BEST APPROXIMANTS FROM CERTAIN SUBSETS OF BOUNDED FUNCTIONS

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ABSTRACT. Let A be a subalgebra of $C(T, \mathbb{R})$, where T is a compact Hausdorff space. It is well known that the uniform closure of A is proximinal in $C(T,\mathbb{R})$ equipped with the sup-norm. In this paper we show that the uniform closure of $A^+ := \{f \in A; f \geq 0\}$, say V, is proximinal too. Moreover, for any bounded non-empty subset $B \subset C(T,\mathbb{R})$, the set $\text{cent}(B;V)$ of relative Chebyshev centers of B (with respect to V) is non-empty. The proof relies on a generalization of Bernstein's Theorem on approximation of a positive continuous function f on $[0,1]$ by its Bernstein polynomials $B_n(f)$.

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Let T be a topological space and let $C_b(T;\mathbb{R})$ be the Banach space of all bounded continuous real-valued functions on T, equipped with the sup-norm, $\|f\| = \sup\{|f(t)|; t \in T\}$. When T is compact, $C_b(T;\mathbb{R}) = C(T;\mathbb{R})$, the space of all continuous real-valued functions on T. In this case it is well known that any closed subalgebra A of $C(T;\mathbb{R})$ is proximinal and several proofs have been presented. Smith and Ward [15] extended this result by proving that any closed subalgebra A of $C(T;\mathbb{R})$, for compact T, has the relative Chebyshev center property in $C(T;\mathbb{R})$. Applying this result to the algebra $A = C(T;\mathbb{R})$ one gets that $C(T;\mathbb{R})$, for compact T, admits Chebyshev centers. This result was obtained for $T = [a,b]$ by Kadets and Zamyatin [7], and for any compact T by Garkavi [6]. It was extended by Mach [9]: indeed, it follows from Theorems 3 and 4 of Mach [9] that, for any topological space T, the algebra $C_b(T;\mathbb{R})$ has the relative Chebyshev center property in $\ell_\infty(T;\mathbb{R})$ and the map $B \to \text{cent}(B;C_b(T;\mathbb{R}))$ is lower semicontinuous. See also Mach [8]. The result that $C_b(T,\mathbb{R})$ admits Chebyshev centers, for any topological space T, was also noticed Franchetti and Cheney [5]. All these results were generalized and extended by Prolla, Chiacchio and Roversi [11], who showed that any closed subalgebra $A \subset C_b(T;\mathbb{R})$, for an arbitrary topological space T, has the relative Chebyshev center property in both $C_b(T;\mathbb{R})$ and $\ell_\infty(T;\mathbb{R})$ and the map $B \to \text{cent}(B;A)$ is Lipschitz continuous in the Hausdorff metric d_H with Lipschitz constant not greater than 2. This result was proved using among other things the Stone-Weierstrass Theorem. Since we extended recently this theorem to a description of the closure of A^+, for compact T (see [12] or [13]), it is natural to atempt to extend this result of [11] to the uniform closure of A^+. Our Theorem 3 below achieves this objective, even for $A \subset \ell_\infty(T;\mathbb{R})$.

Let us explain our notation and terminology. For any Banach space E, the open and closed balls of center a and radius r are denoted, respectively, by $B(a;r)$ and $\overline{B}(a;r)$. If V is any non-empty subset of E and $a \in E$, then

$$\text{dist}(a;V) := \inf\{\|a - v\|; v \in V\}.$$

We denote by $P_V(a)$ the set of all best approximants to a from V, i.e.,

$$P_V(a) := \{v \in V ; \|v - a\| = \mathrm{dist}(a;V)\}.$$

If $P_V(a) \neq \phi$ for all $a \in E$, we say that V is *proximinal* in E. If $B \subset E$ is any bounded non-empty subset, then

$$\mathrm{rad}(B;V) := \inf \{ \sup_{f \in B} \|f - v\| ; v \in V\}$$

is called the *relative Chebyshev radius of* B *with respect to* V. When $V = E$ we write simply $\mathrm{rad}(B)$ and call it the *Chebyshev radius of* B. An element $v \in V$ such that

$$\sup_{f \in B} \|f - v\| = \mathrm{rad}(B;V)$$

is called a *relative Chebyshev center of* B *with respect to* V, and we denote by $\mathrm{cent}(B;V)$ the set of all such elements. When $V = E$, we write simply $\mathrm{cent}(B)$ and its elements are called the *Chebyshev centers* of B.

When $\mathrm{cent}(B;V) \neq \phi$ for any bounded $B \subset E$, we say that V has the *relative Chebyshev center property* in E. When $\mathrm{cent}(B) \neq \phi$ for all such B, we say that E *admits Chebyshev centers*.

If T is any non-empty set, we denote by $\ell_\infty(T,\mathbb{R})$ the vector space of all bounded real-valued functions defined on T. When we equip $\ell_\infty(T;\mathbb{R})$ with the sup-norm

$$\|f\| = \sup \{|f(t)| ; t \in T\}$$

it becomes a Banach space. When T is a topological space, then the vector subspace of all elements of $\ell_\infty(T;\mathbb{R})$, wich are continuous on T, is denoted by $C_b(T;\mathbb{R})$. Since is is closed in $C_\infty(T;\mathbb{R})$, it is a Banach space too. When T is compact, then all continuous real-valued functions on T are bounded, i.e., $C(T;\mathbb{R}) = C_b(T;\mathbb{R})$ for compact T. For any topological space T, the space $C_b(T,\mathbb{R})$ is isometrically, algebraically and lattice isomorphic to $C(K;\mathbb{R})$ for some compact Hausdorff space K. When T is a completely regular Hausdorff space, then we may take K to be the Stone-Cech compactification of T. The set of all $f \in \ell_\infty(T;\mathbb{R})$ such that $f(t) \geq 0$, for any $t \in T$, is denoted by $\ell_\infty^+(T;\mathbb{R})$. For any subset $A \subset \ell_\infty(T,\mathbb{R})$, $A^+ := A \cap \ell_\infty^+(T,\mathbb{R})$.

The following definition was introduced in [10].

DEFINITION 1. Let V be a closed non-empty subset of a Banach space E, and let $\mathcal{B}$ be a class of bounded non-empty subsets of E. We say that the pair $(V,\mathcal{B})$ *has property* (C) *in* E, if given $B \in \mathcal{B}$, $w \in V$, $r > 0$ and $\varepsilon > 0$ such that $V \cap \bigcap_{f\in B} \bar{B}(f;r) \neq \phi$ and $\| f - w\| < r + \varepsilon$, for all $f \in B$, there exists $v \in V$ such that $\|v - w\| \leq \varepsilon$ and $\|f - v\| \leq r$ for all $f \in B$.

Let us say that V *has property* (C) *in* E, if the pair $(V,\mathcal{B}(E))$ has property (C) in E, where $\mathcal{B}(E)$ is the class of all bounded non-empty subsets of E. Clearly, if V has property (C) in E, and F is a closed vector subspace such that $V \subset F \subset E$, then V has property (C) in the Banach space F too.

The following result was proved in [11]. (See Proposition 2.2 and Theorem 2.4 of [11].)

THEOREM 1. *Let* V *be a closed non-empty subset of a Banach space* E. *If* V *has property* (C) *in* E, *and* F *is any closed vector subspace of* E *containing* V, *then*

(1) $\text{cent}(B;V) \neq \phi$, *for every bounded and non-empty subset* B *of* F.

(2) *The map* $B \to \text{cent}(B;V)$ *is Lipschitz* d_H*-continuous, with Lipschitz constant not greater than* 2, i.e.,

$$d_H(\text{cent}(K;V), \text{cent}(L;V)) \leq 2\, d_H(K,L)$$

for any pair K,L *of bounded and non-empty subsets of* F.

(3) V *is proximinal in* F.

(4) $d_H(P_V(f),P_V(g)) \leq 2\,\|f - g\|$ *for any pair* f,g *in* F.

(5) *The metric projection* P_V *admits a continuous selection.*

REMARK. The Hausdorff metric d_H is defined as follows:

$$d_H(A,B) = \inf\{r > 0;\, A \subset B + rU\ ,\ B \subset A + rU\}$$

where $U = \{v \in E; \|v\| \leq 1\}$, for any pair A, B of bounded and non-empty subsets of E.

THEOREM 2. *Let* K *be a closed and non-empty subset of* $\ell_\infty(T;\mathbb{R})$ *such that, for any pair* $w, h \in K$ *and* $\varepsilon > 0$ *the function* $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ *belongs to* K. *Let* $T_0 \subset T$, *and let* $K_0 = \{f \in K;\ f(t) = 0$ *for all* $t \in T_0\}$. *If* $V = K$ *or if* $V = K_0$, *if* K_0 *is non-empty, and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem* 1 *are true*.

PROOF. We have to prove that V has property (C) in $\ell_\infty(T;\mathbb{R})$, and it suffices to show that $V = K$ has property (C) in $\ell_\infty(T;\mathbb{R})$. Indeed, K_0 is closed too and if $w, h \in K_0$, then by hypothesis $g = ((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ belongs to K, since w and h belong to K. Now, if we take $t \in T_0$, then $w(t) = h(t) = 0$, and therefore $g(t) = 0$. Hence $g \in K_0$.

We claim that K has property (C) in $\ell_\infty(T;\mathbb{R})$. Indeed, let $B \subset \ell_\infty(T;\mathbb{R})$ be a bounded non-empty subset, let $w \in K$, $r > 0$ and $\varepsilon > 0$ be given with $K \cap \bigcap_{f \in B} \bar{B}(f;r) \neq \phi$ and $\|f - w\| < r + \varepsilon$ for all $f \in B$. Choose $h \in K$ such that $\|f - h\| \leq r$, for all $f \in B$. Let $v = ((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$. Then $v \in K$ and $\|v - w\| \leq \varepsilon$. We claim that $\|f - v\| \leq r$ for all $f \in B$. Indeed, let $x \in T$ and $f \in B$ be given.

CASE 1. $|h(x) - w(x)| \leq \varepsilon$.
Then $v(x) = h(x)$ and $|f(x) - v(x)| = |f(x) - h(x)| \leq r$.

CASE 2. $h(x) - w(x) > \varepsilon$.
Then $v(x) = w(x) + \varepsilon$ and $-r \leq f(x) - h(x) < f(x) - w(x) - \varepsilon < r + \varepsilon - \varepsilon = r$.

CASE 3. $h(x) - w(x) < -\varepsilon$.
Then $v(x) = w(x) - \varepsilon$ and $-r = -(r+\varepsilon) + \varepsilon < f(x) - w(x) + \varepsilon < f(x) - h(x) \leq$ $\leq r$. □

REMARK. It is obvious from the proof of Theorem 2, that whenever a set $K \subset \ell_\infty(T;\mathbb{R})$ is such that, for any pair $w, h \in K$ and $\varepsilon > 0$ the function $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ belongs to K, then the set $K_0 = \{f \in K;\ f(t) = 0$ for all $t \in T_0\}$ has the same property, for any subset $T_0 \subset T$. Hence, to each corollary listed below, there is a corresponding result for K_0, whenever K_o is non-empty. Most of the time we will not state explicitly the corresponding corollary.

COROLLARY 1. *Let* $a,b \in \ell_\infty(T;\mathbb{R})$, *with* $a \leq b$, *let* $V = [a,b] := \{h \in \ell_\infty(T;\mathbb{R})\ ;\ a(x) \leq h(x) \leq b(x)$, *for all* $x \in T\}$ *and let* $E = \ell_\infty(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem 1 are true.*

PROOF. It is easy to see that for any pair w,h in $[a,b]$, the function $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ belongs to $[a,b]$. □

REMARK. Franchetti and Cheney proved the proximinality of any order interval in any Banach lattice (see [5, Lemma 3.5]). Roversi proved that $[a,b] \subset \ell_\infty(T;\mathbb{R})$ has the relative Chebyshev center property in $\ell_\infty(T;\mathbb{R})$ (see [14, Proposition 2.6]). Notice that Corollary 1 applies to the set $V = \{h \in \ell_\infty(T;\mathbb{R})\ ;\ h(T) \subset [a,b]\}$ when $[a,b] \subset \mathbb{R}$. Indeed, this case corresponds to take a and b in Corollary 1 to be constant functions.

COROLLARY 2. *Let* $(T,\leq)$ *be a preordered set and let* V *be the subset of all* $f \in \ell_\infty(T;\mathbb{R})$ *which are non-decreasing (resp. non-increasing) on* T, *and let* $E = \ell_\infty(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem 1 are true.*

PROOF. The set V is a closed sublattice of $\ell_\infty(T;\mathbb{R})$ and $w \pm \varepsilon$ belong to V, for each $w \in V$ and $\varepsilon > 0$. □

REMARK. When T is a topological space and V is as in Corollary 1 or 2 then $V \cap C_b(T,\mathbb{R})$ has the same property in $\ell_\infty(T,\mathbb{R})$ as V, and analogous results can be formulated. Roversi had proved that the closed sublattice V of Corollary 2 has the relative Chebyshev center property in $\ell_\infty(T;\mathbb{R})$. (See Proposition 2.4 of [14].)

COROLLARY 3. *Let* K *be a closed sublattice of* $\ell_\infty(T;\mathbb{R})$ *such that for any* $w \in K$ *and* $\varepsilon > 0$, *the functions* $w + \varepsilon$ *and* $w - \varepsilon$ *belong to* K. *Let* $T_o \subset T$, *and* $K_o = \{f \in K;\ f(t) = 0$ *for all* $t \in T_o\}$. *If* $V = K$ *or* $V = K_o$ *and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem 1 are true.*

PROOF. Clearly, K satisfies the hypothesis of Theorem 2. □

REMARK. The hyphotesis of Corollary 3 are verified if K is a closed sublattice such that $K + K \subset K$ and K contains the constants; in particular, if K is a closed sublattice containing the constants which is also a convex cone. Hence Corollary 3 is a generalization of an Approximation Theorem of Blatter and Seever [2], [3]. Under the latter hypothesis they proved that K_o is proximinal in $\ell_\infty(T;\mathbb{R})$. Their proof uses their theory of interposition of functions. In [3] they establish a formula for dist(f;K) in terms of the quasi-proximity defined by K on T. The approximation theorem of Blatter and Seever extends an approximation theorem of Nachbin [10, Appendix, § 5, Theorem 6] which proves that any closed lattice cone $K \subset C(T;\mathbb{R})$, containing the constants, is proximinal in $C(T;\mathbb{R})$, for T a compact Hausdorff space. (When $T_o = \phi$, then Blatter and Seever's result follows from Nachbin's). Nachbin also proved a formula for dist(f;K).

In [11] it is considered the case in which V is a closed vector subspace of $\ell_\infty(T;\mathbb{R})$. Then Theorem 2 takes the following simplified form.

THEOREM 2'. *Let* V *be a closed vector subspace of* $\ell_\infty(T;\mathbb{R})$ *such that, for any* $h \in V$ *and* $\varepsilon > 0$, *the function* $(\varepsilon \wedge h) \vee (-\varepsilon)$ *belongs to* V. *Then* V *has property* (C) *in* $\ell_\infty(T;\mathbb{R})$.

Using Theorem 2' the following result was proved in [11].

THEOREM 3. *Let* V *be a closed subalgebra of* $\ell_\infty(T;\mathbb{R})$, *and let* $E = \ell_\infty(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem 1 are true.*

The proof of Theorem 3 is reduced to the case of a closed subalgebra V of $C(K;\mathbb{R})$, where K is a compact Hausdorff space, and in this case the proof that V satisfies the hypothesis of Theorem 2' uses the Stone-Weierstrass Theorem. (See [11].) Since any closed subalgebra of $C_b(T;\mathbb{R})$, is closed in $\ell_\infty(T;\mathbb{R})$, Theorem 3 implies our next result.

COROLLARY 4. *Let* T *be a topological space. Let* V *be a closed subalgebra of* $C_b(T;\mathbb{R})$ *and let* $E = \ell_\infty(T;\mathbb{R})$ *or* $E = C_b(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem* 1 *are true. If* T *is locally compact, and* V *is a closed subalgebra of* $C_o(T;\mathbb{R})$ *and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem* 1 *are true.*

Let us now extend Theorem 3 and Corollary 4 to the uniform closure of the set of positive elements of a given subalgebra A. Firstly, we show that our version of the Stone-Weierstrass theorem ([12] or [13]), describing the uniform closure of A^+, for $A \subset C(K;\mathbb{R})$, K compact, can be used to prove that such a closed convex cone satisfies the hypothesis of Theorem 2.

LEMMA 1. *Let A be a subalgebra of $C(K;\mathbb{R})$, where K is a compact Hausdorff space. For any w and h in the uniform closure of A^+ and any $\varepsilon > 0$, the function $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ belongs to the uniform closure of A^+.*

PROOF. Let $g = ((w+\varepsilon)\wedge h) \vee (w-\varepsilon)$. By Theorem 3, Prolla [12] there exists a point $x \in K$ such that

$$\mathrm{dist}(g;A^+) = \mathrm{dist}(g_{[x]} ; A^+_{[x]})$$

where $[x]$ is the equivalence class of x mod. A^+, i.e., $[x] = \{t \in K; a(t) = a(x) \text{ for all } a \in A^+\}$. Since both w and h belong to the uniform closure of A^+, they are constant on $[x]$. Let w_o and h_o be constant value of w and h, respectively, on the set $[x]$. Notice that $w_o \geq 0$ and $h_o \geq 0$. Hence $((w_o+\varepsilon)\wedge h_o) \vee (w_o - \varepsilon) \geq 0$.

CASE 1. *For any* $a \in A^+$, $a(x) = 0$.

In this case $w_o = h_o = 0$, and $g(t) = 0$ for all $t \in [x]$. Hence $\|g - 0\|_{[x]} = 0$ and, since $0 \in A^+$, $\mathrm{dist}(g_{[x]} ; A^+_{[x]}) = 0$.

CASE 2. *For some* $a \in A^+$, $a(x) > 0$.

Let $f_o = (((w_o+\varepsilon)\wedge h_o) \vee (w_o - \varepsilon))/a(x)$ and $f = f_o a$. Then $f \in A^+$ and $f(t) = g(t)$ for all $t \in [x]$. Hence $\|g - f\|_{[x]} = 0$ and $\mathrm{dist}(g_{[x]} ; A^+_{[x]}) = 0$.

In both cases, $\mathrm{dist}(g; A^+) = 0$ and therefore g belongs to the uniform closure of A^+ in $C(K;\mathbb{R})$. □

THEOREM 4. *Let A be a subalgebra of $\ell_\infty(T;\mathbb{R})$, let V be the uniform* *closure of* A^+ *in* $\ell_\infty(T;\mathbb{R})$ *and let* $E = \ell_\infty(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem 1 are true.*

PROOF. Let K be the Stone-Cech compactification of T equipped with the discrete topology. Then $\ell_\infty(T;\mathbb{R})$ is isometrically, algebraically and lattice isomorphic to $C(K;\mathbb{R})$. The result now follows from Lemma 1 and Theorem 2. □

COROLLARY 5. *Let* T *be a topological space and let* A *be a subalgebra of* $C_b(T;\mathbb{R})$. *If* V *denotes the uniform closure of* A^+ *in* $C_b(T;\mathbb{R})$ *and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem 1 are true.*

PROOF. The algebra A is a subalgebra of $\ell_\infty(T;\mathbb{R})$ and the uniform closure of A^+ in $C_b(T;\mathbb{R})$ is the same as the uniform closure of A^+ in $\ell_\infty(T;\mathbb{R})$, since $C_b(T;\mathbb{R})$ is closed in $\ell_\infty(T;\mathbb{R})$. □

COROLLARY 6. *Let* T *be a locally compact space and let* A *be a subalgebra of* $C_o(T;\mathbb{R})$. *If* V *denotes the uniform closure of* A^+ *in* $C_o(T;\mathbb{R})$ *and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem 1 are true.*

PROOF. The algebra A is a subalgebra of both $C_b(T;\mathbb{R})$ and $\ell_\infty(T;\mathbb{R})$ and the uniform closure of A^+ in $C_o(T;\mathbb{R})$ is the same as the uniform closure of A^+ in $C_b(T;\mathbb{R})$ and in $\ell_\infty(T;\mathbb{R})$, since $C_o(T;\mathbb{R})$ is closed in both $C_b(T;\mathbb{R})$ and $\ell_\infty(T;\mathbb{R})$. □

COROLLARY 7. *Let* T *be a topological space (resp. a locally compact space), let* $V = C_b^+(T;\mathbb{R})$ (resp. $V = C_o^+(T;\mathbb{R})$), *and let* $E = \ell_\infty(T;\mathbb{R})$. *Then* (1)-(5) *of Theorem 1 are true.*

COROLLARY 8. *Let* $V = \ell_\infty^+$ (resp. c_o^+) *and let* $E = \ell_\infty$. *Then* (1) - (5) *of Theorem 1 are true.*

PROOF. In Corollary 7 take $T = \mathbb{N}$ with the discrete topology. □

COROLLARY 9. *Let* $\varphi \geq 0$ *be defined on* $T \times T$, *and let* $V = \{f \in \ell_\infty(T;\mathbb{R});\ |f(t) - f(u)| \leq \varphi(t,u)$ *for all* $(t,u) \in T \times T\}$. *Then* (1)-(5) *of Theorem 1, are true.*

PROOF. It is easily seen that $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)$ belongs to V, whenever w and h belong to V. Indeed, V is a lattice containing $w \pm \varepsilon$ for any $w \in V$ and $\varepsilon > 0$. □

As an example of application of Corollary 9, assume that (T,d) is a metric space (or even a pseudo-metric space) and let $\varphi(t,u) = Md(t,u)^\alpha$ for some fixed $M > 0$ and $\alpha \in \mathbb{R}$. Then V consists of all $f \in \ell_\infty(T;\mathbb{R})$ such that $|f(t) - f(u)| \leq Md(t,u)^\alpha$ for all $(t,u) \in T \times T$, i.e., all $f \in Lip_\alpha$ with Lipschitz constant not greater than M.

In order to state our last corollary let us recall the definition of $C_{\mathbb{R}}(T;\mathbb{R})$ when T is a locally compact Hausdorff space. For any $f \in C_b(T;\mathbb{R})$ and $v \in \mathbb{R}$, we say that $\lim_{t\to\infty} f(t) = v$ if, given $\varepsilon > 0$ there exists a compact subset $K \subset T$ such that $|f(t) - v| < \varepsilon$ for all $t \in T$, $t \notin K$. Following Amir and Deutsch [1], $C_{\mathbb{R}}(T;\mathbb{R})$ denotes the closed subalgebra of $C_b(T;\mathbb{R})$ of all functions that have "limit at infinity". When $T = \mathbb{N}$ with the discrete topology, we write $c = C_{\mathbb{R}}(\mathbb{N};\mathbb{R})$.

COROLLARY 10. *Let* T *be a locally compact Hausdorff space* (resp. $T = \mathbb{N}$ *with the discrete topology*), *let* $V = C^+_{\mathbb{R}}(T;\mathbb{R})$ (resp. $V = c^+$), *and let* $E = \ell_\infty(T;\mathbb{R})$ (resp. $E = \ell_\infty$). *Then* (1)-(5) *of Theorem* 1 *are true.*

REMARK. Theorem 2 remains true when $K \subset \ell^+_\infty(T;\mathbb{R})$, if we assume that K satisfies the hypothesis $((w+\varepsilon) \wedge h) \vee (w-\varepsilon)^+ \in K$ whenever $w, h \in K$ and $\varepsilon > 0$, where $(w-\varepsilon)^+ := (w-\varepsilon) \vee 0$. Hence, Corollary 3 is true, if $K \subset \ell^+_\infty(\mathbb{R})$ is a closed sublattice such that $w + \varepsilon$ and $(w-\varepsilon)^+$ belong to K, whenever $w \in K$ and $\varepsilon > 0$. In particular, Corollary 3 is true, if $K \subset \ell^+_\infty(T;\mathbb{R})$ is a convex conic lattice, containing the positive constants and $(w-\varepsilon)^+$ whenever $w \in K$ and $\varepsilon > 0$. For example, K may be the set of all positive non-decreasing (resp. non-increasing) bounded functions on a preordered set $(T, \leq)$, or K may be the set of all positive non-decreasing (resp. non-increasing) bounded continuous functions on a preordered topological space $(T, \leq)$. We can apply this remark to prove that certain semi-algebras are proximinal.

We recall that a non-empty set $A \subset \ell_\infty(T;\mathbb{R})$ is called a *semi-algebra* according to Bonsall [4], if $f + g$, λf and $fg \in A$ whenever $f, g \in A$ and $\lambda \geq 0$. A semi-algebra is called a *semi-algebra with identity* if it contains the positive constants. Given a non-negative integer n, a semi-algebra A is said to be of *type* n if $f^n/(1 + f)$ belongs to A whenever $f \in A$. If A is of type n, then it is of type $n + 1$. Every semi-algebra of type 0 is a semi-algebra with identity, and if A is a semi-algebra of type n $(n \geq 0)$, then $A \subset \ell_\infty^+(T;\mathbb{R})$.

THEOREM 5. *Let* $A \subset \ell_\infty^+(T;\mathbb{R})$ *be a closed semi-algebra with identity, of type* 0 *or* 1. *If* $V = A$ *and* $E = \ell_\infty(T;\mathbb{R})$, *then* (1)-(5) *of Theorem* 1 *are true.*

PROOF. (1) Assume that A is of type 0. By Theorem 4 of Bonsall [4], $A = B^+$, where B is the closed subalgebra of all functions $f \in \ell_\infty(T;\mathbb{R})$ such that $f(t) = f(t')$ whenever $g(t) = g(t')$ for all $g \in A$. By Theorem 4 above applied to $A = B^+$, (1)-(5) of Theorem 1 are true.

(2) If A is a semi-algebra with identity, then by Theorem 7, Bonsall [4], A is the class of all elements of $\ell_\infty^+(T;\mathbb{R})$ that are non-decreasing when T is preordered by taking $t \leq t'$ whenever $g(t) \leq g(t')$ for all functions $g \in A$. By the preceding Remark, the conclusion follows.

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