MAGNETIC MONOPOLES WITHOUT STRINGS BY KÄHLER–CLIFFORD ALGEBRAS

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ABSTRACT: In substitution for Dirac monopoles with string, we have recently introduced "monopoles without string" on the basis of a generalizes potential, the sum of a vector A and a pseudovector $\gamma_5 B$ potential. By making recourse to the (graded) Clifford algebra, which just allows adding together tensors of different rank (e.g., scalars + pseudoscalars + vectors + pseudovectors +...), in a previous paper we succeeded in constructing a lagrangian and hamiltonian formalism for interacting monopoles that can be regarded as satisfactory from various points of view. In the present note, after having completed that formalism, we put forth a *purely geometrical* interpretation of it within the Kähler algebra on differential forms, essential ingredients being the natural introduction of a "generalized curvature" and the Hodge decomposition. We thus pave the way for the extension of our "monopoles without string" to non-abelian gauge groups. The analogies of this approach with supersymmetric theories are apparent.

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MAGNETIC MONOPOLES WITHOUT STRINGS BY KÄHLER-CLIFFORD ALGEBRA: GEOMETRICAL INTERPRETATION OF A SATISFACTORY FORMALISM.

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It is wellknown that, when describing the electromagnetic field $F^{\mu\nu}$ produced by a Dirac monopole [1] in terms of one potential A^{μ} only, such a potential has to be singular along an arbitrary line starting from the monopole and going to infinity. This "string" has been considered —since long [2]— as unphysical, since the singularity in A^{μ} does not correspond to any singularity in $F^{\mu\nu}$.

Moreover, in the context of the ordinary U(1) gauge theory of electromagnetism, the string may be regarded as originating from the fact that the monopole charges are to be identified [3] with the Chern numbers characterizing the principal U(1) bundles over the base-space $R^2 \times S^2$. And $R^2 \times S^2$ is homeomorphic just to R^4-{line}. This circumstance sheds light on another unphysical feature of the Dirac monopoles, i.e. on the fact that the topology of the base-space is modified by the presence of magnetic charges [3,4]. And such a topology has to become even more exotic when generalized [5] Dirac monopoles are present.

A way out has been looked for by many authors [2,6] via the introduction of a second potential B^{μ}. But they did not completely succeed in dispensing with an exotic base-space, whenever they wanted to stick to the ordinary vector-tensor algebra. However (just on the basis of both a vector potential A and a pseudovector potential $\gamma_5 B$), we recently constructed [7] a rather satisfactory formalism for magnetic monopoles without string (i.e., living in the ordinary Minkowski base-space R^4), by making recourse to Clifford algebra: that is to say, to a (graded) algebra sufficiently powerful to allow adding together tensors of different rank (scalars, pseudoscalars, vectors, pseudovectors, etc.). In ref. [8], for example, both the electric and the magnetic current are vectorial, whilst in our approach they are represented by a vectorial and a pseudovectorial current, respectively (and, nevertheless, we can add them together [7]). Our formalism can be considered satisfactory for the reasons we shall see below. Some analogous results have been got in refs. [9,10].

From Clifford to Kähler. In this paper we want, first of all, to pass from the Clifford language, used in ref. [7], to the language of differential forms, which is the popular one in fiber bundle and gauge theories [11]. This will pave the way, incidentally, for a generalization of our "monopoles without string" to non-abelian gauge groups.

The new language will allow approaching the question of a suitable formalism for interacting monopoles from a purely geometrical point of view. The algebraic structure is now the Kähler-Dirac--Atiyah (or simply Kähler) one [12], since Kähler algebra —acting on the differential forms— is isomorphic to Clifford's —acting, as known, on the "multivectors" [13]—. More precisely, Kähler algebra is isomorphic [14,15] to the "space-time≠1 (Clifford) algebra" $\mathbb{R}_{1,3}$. Notice that we are always confining ourselves to the (16--dimensional) algebras Λ built on the four-dimensional space-time M^4; in such a way that

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^4 \,, \qquad (1)$$

where, as is known, Λ^k is the $\binom{4}{4-k}$-dimensional space of the k--forms (or of the Clifford k-vectors [13]). In other words, Λ^0 and Λ^4 are the one-dimensional spaces of the scalars (0-forms) and pseudoscalars (4-forms), respectively; Λ^1 and Λ^3 the four-dimensional spaces of the vectors (1-forms) and pseudovectors (3-forms), respectively; and Λ^2 is the six-dimensional space of the space--time bivectors (2-forms).

A dictionary is already available from Clifford to Kähler algebra [14÷16]. Let us first recall that, out of the vector space of the differential forms (over the real field), one obtains: (i) the exterior (or Grassmann-Cartan) algebra, by adding the antisymmetric wedge-product ($\wedge$); (ii) the Hodge algebra, by adding also the symmetric inner-product ($\cdot$); (iii) the Kähler algebra, by adding moreover a Clifford product ($\vee$) [17], which —in the simple case when α, or β, is a 1-form— reduces to $\alpha\vee\beta = \alpha\cdot\beta + \alpha\wedge\beta$. Notice that, if α is a i-form and β a j-form, then $\alpha\cdot\beta$ is a $|i-j|$-form and $\alpha\wedge\beta$ is a (i+j)-form.≠2

Such a dictionary is easily obtained by associating the basis $\{dx^\mu\}$ of the algebra of differential forms with the basis $\{\gamma^\mu\}$ of the space-time Clifford algebra. As a consequence, the Hodge star operator (*), acting on the forms, corresponds to Clifford's γ_5 duality operator[7], and the dictionary essentially reduces to:

CLIFFORD		KÄHLER	"DICTIONARY"	
$\gamma_5 f$	$\longrightarrow$	$*f$;	$\Longrightarrow \gamma_5 \simeq *$	(2)

$$\partial\!\!\!/ f \equiv \partial\!\!\!/ \cdot f + \partial\!\!\!/ \wedge f \longrightarrow (d+\delta)\vee f; \Longrightarrow \partial\!\!\!/ \simeq d+\delta \ , \qquad (3)$$

where f is a Clifford number (or a sum of different-degree forms: i.e., an element of Kähler algebra), and the vector derivative $\partial\!\!\!/$ is the Dirac operator[7] ($\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$, when acting on Clifford numbers, or in particular on Dirac matrices). In Kähler algebra, d is the exterior derivative and $\delta \equiv *d*$ is the Hodge co-derivative; with reference to eq.(1), let us observe for future convenience that d transforms for instance one-forms $f_1 \in \Lambda^1$ into two-forms $f_2 \in \Lambda^2$, while δ transforms for instance three-forms $f_3 \in \Lambda^3$ into two-forms $f_2 \in \Lambda^2$. Notice, moreover, that in tensorial language the eq.(3) corresponds to nothing but the decomposition of the vector derivative into divergence and curl; so that $df \simeq \partial\!\!\!/ \wedge f \simeq \text{rot}\, f$, and $\delta f \simeq \partial\!\!\!/ f \simeq \text{div}\, f$. At last, both $\partial\!\!\!/$ and $(d+\delta)$ are the square-root[7,13,14] of the D'Alambertian operator $\Box$; and $d^2 = \delta^2 = 0$.

Generalized potential and field: a satisfactory formalism. Before going on, let us recall from our previous work[7,18] in Clifford (space-time) algebra that the "completed" Maxwell equations wrote

$$\partial\!\!\!/ F = \bar{J} \ ; \qquad \text{with} \qquad \bar{J} \equiv J_e + \gamma_5 J_m \ , \qquad (4)$$

where the space-time "bivector" $F = \frac{1}{2} F^{\mu\nu} \gamma_{\mu\nu} \equiv \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$ is the sum of the Pauli vector $\vec{E}$ and of the Pauli pseudovector $\gamma_5 \vec{H}$; and $\bar{J}$ is the sum of a space-time vector J_e and a space-time pseudovector (= "trivector") $\gamma_5 J_m$. Or, alternatively $[\partial\!\!\!/^2 \equiv \Box]$:

$$\partial\!\!\!/^2 \bar{A} = \bar{J}; \quad \partial\!\!\!/ \cdot \bar{A} = 0, \qquad \text{with} \qquad \bar{A} \equiv A + \gamma_5 B \ . \qquad (4')$$

For an exploitation of the role of the pseudoscalar unit $\gamma_5 \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3$, and the edification of a lagrangian and hamiltonian formalism for interacting monopoles, we just refer to [7].

Here we want to add the following. Our lagrangian and hamiltonian formalism for electromagnetism with monopoles is satisfactory since: (a) we derived the correct field equations (4) from a lagrangian[7] in which also the crossed interactions between J_e and B, and between J_m and A, do explicitly appear, without violating the gauge symmetries (in other words, in our theory a unique type of photon does exist, consistently with the fact that the unique physical field is F); (b) our hamiltonian forwards, among others, the correct expressions for the field energy-density, the Poynting vector, etc. Our approach, however, cannot overcome the "no-go theorems" by Rosenbaum et al.[8]; for instance, Rohrlich[8] showed that a single lagrangian can yield both the field equations and the charge motion-equations only[6] in the trivial case when J_m is proportional to J_e $\left(\text{that is, } -J_e\cdot(\gamma_5 J_m) = J_e \wedge J_m = 0\right)$. Nevertheless, in our theory we need applying the variational principle just once, since our single lagrangian[7] implies even the correct couplings of the currents with the field. In the sense that the field equations (4) imply, if $S^\mu \equiv \frac{1}{2}\bar{F}\gamma^\mu F$, that: $\partial_\mu S^\mu = J_e\cdot F + J_m\cdot(\gamma_5 F)$, where $S^\mu\cdot\gamma^\nu \equiv E^{\mu\nu}$ is the field energy-momentum tensor. As we shall show elsewhere, by projecting into the Pauli algebra and calling $K_e \equiv F\cdot J_e = -J_e\cdot F$ and $K_m \equiv (\gamma_5 F)\cdot J_m = -J_m\cdot(\gamma_5 F)$, one does consequently find the expected expressions for the forces (in particular the Lorentz forces) acting on a charge and a monopole:

$$\vec{K}_e = \rho_e\vec{E} + \vec{J}_e\times\vec{H} \tag{5a}$$

$$\vec{K}_m = -\rho_m\vec{H} + \vec{J}_m\times\vec{E} \tag{5b}$$

Generalized connection and curvature. As is wellknown, given (R^4, g), quantity g being the Minkowski metric with signature -2, the potentials are connections in principal fiber bundles and the associated field is the connection curvature. In the ordinary language of differential forms the field F is a 2-form derived from the potential A (a 1-form); for instance in the electromagnetic U(1) theory:

$$F = dA \ . \tag{6}$$

However, the Hodge decomposition theorem[19] assures us that more generally, if $F \in \Lambda^2$ is a 2-form, then there always exist a

1-form A, a 3-form $\hat{B} \equiv *B$ and a harmonic 2-form C (with $dC = \delta C = 0$ $\longleftrightarrow \Box C = 0$) such that F can be uniquely decomposed into

$$F = dA + \delta\hat{B} + C \equiv dA + \delta * B + C \ . \qquad (7)$$

[In the particular U(1) case of electromagnetism, the principal fiber bundle[3,11] is π: $P \to R^4$, with $P = R^4 \times U(1)$; and, if C is the space of the connections in P, then the elements of C assume values in the (commutative) Lie algebra of U(1)].

The Hodge decomposition naturally suggests assuming as <u>generalized connection</u>

$$\bar{A} = A + \hat{B} \equiv A + *B \quad \in \Lambda^1 \oplus \Lambda^3 \qquad (8)$$

and as <u>generalized curvature</u> $F = \partial\!\!\!/ \bar{A}$ the quantity

$$F = (d + \delta)\bar{A} \ , \qquad (9)$$

which yields $F = dA + \delta * B + d * B + \delta A$. If we want F to be still a 2-form, then the last two addenda have to vanish, and we <u>automatically</u> end up with the Lorentz gauge condition

$$d * B = \delta A = 0 \ , \qquad (10)$$

which in tensorial language reads $\partial^\mu B_\mu = \partial^\alpha A_\alpha = 0$ and in Clifford language[7] $\partial\!\!\!/ \cdot \bar{A} = 0$. We are thus left with

$$F = dA + \delta * B \ . \qquad (9')$$

The geometrical meaning of eq.(9') is particularly interesting and transparent: with reference to eq.(1) it is evident that a 2-form $\in \Lambda^2$ can be obtained <u>both</u> by applying d to a 1--form $A \in \Lambda^1$, <u>and</u> by applying δ to a 3-form $\hat{B} \equiv *B \in \Lambda^3$. The field equations, at last, are got by evaluating $\partial\!\!\!/ F$, with $\partial\!\!\!/ \equiv d + \delta$:

$$(d + \delta)(dA + \delta * B) = \partial\!\!\!/^2 A + \partial\!\!\!/^2 * B \ , \qquad [\partial\!\!\!/^2 \equiv \Box]$$

which writes:

$$\partial\!\!\!/ F = J_e + *J_m = \bar{J} \qquad (11)$$

once[7,9] the 1-form $\partial\!\!\!/^2 A$ is called J_e and the 3-form $\partial\!\!\!/^2(*B)$ is called $*J_m$.

Equations (8÷11) are nothing but the Maxwell equations with monopoles, i.e. our eqs.(4)-(4'), now deduced within a purely geometrical context, via a natural generalization of the definitions of connection and of curvature; a generalization inspired by the "correspondences" (2)-(3) and by the Hodge decomposition theorem for differential forms.

Further remarks: (i) A rather interesting consequence of this geometrical interpretation of our "completed" Maxwell equations is that eq.(9) can be assumed as a new definition of "generalized curvature" F, without imposing any longer the Lorentz gauge (10). Instead of requiring the curvature (= the field) to be a 2-form, we can let it be an element of the even part $\mathbb{R}^+_{1,3} \simeq \mathbb{R}_{3,0}$ of the space-time algebra $\mathbb{R}_{1,3}$; so that $F \in \Lambda^2 \oplus \Lambda^0 \oplus \Lambda^4$:

$$F = \not{\partial}\bar{A} = (dA + \delta * B) + d*B + \delta A \,. \qquad (9bis)$$

In fact, even with such a generalized definition (i.e., without imposing —let us repeat— the Lorentz gauge), the field equations result to be the correct, ordinary ones:

$$\not{\partial}^2\bar{A} = \not{\partial}F = \not{\partial}^2 A + \not{\partial}^2 * B \equiv J_e + *J_m \,, \qquad (11')$$

since $d^2A = d^2*B = \delta^2 A = \delta^2 * B = 0$. Equations (11') are equivalent, of course, to the couple [2,18] of equations $\Box A = J_e$; $\Box B = J_m$.

(ii) For future convenience, let us notice that the Minkowski metric g induces [20] a "dual" metric g' in the space of the dual differential forms:

$$g'(\phi_1, \phi_2)\omega = \phi_1 \wedge * \phi_2 \qquad (12a)$$

where $\phi_1, \phi_2 \in \Lambda^k$ and ω is the volume element in R^4. In the particular case when $\phi_1 = \phi_2 = \phi \in \Lambda^1$, then:

$$g'(*\phi, *\phi) = - g'(\phi, \phi) \,. \qquad (12b)$$

When, more generally, we deal with quantities such as $J_e + \hat{J}_m \equiv J_e + *J_m \in \Lambda^1 \oplus \Lambda^3$ and define $g'(J_e + \hat{J}_m, J_e + \hat{J}_m) \equiv g'(J_e, J_e) + g'(\hat{J}_m, \hat{J}_m)$, then we obtain that:

$$g'(J + \hat{J}, J + \hat{J}) = 0$$

whenever $J_e = J_m = J$.

(iii) The introduction of our "monopoles without string" for the more general case of non-abelian gauge groups will appear elsewhere. Here let us emphasize once more that, for our aims, the ordinary tensorial formalism is too poor, since —among others— it does not satisfactorily distinguish between scalar and pseudoscalar quantities, as sometimes it is strictly required by physics. For instance, it is an essential characteristic of the lagrangian density in eq.(14) of ref.[7] to be the sum of a scalar and a pseudoscalar part.[7,18]

(iv) At last, let us take advantage of the present opportunity for pointing out some misprints appeared in the previous paper[7], and that might make difficult for the interested reader to re-derive those results of ours: (1) at page 234, column 2, line 18: the two expressions $\bar{\bar{J}}\cdot\bar{J}$ ought rather to write $\bar{\bar{J}}\circ\bar{J}$; (2) at page 235, eqs.(14) and (15): all the three expressions $\bar{\bar{J}}\cdot\bar{A}$ should be written $\bar{\bar{J}}\circ\bar{A}$; (3) at page 235: the last term in the r.h.s. of eq.(17) ought to be eliminated; (4) at page 236, column 1, line 22: "pseudoscalars" should be corrected into "pseudovectors". Let us stress that the "ball-product" ($\circ$) is not a new fundamental product, since in terms of the Clifford product it defines $A\circ B = \frac{1}{2}(A\bar{B} + B\bar{A})$.

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FOOTNOTES

#1 By adopting Hestenes' notations (cf. the second one of refs. [13]), we call "space-time algebra" the Clifford algebra $\mathbb{R}_{1,3}$ that we called "Dirac algebra" in ref. [7]. More correctly we shall reserve the name of Dirac algebra for $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. Notice, incidentally, that the "Majorana algebra" $\mathbb{R}_{3,1}$ is quite different from $\mathbb{R}_{1,3}$, so that two algebras [$\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, and $\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$] can be naturally associated with Minkowski space-time; and this can have a bearing on physics (even for the mathematical problems with tachyons, for instance). At last, the Pauli algebra is $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$.

#2 Recall, however, that within Clifford algebra no special symbol is associated with the Clifford product.

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