CODIMENSION TWO PRODUCT SUBMANIFOLDS WITH NON–NEGATIVE CURVATURE

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ABSTRACT. We study an isometric immersion $f: M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, of a complete, non-compact Riemannian manifold with non-negative sectional curvatures when M is a Riemannian product $M_1^{n_1} \times M_2^{n_2}$ of two irreducible manifolds with $n_i \geq 2$. We prove that either M is homeomorphic to $S^{n_1} \times \mathbb{R}^{n_2}$ or one of the factors is flat and f is cylindrical.

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Abstract : We study an isometric immersion $f: M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, of a complete, non-compact Riemannian manifold with non-negative sectional curvatures when M is a Riemannian product $M_1^{n_1} \times M_2^{n_2}$ of two irreducible manifolds with $n_i \geq 2$. We prove that either M is homeomorphic to $S^{n_1} \times \mathbb{R}^{n_2}$ or one of the factors is flat and f is cylindrical.

1. Introduction: A complete Riemannian manifold M with non-negative sectional curvatures is diffeomorphic to the total space of a vector bundle over a compact submanifold, its soul (Cheeger-Gromoll, [4]). It is an interesting problem to know under which conditions M turns out to be a trivial bundle over its soul. Some results in this direction were obtained in [1] and [7]. In this note we prove the following:

Theorem: Let M^n be a Riemannian product $M_1^{n_1} \times M_2^{n_2}$, where for each $i=1,2$, $M_i^{n_i}$ is a n_i-dimensional ($n_i \geq 2$) complete, non-compact, irreducible Riemannian manifold with non-negative sectional curvatures. Suppose M non-flat with non-trivial soul. If $f: M^n \to \mathbb{R}^{n+2}$ is an isometric immersion then either

(a) one of $M_i^{n_i}$ is flat and f is n_i - cylindrical; or

(b) M is homeomorphic to the product $S^{n_1} \times \mathbb{R}^{n_2}$ and f is a product of hypersurface immersions.

2. Proof of Theorem :

First we prove that if both M_1 and M_2 are not flat then f is not cylindrical.

In fact, if f is cylindrical we consider an isometric splitting of M as $\overline{M}^m \times \mathbb{R}^{n-m}$, $m \geq 2$, so that f is (n-m)-cylindrical ([5]). By a Theorem of Bishop ([2]) the holonomy algebra of M, $h(M) = h(\overline{M})$ is one of the following possibilities:

$$h(M) = h(\overline{M}) = \begin{cases} o(m); \\ o(r) + o(m-r), \; r \neq 0; \\ u(2), \text{ the unitary algebra of some complex structure} \\ \qquad \text{on } T\overline{M} \text{ if } m=4. \end{cases}$$

By the other hand, we have also $h(M) = h(M_1) + h(M_2)$. For each $(x_1,x_2) \in M_1 \times M_2$ let $j_1: M_1 \to M_1 \times \{x_2\}$ and $j_2: M_2 \to \{x_1\} \times M_2$ be respectively the copies of M_1 and M_2 through (x_1,x_2). Since M_i are irreducible, we observe that $f_i = f \circ j_i$ cannot be cylindrical. Therefore, we choose x_i in each M_i so that there is no relative nullity directions of f_i at x_i. Each holonomy algebra $h(M_i)$ at x_i is again one of the possibilities: $o(n_i)$; $o(r_i) + o(n_i - r_i)$, $r_i \neq 0$; $u(2)$, the unitary algebra of some complex structure on TM_i if $n_i = 4$.

Since both M_i are not flat, it is obvious that at $(x_1,x_2) \in M^*$ chosen as above, the only possibility for the holonomy is $o(n_1 - 1) + o(n_2 - 1)$ with

$\partial(n_i - 1) = 0$, $i=1,2$. In this case, Theorem 1' of Bishop ([2]) implies that the normal curvature of f at (x_1,x_2) vanishes, that is . there is a choice of tangent and normal frame at (x_1,x_2) such that the matrices for the second fundamental operators have the form:

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where each A_i is a $(n_i - 1) \times (n_i - 1)$ non-singular diagonal matrix, $i=1,2$. This means that there must be directions of relative nullity tangent to each M_i at (x_1,x_2), which is a contradiction.

Now, we claim that if $M_2^{n_2}$ is flat then f is n_2-cylindrical.

As above, consider $(x_1,x_2) \in M$ such that for each $i=1,2$, f_i has no relative nullity directions at x_i . M_1 has some non-zero sectional curvature at x_1 . Otherwise, since M_1 has no relative nullity vectors at x_1, we would have $n_1=2$ which implies that M_1 is flat. Then the argument of Lemma 3.3 in [7] implies that the relative nullity subspace of f_2 at x_2 is at least (n_2-1)-dimensional, which is a contradiction.

Therefore, since M_i are both irreducible, we can conclude that at every $(x_1,x_2) \in M$ all the vectors tangent to M_2 are directions of relative nullity, hence f is n_2-cylindrical, as claimed.

To conclude, we prove that if f is not cylindrical then f is a product of hypersurface immersions.

In fact, if f is not a product of hypersurface immersions, Theorem 2 of

Moore ([6]) implies that there is a complete geodesic carried by f onto a straight line in $\mathbb{R}^{n+2}$. This geodesic must be a line, that is, a geodesic each segment of which realizes the distance between its end points. Then Toponogov's Theorem ([9],[4]) allows us to consider the isometric splitting of M as $\overline{M}^m \times \mathbb{R}^{n-m}$ with $\overline{M}$ without lines and $\mathbb{R}^{n-m}$ flat. The claim above implies that f is cylindrical, a contradiction.

Therefore, there are isometric immersions $f_1: M_1^{n_1} \to \mathbb{R}^{n_1+1}$ and $f_2: M_2^{n_2} \to \mathbb{R}^{n_2+1}$ such that $f = f_1 \times f_2$. Furthermore, since both M_1 and M_2 are irreducible, there are points x_1 and x_2 respectively in M_1 and M_2 with no relative nullity directions. For each $i=1,2$, all the sectional curvatures of M_i at x_i are positive, hence $M_i^{n_i}$ is either homeomorphic to S^{n_i} or $\mathbb{R}^{n_i}$ ([3], [8]). As we supposed M with non-trivial soul, we must have M homeomorphic to $S^{n_1} \times \mathbb{R}^{n_2}$.

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