GREEN'S FUNCTION FOR PEDESTRIANS

E. Capelas de Oliveira

RELATÓRIO TÉCNICO Nº 17/86

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Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Ciência da Computação
IMECC – UNICAMP
Caixa Postal 6065
13.100 - Campinas, SP
BRASIL

Abril – 1987

GREEN'S FUNCTION FOR PEDESTRIANS

E. Capelas de Oliveira
Departamento de Matemática Aplicada
IMECC - UNICAMP - São Paulo - Brasil
Caixa Postal 6065

ABSTRACT. We introduce the Green's function in a pedestrian way for Jacobi differential equation by means of the method of variation of parameters.

KEY WORDS. Green's Function; Jacobi Differential Equation; Orthonormal Polynomials; Variation of Parameters; Hypergeometric Equation.

0. INTRODUCTION.

Many second order differential equations which appear in the context of mathematical physics can be identified with a hypergeometric differential equation. This differential equation can be written in terms of the Jacobi differential equation which is the general case of a equation having as solutions the orthonormal polynomials, particular cases being Gegembauer, Legendre and Chebycheff polynomials. These polynomials, commonly called classical polynomials, appear frequently in Quantum Mechanics(1) and Lie Theory(2).

In this paper we discuss the hypergeometric differential equation in terms of the Jacobi polynomials. By knowing the parameters we obtain the particular cases of this differential equation.

The treatment presented in this paper is via Green's function because we know two linearly independent solutions of the homogeneous differential equation and to calculate the Green's function we use the method of variation of parameters.

This paper is organized as follows: in the first section we discuss the differential equation for the hypergeometric functions; in the second section we obtain the Green's function for the non-homogeneous differential equation; in the third section we present an exemple involving boundary conditions (Dirichlet conditions) and in the last section we present our conclusions.

1. THE HYPERGEOMETRIC EQUATION

The second order differential equation[3]

$$x(1-x)\frac{d^2z}{dx^2} + [c-(a+b+1)x]\frac{dz}{dx} - abz = 0 \qquad (1)$$

where $z = z(x)$ and $\underline{a}$, $\underline{b}$, $\underline{c}$ are constant parameters, called the hypergeometric differential equation has three singular points $x = 0,1,\infty$. These are regular singular points of the differential equation and the pairs of expoents at these points are

$$\gamma^{(0)} = 0;1-c \qquad \gamma^{(1)} = 0;c-a-b \qquad \gamma^{(\infty)} = a;b$$

respectively[4]. From the general theory of the differential equations one distinguishes the following cases:

i) None of the numbers c; $c-b$; $c-a-b$ is equal to an integer.

In this case a system of two linearly independent solutions of the equation in the vicinity of the singular points $x = 0,1,\infty$ are given by Kummer's relations[3].

ii) One of the numbers $a, b, c-a, c-b$, is an integer.

In this case one of the hypergeometric series terminates and the corresponding solution is of the form

$$z(x) = x^{\lambda}(1 - x)^{\mu}P_n(x)$$

where $P_n(x)$ is an n^{th} degree polynomial in x. This is the degenerate case of the hypergeometric differential equation:

iii) $c - a - b$ is an integer and c is not an interger; $c=1$; $c = m+1$ with $m = 1,2,3,...$ and $c = 1 - m$ with $m = 1,2,3...$.

In this case a fundamental systems of solutions involve a logarithm term.

In this paper we fix our attention in the degenerate case where we have a polynomial solution.

Introducing a change of variable $2x = 1 + t$ and defining the parameters $a = n+1+\alpha+\beta$; $b = -n$; $c = \beta+1$, in the Eq. (1) we obtain the following differential equation

$$(1 - x^2)\frac{d^2z}{dx^2} + [(\beta - \alpha) - (\alpha + \beta + 2)x]\frac{dz}{dx} + n(n+1+\alpha+\beta)z = 0 \qquad (2)$$

called the differential equation of Jacobi polynomials.

The Jacobi polynomials, denotaded by $P_n^{(\alpha,\beta)}(x)$, are orthonormal polynomials over the interval $(-1, 1)$ with the weight function $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ and normalized by the relation $P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(\alpha+1)\Gamma(n+1)$. The particular cases are Gegembauer, Legendre and Chebycheff polynomials.

The two linearly independent solutions of Eq. (2) are

$$z_1(x) = P_n^{(\alpha,\beta)}(x) \quad \text{and} \quad z_2(x) = Q_n^{(\alpha,\beta)}(x)$$

where the second solution of this differential equation is called the Jacobi function of the second kind and which is defined[4] by

$$(x-1)^{\alpha}(x+1)^{\beta}Q_n^{(\alpha,\beta)}(x) = 2^{-n-1}\int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}(x-t)^{-n-1}dt \quad (3)$$

where $\alpha > -1$, $\beta > -1$ and $n \geq 0$. Excluding the case $n = 0$, $\alpha + \beta + 1 = 0$ this solution is also linearly independent of $P_n^{(\alpha,\beta)}(x)$. In this exceptional case we have $Q_o^{(\alpha,\beta)}(x) = \text{constant}$.

2. GREEN'S FUNCTION

The non-homogeneous differential equation of Jacobi polynomials is

$$(1-x^2)\frac{d^2z(x)}{dx^2} + [(\beta-\alpha) - (\alpha+\beta+2)x]\frac{dz(x)}{dx} + n(n+1+\alpha+\beta)z(x) = -f(x) \quad (4)$$

where $f(x)$ is a forcing term and $\alpha > -1$, $\beta > -1$ and $n > 0$.

The Green's function associated to this differential equation satisfy

$$(1-x^2)\frac{d^2}{dx^2}G(x,x') + [(\beta-\alpha) - (\alpha+\beta+2)x]\frac{d}{dx}G(x,x') + n(n+1+\alpha+\beta)G(x,x') = -\delta(x-x') \quad (5)$$

where a delta function introduce a impulse applied at the point x'.

The solution of Eq.(4) in terms of the Green's function is given by

$$z(x) = \int_a^b f(x')G(x,x')dx' \quad (6)$$

where the boundary conditions are carried by the Green's function. Once we know the two linearly independent solutions we use the method of the variation of parameters [5].

The solution of Eq.(5) is written in terms of solutions to the corresponding homogeneous differential equation as

$$G(x,x') = \phi(x,x')P_n^{(\alpha,\beta)}(x) + \psi(x,x')Q_n^{(\alpha,\beta)}(x) \tag{7}$$

where the functions $\phi(x,x')$ and $\psi(x,x')$ are two functions to be determined.

We have for the first derivative term

$$\frac{d}{dx} G(x,x') = \frac{d}{dx} \phi(x,x')P_n^{(\alpha,\beta)}(x) + \phi(x,x')\frac{d}{dx} P_n^{(\alpha,\beta)}(x) + $$
$$+ \frac{d}{dx} \psi(x,x')Q_n^{(\alpha,\beta)}(x) + \psi(x,x')\frac{d}{dx} Q_n^{(\alpha,\beta)}(x)$$

and with the usual constrain

$$P_n^{(\alpha,\beta)}(x)\frac{d}{dx} \phi(x,x') + Q_n^{(\alpha,\beta)}(x)\frac{d}{dx} \psi(x,x') = 0 \tag{8}$$

we obtain for the second derivative term, already substituing, in Eq.(4) the following equation:

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x)\frac{d}{dx} \phi(x,x') + \frac{d}{dx} Q_n^{(\alpha,\beta)}(x) \frac{d}{dx}\psi(x,x') = -\frac{1}{1-x^2}\delta(x-x'). \tag{9}$$

Solving the system of Eqs.(8) and (9) for $\frac{d}{dx} \phi(x,x')$ and $\frac{d}{dx} \psi(x,x')$ we obtain

$$\frac{d}{dx} \phi(x,x') = \frac{1}{W(x)} (1 - x^2)^{-1} Q_n^{(\alpha,\beta)}(x)\delta(x - x')$$
$$\tag{10}$$
$$\frac{d}{dx} \psi(x,x') = - \frac{1}{W(x)}(1 - x^2)^{-1} P_n^{(\alpha,\beta)}(x)\delta(x - x').$$

Where $W(x)$ is the wronskian which is given by the following relation

$$W(x) = P_n^{(\alpha,\beta)}(x)\frac{d}{dx} Q_n^{(\alpha,\beta)}(x) - Q_n^{(\alpha,\beta)}(x)\frac{d}{dx} P_n^{(\alpha,\beta)}(x)$$

which is always non zero, because $P_n^{(\alpha,\beta)}(x)$ and $Q_n^{(\alpha,\beta)}(x)$ are two linearly independent solution of the homogeneous differential equation.

The integration of Eq.(10) results in

$$\phi(x,x') = A - \begin{cases} 2^{-\alpha-\beta} \dfrac{n!\,\Gamma(n+1+\alpha+\beta)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} Q_n^{(\alpha,\beta)}(x') & x > x' \\ 0 & x < x' \end{cases}$$

$$\psi(x,x') = B + \begin{cases} 2^{-\alpha-\beta} \cdot \dfrac{n!\,\Gamma(n+1+\alpha+\beta)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha,\beta)}(x') & x > x' \\ 0 & x < x' \end{cases} \tag{11}$$

where A and B are arbitrary constants.

To obtain the Green's function we substitue the above set of equations in Eq.(7), then

$$G(x,x') = AP_n^{(\alpha,\beta)}(x) + BQ_n^{(\alpha,\beta)}(x) + 2^{-\alpha-\beta} \frac{n!\,\Gamma(n+1+\alpha+\beta)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \cdot \begin{cases} -P_n^{(\alpha,\beta)}(x)Q_n^{(\alpha,\beta)}(x') + Q_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x') & x > x' \\ 0 & x < x' \end{cases} \tag{12}$$

where A and B are arbitrary constants determined by boundary conditions.

To determine the solution of Eq.(4) we introduce Eq.(12) in Eq.(6).

3. SPECIFIED BOUNDARY CONDITIONS

In this section we take the Green's function with the following conditions: G(0,x') and G(a,x') are specified.

This problem, where we specify the boundary conditions in the Green's function appear in many problems of mathematical physics[6].

We consider a case where α and β are integer number with $\alpha = \beta = 0$. In this case we obtain the Green's function for the Legendre differential equation which appear in many problems of quantum mechanics, associated, for exemple, with Laplace and Klein-Gordon differential equations after the separation of variables[6].

Substituing $\alpha = \beta = 0$ in Eq.(12) we obtain

$$G(x,x') = AP_n(x) + BQ_n(x) + \\ + \begin{cases} -P_n(x)Q_n(x') + Q_n(x)P_n(x') & x > x' \\ 0 & x < x' \end{cases} \tag{13}$$

where $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of Legendre differential equation.

A second solution is given by $Q_n(x)$, Legendre function of the second kind. The function $Q_n(x)$ is not a polynomial and is defined in the complex x-plane cutted along the real axis from -1 to $+1$.

In terms of the hypergeometric series the function $Q_n(x)$ is given by

$$2^{-n}\frac{(2n+1)!}{n!\,n!}Q_n(x) = (x-1)^{-n-1}{}_2F_1(n+1, n+1; 2n+2; \frac{2}{1-x}).$$

To determine the constant parameters A and B in Eq.(13) we use, to exemplify, the Dirichlet boundary conditions

$$G(0,x') = 0 \quad \text{and} \quad G(a;x') = 0.$$

Introducing this boundary conditions for the Eq.(13) we obtain

$$G(x,x') = \frac{1}{P_n(a)} \begin{cases} [P_n(a)Q_n(x) - Q_n(a)P_n(x)]\, P_n(x') & x > x' \\ [P_n(a)Q_n(x') - Q_n(a)P_n(x')]\, P_n(x) & x < x'. \end{cases} \tag{14}$$

Now knowing the function $f(x)$ we obtain the solution of Eq.(4) for $\alpha = \beta = 0$ by means of

$$z(x) = \int_0^a f(x')G(x,x')dx'$$

where $G(x,x')$ is given by Eq.(14).

4. CONCLUSIONS

In this paper we present in the first section the discussions of the hypergeometric differential equation because this equation is a general second order linear differential equation which appear in many problems of mathematical physics being particular cases, for exemple, the Gegembeuer and the Legendre differential equation.

In section two we calculate the Green's function for Jacobi differential equation by means of the method of variation of parameters. We use Dirichlet boundary conditions in section three, to discuss the Green's function for Legendre differential equation.

In this paper we presented the Green's function technic to solve second order linear differential equation via the method of variation of parameters. This result is important from the pedagogical point of view since students can manipulate the Jacobi differential equation with its variations. For exemple, Legendre differential equation appear in many problems of mathematical physics and also engineering. We belive that this technic is a very powerfull tool to solve linear differential equation with boundary conditions without the need of knowing so many concepts.

We use in the example of section three the Dirichlet boundary conditions but this, of course, is not necessary. We can use, for exemple, Neumann or mixed boundary conditions depending of the problem.

We note the important fact that the boundary conditions are carried by Green's function and then knowing the Green's function the problem transforms in the calculation an integral.

ACKNOWLEDGES

The author thanks very much Prof. W. A. Rodrigues Jr. for many useful discussions.

We are gratefull to CNPq for a research grant.

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