\nu(S)$. We can define for each $\bar{x}\in\bar{S}$ such that $\bar{x}=\exp_x \bar{t}Z_1$ with $x\in S$, $\bar{Z}_1(\bar{x})$ by $\gamma'(\bar{t})$ where $\gamma(t)=\exp_x tZ_1$. Then $\bar{Z}_1$ is a parallel section of $\nu(\bar{S})$ by construction. We want to prove that we can construct m linearly independent sections in $\nu(\bar{S})$ where m is the codimension of the soul. We fix $x\in S$ and if $Z_2,\dots,Z_m$ are unit orthogonal vectors to $Z_1(x)$, we define $\bar{Z}_2,\dots,\bar{Z}_m$ at $\bar{x}$, by parallel transportation along the geodesic γ. We claim that $\bar{Z}_2,\dots,\bar{Z}_m$ belong to the normal space to $\bar{S}$, denoted by $T_{\bar{x}}\bar{S}^\perp$. In fact, consider $\bar{y}\in\bar{S}$ such that $\bar{y}=\exp_y \bar{t}Z_1$, $y\in S$, and the curves c from x to y and $\bar{c}$ from $\bar{x}$ to $\bar{y}$ respectively. Let us consider the rectangle $f:[0,a]\times[0,\bar{t}]\to M$ defined by $f(s,t)=\exp_{c(s)}tZ_1(s)$. We have

$$(\partial f/\partial s)(s,t)=\lambda X \qquad (\partial f/\partial t)(s,t)=\mu Z_1$$

with $X(s,t)$, $Z_1(s,t)$ having unit length and $Z_1(s,\bar{t})=\bar{Z}_1(s)$. Since the Lie bracket $[\partial f/\partial s,\partial f/\partial t]=0$, this will be

$$\lambda X(\mu)Z_1 + \lambda\mu\nabla_X Z_1 - \mu Z_1(\lambda)X - \mu\lambda\nabla_{Z_1}X = 0. \tag{2.5}$$

We have omitted t for brevity. We see that:

i) $\nabla_X Z_1=0$, since Z_1 is parallel

ii) $\langle\nabla_{\bar{Z}_1}X,Z_1\rangle=0$ because $\varphi_{Z_1}(S)$ is a product

iii) $\langle\nabla_{Z_1}X,X\rangle=0$ because X is unitary.

Then, (i),(ii) and (iii) imply in (2.5) that $\nabla_{\bar{Z}_1}X=0$. This implies that $T_x\bar{S}_t$ is parallel along γ and then if $Z_2,\dots,Z_m\in T_xS^\perp$, $\bar{Z}_2,\dots,\bar{Z}_m\in T_{\bar{x}}\bar{S}^\perp$.

Now we make a parallel transportation of $\bar{Z}_2,\dots,\bar{Z}_m$ along $\bar{c}$ and we write the expression for $R(X,\bar{Z}_1)\bar{Z}_i$, $i\geq 2$, which is zero by (2.2):

$$R(X,\bar{Z}_1)\bar{Z}_i = \nabla_X\nabla_{\bar{Z}_1}\bar{Z}_i - \nabla_{\bar{Z}_1}\nabla_X\bar{Z}_i - \nabla_{[X,\bar{Z}_1]}\bar{Z}_i = \nabla_X\nabla_{\bar{Z}_1}\bar{Z}_i = 0,$$

since $\nabla_X\bar{Z}_i=0$ and $[X,\bar{Z}_1]=0$ by (i),(ii),(iii) and (2.5). It follows that

$$\partial(\langle\nabla_{\bar{Z}_1(s)}\bar{Z}_i(s),\nabla_{\bar{Z}_1(s)}\bar{Z}_i(s)\rangle)/\partial s = 0. \tag{2.6}$$

But $\bar{c}(0)=\bar{x}$ and $\nabla_{\bar{Z}_1(0)}\bar{Z}_i(0)=0$. So, (2.6) implies $\nabla_{\bar{Z}_1(s)}\bar{Z}_i(s)=0$ for each s. This means that the vectors $\bar{Z}_2,\dots,\bar{Z}_m$ obtained along $\bar{c}$ by parallel

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transportation are the same vectors that we would obtain making parallel transportation of $Z_2,\dots Z_m$ from x to y along c and then along the geodesic $\psi(t)=\exp_y tZ_1$. Since by Proposition (2.1), the parallel transportation in S does not depend on the curve c joining x to y, the parallel transportation in $\bar{S}$ from $\bar{x}$ to $\bar{y}$ will not depend on the curve $\bar{c}$ joining $\bar{x}$ to $\bar{y}$ either. This implies the proposition.

(2.7)REMARK: We observe that the above proof also shows that the isometric immersion φ_{Z_1} is totally geodesic. Since for each $x\in S$, $\exp_x tZ_1$ is a geodesic in M, all we need is to prove that for each t, $\bar{S}_t$ is a totally geodesic submanifold of M. Then, if X(t) and Y(t) are vector fields tangent to $\bar{S}_t$, we have for every i

$$(2.8)\qquad \frac{d}{dt}\langle \nabla_X Y, Z_i\rangle = \langle \nabla_{Z_1}\nabla_X Y, Z_i\rangle + \langle \nabla_X Y, \nabla_{Z_1} Z_i\rangle = 0$$

because $R(Z_1,X)Y=0$ and $[Z_1,X]=0$ imply that $\nabla_{Z_1}\nabla_X Y = \nabla_X \nabla_{Z_1} Y = 0$ and Z_i is parallel along γ. Since for $t=0$ we have $\langle \nabla_X Y, Z_i\rangle=0$ because the soul is totally geodesic, (2.8) implies that $\bar{S}_t$ is also totally geodesic.

(2.9)PROPOSITION: Let M be a manifold as in Proposition (2.4) Then there exists a smooth foliation of M by totally geodesic manifolds isometric to S.

Proof: First, we prove that for each point $x\in M$ there exists a totally geodesic manifold $\bar{S}$ isometric to S such that $x\in\bar{S}$. For that, consider $\gamma:[0,a]\to M$ the minimal connection from x to S. $\gamma'(a)\in\nu(S)$. Let Z be the parallel normal field defined on S such that $Z(\gamma(a))=\gamma'(a)$. Then we have a pseudo-soul $\bar{S}=\exp_S aZ$ and $x\in\bar{S}$.

We claim that there exists only one totally geodesic manifold $\bar{S}$ such that $x\in\bar{S}$ and $\bar{S}$ is isometric to S. Suppose that there exits $\tilde{S}$ with the same conditions and $x\in\tilde{S}$. Let $\bar{S}$ be a pseudo-soul containing x. If $X\in T_x\tilde{S}$ and $X\notin T_x\bar{S}$, we consider $\bar{X}$ the unitary orthogonal projection of X on $T_x\bar{S}^{\perp}$. Since $\bar{S}$ has flat normal bundle we take the parallel transportation of $\bar{X}$ along $\bar{S}$. Let us call $\bar{M}=\exp_{\bar{S}} t\bar{X}$. $\bar{M}$ has $\bar{S}$ as a soul, since $\bar{M}$ is isometric to $\bar{S}\times\mathbb{R}$. The vector X belongs to $T_x\bar{M}$ and is transversal to $\bar{S}$. By Theorem (5.1) of [CG], the geodesic $\sigma(t)=\exp_x tX$ must go to infinity. Since $\bar{M}$ and $\tilde{S}$ are totally geodesic, σ is a geodesic in M and $\tilde{S}$ going to infinity, contradicting the fact that $\tilde{S}$ is compact.

This shows that the foliation is well defined. We need to prove the smoothness of the foliation. Let $\bar{S}$ be the leaf containing x. Let us take ε smaller than the injectivity radius of $\nu(\bar{S})$. Now we exponentiate the global sections of $\nu(\bar{S})$ at distances smaller than ε and we get totally geodesic manifolds isometric to $\bar{S}$ which coincide with the leaves by uniqueness.

3.PROOF OF THE THEOREM

By the proposition (2.9) we have two differentiable distributions defined on M. The first one D_1, given by the tangent vectors to the leaves of the foliation F and the second $D_2 = D_1^{\perp}$. We will prove that D_1 and D_2 are involutive and parallel and the theorem will follow by Frobenius.

In order to prove this, notice that the leaves of F are equidistant and simply connected. Then we can apply the Theorem of R.Hermann in [H] which says that M/F is a smooth manifold and admits a Riemannian metric for which the projection $\Phi: M \to M/F$ is a Riemannian submersion. We see that for this submersion, horizontal vectors are orthogonal to the pseudo-souls and vertical vectors are tangent to the pseudo-souls. Now, it is easy to calculate the O'Neill tensors (see [O]). With $\mathcal{H}$ and $\mathcal{V}$ denoting the projections onto the horizontal and vertical subspaces and X and V being horizontal and vertical vectors respectively, we have

$$T_V X = \mathcal{V}(\nabla_V X) \qquad A_X V = \mathcal{H}(\nabla_X V)$$

T is zero because the pseudo-souls are totally geodesic. Then it will be enough to prove that A is zero. By the Corollary 1 of [O] we have for the sectional curvature of the plane spanned by X and V

$$K(X,V) = \langle (\nabla_X T)_V V, X\rangle + \| A_X V\|^2 - \|T_V X\|^2$$

But $K(X,Y)=0$ and $T_V X=0$. Then, all we need is to prove that

$$\langle (\nabla_X T)_V V, X\rangle = \langle \nabla_X T_V V, X\rangle - \langle T_{\nabla_X V} V, X\rangle - \langle T_V \nabla_X V, X\rangle = 0$$

In fact, using again that the pseudo-souls are totally geodesic we have

$$T_V V = \mathcal{H}(\nabla_V V) = 0$$

$$\langle T_{\nabla_X V} V, X\rangle - \langle T_V \nabla_X V, X\rangle =$$

$$= \langle \mathcal{H}(\nabla_{\mathcal{V}(\nabla_X V)} V) - \mathcal{H}(\nabla_V \mathcal{V}(\nabla_X V)) - \mathcal{V}(\nabla_V \mathcal{H}(\nabla_X V)), X\rangle = 0$$

Hence, it follows that $A_X V = 0$.

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4. PROOF OF THE COROLLARY.

Let us consider S the soul of M and $\tilde{S}$ and $\tilde{M}$ the respective universal coverings. By Theorem 9.1 of [CG], $\tilde{S}$ is isometrically diffeomorphic to $S_0\times\mathbb{R}^m$ with S_0 compact and the splitting is in the sense of Toponogov [T]. Then these lines in $\tilde{S}$ must split off in $\tilde{M}$ too and hence $\tilde{M}$ is isometrically diffeomorphic to $M_0\times\mathbb{R}^m$. But M_0 is simply connected and by the previous theorem, $M_0=S'\times\mathbb{P}^r$, where S' is the soul of M_0.

We claim that $S_0=S'$. For that, consider $X\in T_xS_0$. Suppose that $X\notin T_xS'$ and take the geodesic $\sigma(t)=\exp_x tX$. This geodesic, again by Theorem 5.1 of [CG], must go to the infinity contradicting the compactness of S_0. Then $S_0\subset S'$. Since S is totally convex, $\tilde{S}$ and S_0 are totally convex. Now we have S_0 and S' compact, totally convex and without boundary. Applying Theorem 2.1 of [CG], we see that S_0 and S' have the same homotopy type. Since $S_0\subset S'$ and both are compact we have the claim.

Now, we have the following diagram:

$$\begin{array}{ccc} S_0\times\mathbb{P}^r\times\mathbb{R}^m & \xrightarrow{\ \Pi\ } & M \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ S_0\times\mathbb{R}^m & \xrightarrow{\ \Pi\ } & S \end{array}$$

where Π is the covering map, P_1 the projection onto the first factor. Since Π is a local isometry and the fundamental group preserves the splitting $S_0\times\mathbb{P}^r\times\mathbb{R}^m$, P_1 induces a submersion P_2: $M\to S$, which is a local product.

In particular, if there is a point such that the curvature operator is positive, S must be a point and the corollary follows.

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