STABILITY CONDITIONS FOR DISCONTINUOUS VECTOR FIELDS

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SUMMARY. In this work we describe the qualitative behavior of generic discontinuous vector fields around a point $p \in \mathbb{R}^3$.

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§0 – Introduction

Some problems in Control Theory ([3], [4]), Economy [5] and Nonlinear Oscillations [6] lead to consideration of differential equations whose right hand terms are defined by descontinuous vector fields. We are assuming that those discontinuities are of 1^{th} kind and they are concentrated on a C^∞ 2–dimensional surface M contained in $\mathbb{R}^3$. In this work we describe the local behavior of discontinuous vector fields in neighborhoods of points $p \in M$. The main goal of this paper is firstly to classify, via structural stability, such vector fields. The most interesting part of this work is the study of the dynamic properties of the vector field on a region in M called Sliding Region (to be defined later). Roughly speaking, when the trajectory of the vector field meets the Sliding Region it remains tangent to M for all positive time. We mention that we also give some results concerning asymptotic stability.

The paper is structured as follows:

In Section 1 and 2 we give some preliminaries, definitions and establish the notation. We also give the rules for defining the solutions curves of the vector fields at points of M.

Section 3 contains the statements of the main results. We also discuss properties of some "singularities" of the vector fields.

In Section 4 we prepare the way for the construction of equivalence between vector fields by defining a suitable stratification in (M, p).

Section 5 is devoted to the construction of a first return mapping associated to a special singularity of the vector field.

In Section 6 we study the so called Sliding Vector Field.

in Section 7 we prove the main theorem concerning structural stability.

Finally in Section 8 the asympotic stability is studied.

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§1 – Preliminaries

Definition. Let $p \in M$ and $f : (\mathbb{R}^3, M) \to (\mathbb{R}, 0)$ be a C^∞ local implicit representation of M at p with $df(p) \neq 0$. The surface M represents the common boundary separating the domains $M^+ = \{f > 0\}$ and $M^- = \{f < 0\}$. We may so. via f, give an orientation to any curve in $(\mathbb{R}^3, p)$ crossing M.

Denote by $\mathcal{X}^r$ the set of all germs in p of C^r vector fields on $(\mathbb{R}^3, p)$, endowed with the C^r topology with r big enough for our purposes.

Let G^r be the set of all germs in p pf vector fields Z on $\mathbb{R}^3$ satisfying

$$Z(q) = \begin{cases} X(q) & \text{if } \quad q \in M^+ \\ Y(q) & \text{if } \quad q \in M^- \end{cases} \quad \text{where } \; X, Y \in \mathcal{X}^r$$

and the rules for defining the solutions curves of Z at points of M, are due to Gantmaher–Filipov. and in what follows they will be briefly recalled. For their mathematical justification, the reader is refered to [3], [4] and [5].

We may considered $G^r = \mathcal{X}^r \times \mathcal{X}^r$. So we denote any element in G^r by $Z = (X, Y)$.

Given $Z = (X, Y)$ in G^r we distinguish the following regions in M.

a) Sewing Region (SWR): in this case, the vector fields X and Y are directed with the same sense away from M (Figura 1.1). If at a point of the phase space which is moving in an orbit of Z falls onto M then it moves and crosses M over to another part of space.

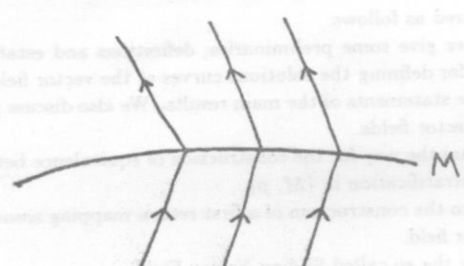

Figure 1.1 — The SW — Region

b) Escaping Region (ESR): the vector fields X and Y point inward M^+ and M^- respectively. The solution throught a point $p \in M$ follows the orbit of either X or Y according to which has the largest normal (respect to M) component (Figura 1.2)

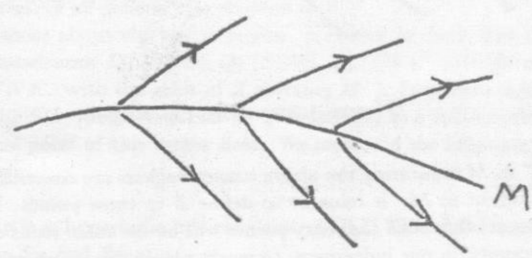

Figure 1.2 — The ES — Region

1.1. Remark. Observe that in this case the field $Z = (X, Y)$ could not be defined anywhere in (M, p). So if at $q \in M$ the normal components of X and Y coincide then we adopt by simplicity that $Z(q) = X(q)$.

c) Sliding Region (SLR): the vector fields X and Y point outward M^+ and M^- respectively. In this case the solution of Z through points of M follows the orbit, of the vector field $F = F(X, Y)$ (called SL–vector field associated to $Z = (X, Y)$). Such F is tangent to M and defined at $p \in m$ by the vector $F(p) = m - p$ where m is the point where the segment joining $p + X(p)$ and $p + Y(p)$ is tangent to M (Figure 1.3). Observe that if $X(p)$

and $Y(p)$ are linearly dependent then p is a critical point of F.

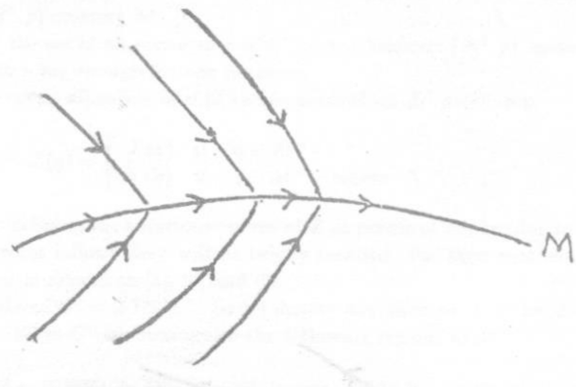

Figure 1.3 — The SL–Region.

All "curves" in M separating the above named regions are constituted by points where X or Y are tangent to M. It remains to define Z at these points. Neverthless one can check in the sequel that such tangency points will be the main focus of our attention and there is no necessity, in our judgement, to worry about this question at this moment.

We mention that we might, for our purposes consider Z at points of M defined as a multivalued mapping. In this way the results in question would not change at all.

§2 – Singularities of Z

2.1. Definition. Let Z, $\tilde{Z} \in G^r$. We say that Z and $\tilde{Z}$ are C^0 equivalent if there are open neighborhoods U and V of p in $\mathbb{R}^3$ and an M–invariant homeomorphism $h : U \to V$ which sends orbits of Z in orbits of $\tilde{Z}$. From this definition the concept of Structural Stability in G^r is naturally reached.

2.2. Definition. We say that $p \in M$ is an M–singular (resp. M–regular) point of $X \in \mathfrak{X}^r$ if $Xf(p) = 0$ (resp. $Xf(p) \neq 0$). We denote by S_X the singular of X.

2.3. Definition. We say that $p \in M$ is a fold (resp. cusp) point of X if $Xf(p) = 0$ and $X^2f(p) \neq 0$ (resp. $Xf(p) = X^2f(p) = 0$ and $\{df(p), d(Xf)(p), d(X^2f)(p)\}$ are linearly independent).

The cusp points are isolated points located at the extremes of the curves of fold points. In fact by projecting M along the orbits of X onto a surface N transverse to the orbit through a stable singularity, we get a singularity of fold or cuspidal type in the sense of Whitney (see [8]).

2.4. Definition. We say that $p \in M$ is a generic singularity of $Z = (X, Y) \in G^r$ if:

either i) p belongs to the sliding region of Z and it is a hyperbolic critical point of $F(X,Y)$;

or ii) p is a fold or a cusp point of X and Y. If p is a M-singular point of both vector fields X and Y then we have to impose the following extra conditions: a) p is a fold point of X and Y and b) S_X is transverse to S_Y at p.

Call $S(Z)$ the set of all generic singularities of Z.

We must say more about the last situation. It is easy to check that the curves S_X, S_Y determine four quadrants: Q_1 (SLR), Q_2 (SCR), Q_3 (SWR^+ with the orbit of Z pointing M^+) and Q_4 (SWR^- with the orbit of Z pointing M^-). Furthermore it will be proved in this case that the $S.L$-vector field can be C^1-extended to a full neighborhood of p in M and p is a critical point of this vector field. We must add the following extra assumption to such situation:

c) The point p is a hyperbolic critical point of $F(X,Y)$ and its respective eigenspaces are transverse to S_X and S_Y at p.

We still remark that if $p \in S_X$, $p \notin S_Y$ and p is a fold point (resp. cusp point) then $F(X,Y)$ is transverse to S_X at p (resp. $F(X,Y)$ has a quadratic contact with S_X at p).

2.5. Remark. Let $q \in S_X \cup S_Y$. For sake of simplicity we adopt that: i) if $q \in \partial(SLR)$ then the orbit of Z at q follows the orbit of $F(X,Y)$; ii) if $q \notin \partial(SLR)$ and $q \in S_X$ then the orbit of Z at q follows the orbit of Y; iii) if $q \notin \partial(SLR)$ and $q \in S_Y$ then the orbit of Z at q follows the orbit of X.

§3 – Main Results

The following result is an immediate consequence of [8].

3.1. Proposition. If $Z \in G^r$ is structurally stable at p then p is either a regular point or a generic singularity of Z.

For technical reasons we are going to define an U-singularity of $Z \in G^r$ just in §5. Any way we remark that the set U_p of all vector fields $Z \in G^r$ having p as an U–singularity is open in G^r.

Theorem 1. $Z \in G^r$ is structurally stable at $p \in M$ if and only if either i) p is a regular point of Z or ii) p is a generic singularity of Z and $Z \notin U_p$.

3.2. Remark. Next, we list some properties of an U–singularity:

i) If $p \in M$ is an U–singularity of $Z = (X, Y)$ then p is a fold point of both fields X and Y. Moreover $S_X \cap_{\tau} S_Y = \{p\}$.

As it was said in §2. S_X and S_Y determine four quadrants $Q_1(SLR)$, $Q_2(SWR)$, $Q_3(ESR)$ and $Q_4(SWR)$. (See Figure 3.1.)

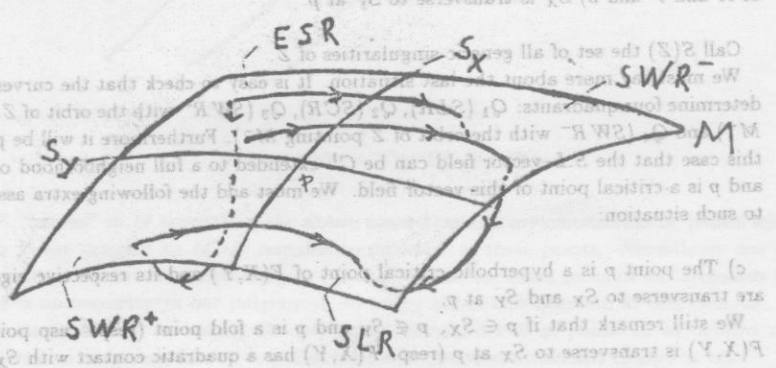

Figure 3.1 — The U—Singularity

iii) In this case we may define around p a 1^{th} return mapping $\varphi_Z : (M,p) \to (M,p)$ associated to Z which belongs to the same class of differentiability of X and Y. Observe that the orbits of Z are spirals around S_X and S_Y. We are going to check in §5 that we may separate U_p in two distinct connected components; in order $U_p = H_p \cup L_p$ with $Z \in H_p$ if p is a hyperbolic fixed point of φ_Z and $L_p = U_p / H_p$. It is convenient to observe that L_p has non empty interior. We add that φ_Z is not hyperbolic if and only if the associated eigenvalues have nonzero imaginary parts or coincide with ± 1.

iv) Let $F = F(X,Y)$ be (the C^1-extension of the sliding vector field associated to $Z = (X,Y)$ and $p \in M$. The following proposition is immediate:

3.3. Proposition. If φ_Z or $F(X,Y)$ is not hyperbolic at p then Z is not structurally stable at p.

A complete answer, in this case, to the structural stability in G^r is then carried out to study the simultaneous behavior of $F(X,Y)$ (restricted to SLR) and φ_Z. We recall that the sector SLR is a distinguished set, as well as its boundary, for the stability of Z.

3.4. Definition. We say that $p \in M$ is an S–singularity of Z if: i) $Z \in L_p$; moreover the rotation number associated to φ_Z is irrational; ii) The eigenvalues of dF_p are real, negative and distinct; moreover the associated eingenspaces are transverse to S_X and S_Y; and iii) the eigenspace associated to the eigenvalue of small absolute value does meet the SLR and the other eigenspace does not meet SLR.

We have the following result:

Theorem 2. $Z = (X,Y) \in G^r$ **is asymptotically stable at** p **provided that one of the following conditions is satisfied:**

i) $p \in SLR$ **and** $F(X,Y)$ **is asymptotically stable at** p**, or ii)** p **is an** S**–singularity of** Z.

§4 — Stratification of M

Let $p \in M$ be a generic singularity of $Z = (X,Y) \in G^r$.

Our goal in this section is to list some submanifolds of M which are distinghished due to their persistance for small perturbations of Z. Moreover any equivalence between Z and $\tilde{Z}$ must necessarily preserve the correspondent such submanifolds. They also stratify M following the Whitney's conditions.

We separate the cases:

i) $p \in \mathrm{Int}(SLR)$.

In this case we distinguish $\{p\}$ and the saddle separatrices of $F(X,Y)$.

ii) p is a singularity of Z of tangencial type and the connected component $C(p)$ of $S(Z)$ is a regular curve (i.e. $C(p)$ is either S_X or S_Y). In this case we list only $C(p)$.

iii) $\{p\} = S_X \cap S_Y$.

In this case we list $\{p\}$, $S_X - \{p\}$ and $S_Y - \{p\}$. We add to this list the saddle separatrices of $F = F(X,Y)$ if they are contained in SLR. If the eigenvalues of dF_p are real and have the same signal then we have to distinguish the strong invariant manifold of dF_p if it is contained in SLR.

4.1. Remark. In [9 section 4] is shown that the above sets are invariant for equivalence between the vector fields in G^r.

§5 — The First Return Mapping

First of all, let $X \in \mathcal{X}^r$, $p \in M$ with $X(p) \neq 0$ and $f : (\mathbb{R}^3, M) \to (\mathbb{R}, 0)$ be any C^∞ local implicit representation of M at p. Assume that $Xf(p) = 0$. Choose condinates $x = (x_1, x_2, x_3)$ around p in $\mathbb{R}^3$ such that $X = \frac{\partial}{\partial x_1}$. Let $x_3 = g(x_1, x_2)$ be a C^∞ solution of $f(x_1, x_2, x_3) = 0$ with $g(0,0) = 0$.

Fix $N = \{x_1 = 0\}$ as being the section transverse to X at p.

One sees that the projection $G_X : (M, p) \to (N, p)$ of M, along the orbits of X, onto M is given by

$$G_X(x_1, x_2, g(x_1, x_2)) = (0, x_2, g(x_1, x_2)).$$

Moreover, we have that

$$Xf(x) = \frac{\partial g}{\partial x_1}(x), \quad X^2 f(x) = \frac{\partial^2 g}{\partial X^2}$$

and

$$X^3 f(x) = \frac{\partial^3 g}{\partial x^3}(x).$$

When p is a fold singularity of G_X then there exists a C^r–diffeomorfism $\varphi_X : (M, p) \to (M, p)$, called the symetric associated to G_X. The mapping φ_X satisfies $\varphi_X(p) = p$, $G_X \circ \varphi_X = G_X$, $\varphi_X^2 = \text{Id}$. We observe that φ_X is C^∞ conjugate to $\varphi_0(x_1, x_2) = (x_1, -x_2)$, and $S_X = \text{Fix}\varphi_X$. Moreover if $q \notin S_X$ then $\varphi_X(q)$ is the point where the trajectory of X possing through q meets M.

Let $Z = (X, Y) \in G^r$ and assume now that: i) p is a fold point of both fields X and Y; ii) $S_X \pitchfork_p S_Y$ and iii) $X^2 f(p) < 0$ and $Y^2 f(p) > 0$. (See Figure 5.1). We say in this case that p is a focus of $Z = (X, Y)$.

If we just look at the foliation generated by the orbits of Z then it is easy to recognise that the diffeomorphism $\varphi = \varphi_X \circ \varphi_Y$ works as a first return mapping of Z at p, with $\varphi(p) = p$.

Given φ_X and φ_Y as above we may choose coordinates (x, y) around p in M such that $\varphi_X(x, y) = (x,, -y)$. Then for some a, b, $c \in \mathbb{R}$ we have

$$\varphi_Y'(0) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a^2 + bc = 1$.

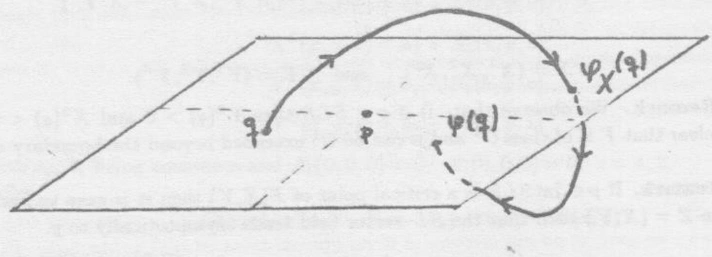

Figure 5.1 — The first return mapping

The following result is proved in [10].

5.1. Lemma. p is a hyperbolic fixed point of $\varphi = \varphi_X \circ \varphi_Y$ if and only if $a^2 > 1$.

We remark that the eigenvalues of $\varphi'(p)$ are $\lambda = a \pm (a^2 - 1)^{1/2}$. In this way we see that if $a^2 > 1$ then p is a saddle point of φ and if $a^2 < 1$ then the eigenvalues of $\varphi'(0)$ have non zero imaginary parts. This shows the assertions contained in 3.2.iii).

Now, the following proposition becomes immediate.

5.2. Proposition. There is an open set $\mathcal{O}$ in G^r such that any $Z \in \mathcal{O}$ is structurally unstable.

Now we define an U–singularity.

5.3. Definition. We say that $p \in M$ is an U–singularity of $Z = (X, Y)$ if: a) p is a generic singularity of Z; b) p is a M–singular point of both vector fields X and Y; c) p is a focus of $Z = (X, Y)$.

If p is a hyperbolic fixed point of $\varphi = \varphi_X \circ \varphi_Y$ we may refer to it as a hyperbolic U–singularity of $Z = (X, Y)$.

§6 — Sliding Vector Fields

Let $Z = (X, Y)$.

Let (x, y, z) be a system of coordinates around $p \in M$ such that the function $f : I\!R^3, M \to I\!R, 0$ which represens M is given by $f(x, y, z) = z$. In these coordinates the SL–vector field $F = F(X, Y)$ has the following expression:

$$F = (F^1, F^2) = (Y^3 - X^3)^{-1}(X^1Y^3 - X^3Y^1, X^2Y^3 - X^3Y^2)$$

with

$$X = (X^1, X^2, X^3) \quad \text{and} \quad Y = (Y^1, Y^2, Y^3).$$

6.1. Remark. We observe that: i) if $q \in SLR$ then $Y^3(q) > 0$ and $X^3(q) < 0$; ii) It seems clear that F is of class C^1 and it can be C^1 extended beyond the boundary of SLR.

6.2. Remark. If $p \in \text{Int}SLR$ is a critical point of $F(X, Y)$ then it is easy to find conditions on $Z = (X, Y)$ such that the SL–vector field tends asymptotically to p.

We have the following lemmas:

6.3. Lemma. Assume that $p \in \partial(SLR)$ is a fold point of X and an M–regular point of Y. Then $F(X, Y)$ is transverse to S_X at p.

Proof. In the above coordinates we have

$$Xf = X^3, \quad X^2f = X^1X_1^3 + X^2X_2^3$$

and

$$FX^3 = F^1X_1^3 + F^2X_2^3.$$

Observe now that $F^1(p) = X^1(p)$ and $F^2(p) = X^2(p)$. This finishes the proof. □

In a similar way we prove that

6.4. Lemma. Assume that $p \in \partial(SLR)$ is a cusp point of X and an M–regular point of Y. Then $F(X, Y)$ has a quadratic contact with S_X at p.

6.5. Remark. Assume $p \in \partial(SLR)$ is a generic singularity and is an M–singular of both fields X and Y. In this case $Xf = X^3$, $Yf = Y^3$ and of course the orbits of $F(X, Y)$ in a neighborhood of p coincide with the orbits of the fields $G(X, Y) = (X^1Y^3 - X^3Y^1,\ X^2Y^3 - X^3Y^2)$. So p is a critical point of this vector field.

6.6. Remark. In what follows we are going to exhibit explicitely the linear part of $G(X, Y)$ in these coordinates. As p is a generic singularity we may assume without loss of generality that

$$Xf(x,y,z) = x \quad \text{and} \quad Yf(x,y,z) = y.$$

We have then $X^3(x,y,z) = x$ and $Y^3(x,y,z) = y$. So the region SLR is given by $x < 0$ and $y > 0$.

We have

$$F(x,y,z) = (yX^1 - xY^1,\ yX^2 - xY^2),$$

$$X^1(x,y,z) = a_1 + A_1(x,y,z),$$

$$X^2(x,y,z) = a_2 + A_2(x,y,z),$$

$$Y^1(x,y,z) = b_1 + B_1(x,y,z),$$

$$Y^2(x,y,z) = b_2 + B_2(x,y,z).$$

with a_i, b_i being constants and $A_i(0,0,0) = 0$, $B_i(0,0,0) = 0$, $i = 1,2$.

So

$$G(x,y) = (a_1y - b_1x,\ \ a_2y - b_2x) + H(x,y)$$

with $H(0,0) = (0,0)$.

6.7. Remark. It is convenient, at this moment to give in the above coordinates the expression of the first return mapping $\varphi = \varphi_X \circ \varphi_Y$ associated to $Z = (X,Y)$.

The solutions of X and Y possing through $q = (x,y,0)$ are respectively

$$(6.7.\text{i.}) \quad \begin{cases} u = x + a_1t + u_1 \\ v = y + a_2t + v_1 \\ w = xt + a_1\frac{t^2}{2} + w_1 \end{cases}$$

and

$$(6.7.\text{ii.}) \quad \begin{cases} u = x + b_1t + u_2 \\ v = y + B_2t + v_2 \\ w = yt + B_2\frac{t^2}{2} + w_3 \end{cases}$$

with u_i, v_i, w_i, $i = 1,2,3$ being higher order terms.

One deduces directly that

$$\varphi(x,y) = (-x - \frac{2b_i}{b_2}(-\frac{2a_2}{a_1}x + y),\ \frac{2a_2}{a_1}x + y) + K(x,y),$$

with $K(x,y) = 0(|(x,y)|^2)$.

It is easy to conclude that $a_1 < 0$ and $b_2 > 0$ provided that p is an U–singularity. □

§7 Proof of Theorem 1

In this section we show the non structural stability of an U-singularity and give the proof of Theorem 1.

The following result will be used in the sequel and its proof is a variation of Th. 2 of [7, p. 341].

7.2. Lemma. Let $\varphi, \tilde{\varphi} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be diffeomorfisms such that their eigenvalues are respectively a, b and $\tilde{a}$, $\tilde{b}$ with $|b| < 1 < |a|$ and $|\tilde{b}| < 1 < |\tilde{a}|$. Let L be a C^r curve passing through 0 and transverse to the eigenspaces of φ and $\tilde{\varphi}$ at 0. If there is a C^0 L-preserving conjugacy between φ and $\tilde{\varphi}$ then

$$\frac{\log|a|}{\log|b|} = \frac{\log|\tilde{a}|}{\log|\tilde{b}|}.$$

Proof. We are going to show that $\frac{\log|a|}{\log|b|}$ is an invariant for the $C^0 L$-preserving conjugacy. In this way we may, of course, assume φ in its normal form $\varphi(x, y) = (ax, by)$.

We consider the case $0 < b < 1 < a$. The other cases are treated in a similar way.

As usual we denote W^u, W^s the unstable and stable manifolds of φ, respectively.

We may take L as being the grafic of a C^r-mapping $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.

Fix $\alpha_0 = (x_0, 0)$ and $\beta_0 = (0, y_0)$ with x_0, y_0 very small and positive.

Let $\gamma_1 = (x_i,\ f(x_i))$ be a sequence in L converging to 0.

Choose sequences $(m_i) \to \infty$, $(n_i) \to \infty$, $(\alpha_i) \to \alpha_0$ and $(\beta_i) \to \beta_0$ such that $\varphi^{m_i}(\gamma_i) = \alpha_i$ and $\varphi^{n_i}(\beta_i) = \gamma_i$.

Call $d_i = \operatorname{dist}(\alpha_i, W^s)$ and $\delta_i = \operatorname{dist}(\beta_i, W^u)$. So we have

$$d_i = a^{m_i} x_i, \quad \gamma_i = b^{-n_i} f(x_i) \quad \text{and} \quad d_i = K a^{m_i} b^{n_i} \delta_i$$

where K is constant depending on f.

So $\log d_i = m_i \log a + n_i \cdot \log b + \log K \delta_i$ and we conclude that

$$\lim_{i \to \infty} \frac{n_i}{m_i} = -\frac{\log a}{\log b}.$$

This finishes the proof. □

7.3. Proposition. If p is an U-singularity of $Z \in G^r$ then Z is structurally unstable in G^r.

Proof. Under the hypothesis of proposition there are two possibilities for $Z = (X, Y)$: i) p is not a hyperbolic critical point of $\varphi = \varphi_X \circ \varphi_Y$ and ii) p is a hyperbolic critical point of $\varphi = \varphi_X \circ \varphi_Y$ (i.e. a saddle point). The first possibility is obviously non stable. Otherwise if $\tilde{Z} = (\tilde{X}, \tilde{Y})$ is a small perturbation of Z and equivalent to Z then this equivalence must be a conjugacy between $\varphi = \varphi_X \circ \varphi_Y$ and $\tilde{\varphi} = \varphi_{\tilde{X}} \circ \varphi_{\tilde{Y}}$. Moreover the sectors in $(\mathbb{R}^2, 0)$ determined by the SLR of the fields must be preserved as well as their boundaries. But the boundaries are constituted by regular curves which generically are transverse to the respective eigenspaces of the diffeomorphisms. So the above lemma implies immediately that Z is structurally unstable at p This finishes the proof. □

7.4. Proof of Theorem 1. Let $p \in M$ be a generic singularity of $Z = (X, Y) \in G^r$ and $\tilde{Z} = (\tilde{X}, \tilde{Y})$ be a small perturbation of Z in G^r. In this case the construction of a topological equivalence between the fields is straighforward.

If p is not an U–singularity we proceed as follows to obtain an equivalence between Z and $\tilde{Z}$.

We define the equivalence h in M and then we extend to a full neighborhood of p, in a natural way. We have firstly to define h on the distinghed sets as given in §4. If p belongs to the boundary of SLR we have to begin by constructing a topological equivalence between the respective SL–vector fields on the regions SLR.

The proof follows immediately from Proposition 5.2 and Proposition 7.3. □

§8 — Asymptotic Stability

Let $p \in M$ be an S–singularity of $Z = (X, Y) \in G^r$.

The proofs of the coming lemmas are trivial.

8.1. Lemma. Assume $p \in \partial(SLR)$. If $Z = (X, Y)$ is asimptoticaly stable at p then the SL–vector field $F(X, Y)$ is asimptoticaly stable at p in SLR.

Call λ_1, λ_2 the eigenvalues of the extended SL–vector fields and T_1, T_2 their correspondent eigenspaces.

8.2. Lemma. Assume $p \in \partial(SLR)$. The SL–vector field $F(X, Y)$ is asymptoticaly stable (at p) in SLR if and only if: i) λ_1, λ_2 are real, negative and distinct (say $\lambda_1 < \lambda_2 < 0$); ii) $T_2 \cap SLR \neq \emptyset$ and $T_1 \cap SLR = \emptyset$. (See Figure 8.1)

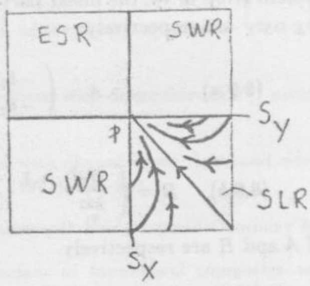

Figure 8.1 — The SL–vector field.

8.3. Remark. We recall that $F(X,Y)$ is transverse to S_X and S_Y off p. The next lemmas are immediate.

8.4. Lemma. Let $p \in SLR$ be a generic singularity of $Z = (X,Y)$. Then Z is asympt-totically stable at p if and only if $F(X,Y)$ is asymptotically stable at p.

8.5. Lemma. Let $p \in \partial(SLR)$ be a generic singularity of $Z = (X,Y)$. If Z is asympoti-cally stable at p then p is a non hyperbolic U–singularity of $Z = (X,Y)$.

Proof of Theorem 2.

Lemma 8.6 implies the first part of proof.

We now proceed the proof of part ii).

Let $q \in M$. It is clear that if $q \in C\ell(SLR)$ then the trajectory of Z passing through q coincides with the trajectory of F and so it converges to p. If $q \notin C\ell(SLR)$) then the (positive) trajectory of Z is governned by either $\varphi = \varphi_X \circ \varphi_Y$ or $\varphi^{-1} = \varphi_Y \circ \varphi_X$. Since by hypothesis the rotation angle of these diffeomorphisms is irrational there will be a positive integer k such that $\varphi^k(q)$ (or $\varphi^{-k}(q)$) falls down in SLR and so the trajectory of Z will converges to p.

Now the conclusion of the proof is obvious. □

8.7. Remark. Observe that any other convention adopted as in Remark 1.1 or Remark 2.5, would not modify any result of the paper.

8.8. Remark. We would get the asymptotic stability of $Z = (X,Y)$ (as in Theorem 2) at p under the follwoing hypothesis: the rotation angle of $\varphi = \varphi_X \circ \varphi_Y$ is rational plus some extra conditions. We add that in the coordinates given in §6 these extra conditions are: the rotation angle must be in the interval $(\frac{\pi}{2}, \frac{\pi}{2})$. The proof follows similarly.

8.9. Remark. Next we are going to show the existence of an S–singularity. Thus is:

In the coordinates system given in §6, the linear parts of the SL–vector fields and of the diffeomorphism $\varphi = \varphi_X \circ \varphi_Y$ are respectively:

$$A = \begin{pmatrix} -b_1 & a_1 \\ -b_2 & a_2 \end{pmatrix} \tag{8.9.a}$$

and

$$B = \begin{pmatrix} \frac{2a_2b_1}{a_1b_2} - 1 & \frac{-2B_1}{b_2} \\ \frac{2a_2}{a_1} & -1 \end{pmatrix} \tag{8.9.b}$$

So the eigenvalues of A and B are respectively

$$\mu = [(b_1 - a_2) \pm ((a_2 + b_1)^2 - 4a_1b_2)^{1/2}]/2$$

and

$$\lambda = \left(\frac{1 - a_2 b_1}{a_1 b_2}\right) \pm \left(\left(\frac{a_2 b_1}{a_1 b_2}\right)^2 - 1\right)^{1/2}.$$

From the hyphoteses and Lemma 8.2 we would look for those a_1, b_1, a_2, b_2 such that

(i) $(\frac{a_2 b_1}{a_1 b_2} - 1)^2 < 1$.

(ii) $(b_1 - a_2) < 0$.

(iii) $|b_1 - a_2| > ((a_2 + b_1)^2 - 4a_1 b_2)^{1/2}$.

We already know that $a_1 b_2 < 0$.
From (i), (ii) and (iii) we have respectively

(i') $2a_1 b_2 < a_2 b_1 < 0$.

(ii') $b_1 < a_2$.

(iii') $a_1 b_2 > a_2 b_1$.

Finally we have to choose a_1, b_1, a_2, b_2 such that $b_1 < a_2$ and $2a_1 b_2 < a_2 b_1 < a_1 b_2 < 0$. □

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