SOME APPLICATIONS OF NON–HERMITIAN OPERATORS IN QUANTUM MECHANICS AND QUANTUM FIELD THEORY

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ABSTRACT: Due to the possibility of rephrasing it in terms of Lie-admissible algebras, some work done in the past in collaboration with A. Agodi, M. Baldo and V. S. Olkhovsky is here reported. Such work led to the introduction of non-Hermitian operators in (classical and relativistic) quantum theory. We deal in particular with: (i) the association of unstable states (decaying "Resonances") with the eigenvectors of non-Hermitian Hamiltonians; (ii) the problem of the four-position operators for relativistic spin-zero particles.

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PART I: UNSTABLE STATES AND NON-HERMITIAN HAMILTONIANS

1. INTRODUCTION

This first Part is based on work done in collaboration with A. Agodi and M. Baldo(1).

In quantum mechanics the "resonance" peaks are generally described as corresponding to unstable states (remember e.g. Schwinger's(2) approach). The present attempt proceeds as follows: (i) singling out *one* state $|\phi\rangle$ in the state space; (ii) finding out the effect of the (internal, virtual) state $|\phi\rangle$ on the transition-amplitude; (iii) finding, in particular, the necessary conditions for $|\phi\rangle$ to be connected with a Resonance in the cross-section. In this way we shall accociate the "resonant states" with the eigenvectors of a non-Hermiatian Hamiltonian (for simplicity, a "*quasi* self-adjoint" Hamiltonian), such eigenvectors being shown to decay in time correctly. We shall adopt the formalism introduced by Akhieser & Gladsman(3), by Lifshitz, by Galinsky & Migdal(4), and by Agodi *et al.*(5).

Chosen a state $|\phi\rangle$, let us define the projectors

$$P \equiv |\phi\rangle\langle\phi|; \quad Q \equiv \mathbb{1} - P. \tag{1}$$

2. PRELIMINARY CASE: TIME-DEPENDENT DESCRIPTION OF POTENTIAL SCATTERING

Let us preliminarly consider the time-dependent description of potential scattering. Quantity V be the potential operator. In the limiting case of plane-waves, the scattering amplitude writes

$$T(\underline{k},\underline{k}') = \langle\underline{k}'|V|\underline{k}\rangle + \langle\underline{k}'|\, V G(E^+) V\,|\underline{k}\rangle \tag{2a}$$

with

$$G(E^+) \equiv (E^+ - H)^{-1}; \quad E^{\pm} \equiv E \pm i\varepsilon\,. \tag{2b}$$

Chosen the *exploring* vector $|\phi\rangle$ and using definitions (1), we have

$$H = \overset{o}{H} + \overset{1}{H}; \qquad (3\underline{a})$$

$$\overset{o}{H} \equiv QHQ; \quad \overset{1}{H} \equiv PHP + PHQ + QHP. \qquad (3\underline{b})$$

By introducing the scattering states $|\overset{o}{\psi}\rangle$ due to $\overset{o}{H}$

$$|\overset{o}{\psi}{}^{(\pm)}_{\underline{k}}\rangle = \left[1 + \frac{1}{E^{\pm} - \overset{o}{H}}(\overset{o}{H} - E)\right]|\underline{k}\rangle, \qquad (4)$$

we obtain

$$S(\underline{k},\underline{k}') \equiv \langle\psi^{(-)}_{\underline{k}'}|\psi^{(+)}_{\underline{k}}\rangle = \langle\overset{o}{\psi}{}^{(-)}_{\underline{k}'}|\overset{o}{\psi}{}^{(+)}_{\underline{k}}\rangle - 2\pi i \cdot \delta(E_{k'} - E_k) \cdot$$

$$\cdot \langle\overset{o}{\psi}{}^{(-)}_{\underline{k}'}|H\, \underbrace{PG(E^{+}_{k})P}\, H|\overset{o}{\psi}{}^{(+)}_{\underline{k}}\rangle, \qquad (5)$$

where the first addendum in the r.h.s. of eq.(5) (let us call it A) is the contribution coming from processes developing entirely in the subspace onto which Q projects, whilst the second addendum (B) is contributed by processes going through the exploring state $|\phi\rangle$ onto which P projects. In other words, the processes with $|\phi\rangle$ as intermediate state correspond to the term

$$[\delta(E_{k'} - E_k)]^{-1} \cdot B = -2\pi i \frac{\langle\overset{o}{\psi}{}^{(-)}_{\underline{k}'}|H|\phi\rangle\langle\phi|H|\overset{o}{\psi}{}^{(+)}_{\underline{k}}\rangle}{E^{+}_{k} - \langle\phi|H|\phi\rangle - \langle\phi|W^{\phi}(E^{+}_{k})|\phi\rangle}; \qquad (6\underline{a})$$

$$W^{\phi}(Z) \equiv PHQ\, \frac{1}{Z - QHQ}\, QHP. \qquad (6\underline{b})$$

Our problem is: under what conditions one (or more) Resonances are actually associated with the chosen $|\phi\rangle$?

Let us notice, in particular, that if $E_\phi \equiv \langle\phi|H|\phi\rangle - \mathrm{Re}\,\langle\phi|W^\phi(E^+)|\phi\rangle$ and $\Gamma_\phi \equiv \mathrm{Im}\,\langle\phi|W^\phi(E^+)|\phi\rangle$ are smooth functions of E, then B gets just the "Breit and Wigner" form:

$$B \simeq -2\pi i \frac{\langle \overset{\circ}{\psi}{}^{(-)}_{\underline{\underline{k}}'} | HPH | \overset{\circ}{\psi}{}^{(+)}_{\underline{\underline{k}}} \rangle}{E - E_\phi + i\Gamma_\phi} .$$

3. CASE OF CENTRAL POTENTIAL AND SPIN-FREE PARTICLES

Let us choose the angular-momentum representation. If $|\phi\rangle$ is assumed to be in particular invariant under O(3), then both terms in which S was split are diagonal. If $\overset{\circ}{\delta}_\ell$ are the phase-shifts due to QHQ and μ is the reduced mass, then

$$S_\ell(k) \equiv \exp[2i\delta_\ell(k)] = \exp[2i\overset{\circ}{\delta}_\ell(k)] \cdot F_\ell(k) \tag{7a}$$

with

$$F_\ell(k) \equiv 1 - \frac{2\pi i\mu}{\hbar^2 k} \cdot \frac{|\langle\phi|H|\overset{\circ}{\psi}{}^{(+)}_{k\ell m}\rangle|^2}{E^+ - \langle\phi|H|\phi\rangle - \langle\phi|W^\phi(E^+)|\phi\rangle} . \tag{7b}$$

Let us observe that the phase-shift of $F_\ell(k)$ crosses the value $\frac{1}{2}\pi$ (with positive slope) when:

$$F_\ell(k) = -1. \tag{8}$$

The conditions for a Resonance to appear are particularly transparent for $\ell = 0$.

$$F_o(k) = \frac{E - E_\phi(k) - i\lambda_o(k)}{E - E_\phi(k) + i\lambda_o(k)} , \tag{9a}$$

when

$$\lambda_o(k) \equiv -\mathrm{Im}\,\langle\phi|W^\phi(E^+)|\phi\rangle = |\langle\phi|H|\overset{\circ}{\psi}{}^{(+)}_{koo}\rangle|^2 \tag{9b}$$

is positive-definite. Namely, the condition $F_o(k) = -1$ yields

$$|1 - S_o(k)|^2 = 4\cos^2 \overset{o}{\delta}_o \, , \tag{8'}$$

with the supplementary conditions $\lambda_o(k) \neq 0$; $\cos\overset{o}{\delta}_o \neq 0$. When $\cos\overset{o}{\delta}_o \simeq 1$ the scattering due to QHQ is negligible, i.e. the scattering proceeds entirely via the intermediate formation of the (quasi-bound) state $|\phi\rangle$; and the possible resonant effects are really related to $|\phi\rangle$. Of course $\cos\overset{o}{\delta}_o \simeq 1$ when, *at the resonance* $[E = E_\phi \, ; F(k) = -1]$, it is $|\overset{o}{\psi}{}^{(\pm)}_{k\ell m}\rangle \simeq |k\ell m\rangle$.

Notice that with every fixed $|\phi\rangle$ *a series* of Resonances (also for different values of ℓ) may be a priori associated, if they are not destroyed by the $\overset{o}{\delta}_o$ behaviour.

4. RESONANCE DEFINITION

It is essential to recognize that the "Resonance condition" $F_\ell(k) = -1$ may be written[1]

$$1 - \alpha(k,\ell)\, \langle \phi_\ell | G(E^+) | \phi_\ell \rangle = 0 \tag{10a}$$

with

$$\alpha(k,\ell) \equiv \frac{i\pi\mu}{\hbar^2 k} \, |\langle \phi_\ell | H | \overset{o}{\psi}{}^{(+)}_{k\ell m} \rangle|^2 . \tag{10b}$$

Let us now study the more general equation

$$\left\{ \begin{array}{l} \boxed{1 - \lambda\, \langle \phi_\ell | G(z) | \phi_\ell \rangle = 0} \, , \\ \text{with } z, \lambda \text{ complex numbers.} \end{array} \right. \tag{11}$$

Of course, a resonance will appear at $\sim \mathrm{Re}\, z$ if z is near the real axis and if

$$\lambda \simeq \alpha(k,\ell),$$

both satisfying eq.(11).

If we introduce now the non-Hermitian Hamiltonian-operator

$$\boxed{\mathcal{H} \equiv H + \lambda P} \qquad \lambda \text{ complex,} \tag{12}$$

whose "resolvent operator" is

$$\mathcal{G}(z) \equiv \frac{1}{z - \mathcal{H}}, \tag{12'}$$

then eq.(11) becomes

$$\frac{\langle \phi_\ell | G(z) | \phi_\ell \rangle}{\langle \phi_\ell | \mathcal{G}(z) | \phi_\ell \rangle} = 0; \tag{13}$$

in other words, studying the (necessary) conditions for resonance-appearing is just equivalent to find out the poles in the diagonal elements of the "resolvent" $\mathcal{G}$-matrix, i.e. the eigenvalues of the *quasi self-adjoint* operator $\mathcal{H}$. Notice that, since

$$\mathcal{G} = G + G \frac{\lambda P}{1 - \lambda \langle \phi_\ell | G | \phi_\ell \rangle} G, \quad [\mathrm{Im}\, \lambda > 0]$$

the difference between the spectra of H and $\mathcal{H}$ is just the presence of complex eigenvalues (corresponding to the solution of our "condition-equation" (13)).

Therefore, in our framework the "resonant (decaying) state" $|\psi\rangle$ is expected to be an eigenvector of $\mathcal{H}$ (notive that it does *not* coincide with the state $|\phi\rangle$ which is not unstable!), corresponding to the complex energy $\mathcal{E}$.

5. APPLICATIONS

Let us confine ourselves to the case $\ell = 0$, and rewrite the non-Hermitian (quasi self-adjoint) Hamiltonian as

$$\mathcal{H} \equiv H + i\alpha_k |\phi\rangle\langle\phi| ; \qquad \alpha_k \equiv - i\alpha(k,0) \tag{14a}$$

where

$$V_\phi \equiv i\alpha_k |\phi\rangle\langle\phi| \tag{14b}$$

is anti-Hermitian. We shall therefore write

$$(H - \mathcal{E})|\psi\rangle = - V_\phi |\psi\rangle \equiv - |\phi\rangle\, i\alpha_k \langle\phi|\psi\rangle , \tag{15}$$

which immediately yields for the eigenvalues the "dispersion-type relation" $[\mathcal{E} \equiv \mathcal{E}_\phi]$:

$$1 + i\langle\phi| \frac{1}{H - \mathcal{E}} |\phi\rangle\, \alpha_k = 0 , \tag{16}$$

and for the eigenvectors the explicit expression

$$|\psi\rangle = - \langle\phi|\psi\rangle\, i\alpha_k \frac{1}{H - \mathcal{E}} |\phi\rangle , \tag{17}$$

where $\langle\phi|\psi\rangle$ is a normalization constant. Notice that to solve eq.(16) we do not need knowing α_k, i.e. the scattering states due to QHQ, since fortunately *at the resonances* it is $[E \equiv E_R]$:

$$\alpha_k \propto |\langle\phi|H|\overset{\circ}{\psi}{}^{(+)}_{koo}\rangle|^2 = |\langle\phi|\psi^{(+)}_E\rangle - \langle\phi|koo\rangle|^{-2} .$$

Notice moreover that the present approach, a priori, allows distinguishing between true Resonances and other effects.

In Ref.(1) the application was considered to the case of scattering by a spherical-well potential $U(r) = U_o \cdot \theta(a - r)$, and as exploring states the class was adapted of the normalized Laurentian wave-packets (good for low energies):

$$\langle k00|\phi\rangle = \sqrt{2b}\;\frac{1}{k^2+b^2} \iff \langle \underline{r}|\phi\rangle = \sqrt{\frac{b}{2\pi}}\;\frac{\exp[-br]}{r}\,.$$

By integration, for low entering energies ($k^2 \ll 2mU_o$) one gets one equation, whose real and imaginary parts forward a system of two equations. The latters individuate $|\phi\rangle$, i.e. the parameter b, for which a series of (true) resonances arises. These resonances are expected to appear for $[k^2 \equiv 2mE;\ K^2 \equiv 2m(E+U_o)]$:

$$\cos Ka = 0 \Rightarrow Ka = (n + \tfrac{1}{2})\,\pi.$$

The system of equations is rather complicated (even when the resonance width is $\gamma < k_o$). But the first equation does not contain γ and yields b. For instance, for $n = 0$ one gets a unique solution ($ab \simeq 0.69$).

6. DECAY OF THE UNSTABLE STATE

We are more interested in the decay in time of the unstable state $|\psi\rangle$

$$\langle\psi|\psi_t\rangle \equiv \langle\psi|U_t|\psi\rangle \equiv \langle\psi|\exp[-i\sigma t]|\psi\rangle. \tag{18}$$

If we assume, as usual, $\sigma = H$, then

$$\langle\psi|\psi_t\rangle \simeq \int_0^\infty dE\,|\langle\psi|\psi_E^{(+)}\rangle|^2 \cdot \exp[-iEt] \tag{19}$$

since the bound-states do not contribute for large t. Moreover, let us remember that

$$|\psi\rangle = -i\alpha_k\,\langle\phi|\psi\rangle\,\frac{1}{H-\mathcal{E}}\,|\phi\rangle.$$

Therefore,

$$|\langle \psi_E^{(+)} | \psi \rangle|^2 = \frac{|\alpha_k|^2}{(\mathrm{Re}\,\mathcal{E} - E)^2 - (\mathrm{Im}\,\mathcal{E})^2} \cdot C; \quad C \equiv |\langle \psi_E^{(+)} | P | \psi \rangle|^2 .$$

The integral (19) can be evaluated following Ref.(4). The expression C contains denominators that - analytically extended - produce *one* pole in $E = \mathcal{E}$. If in the strip $\mathrm{Im}\,\mathcal{E} < \mathrm{Im}\,E < 0$ no other singularities arise from the remaining factors, then we obtain the exponential-type decay

$$\langle \psi | \psi_t \rangle = (C + Dt) \cdot \exp[-(iE_o t + \gamma_o t)] \tag{20}$$

with $E_o \equiv \mathrm{Re}\,\mathcal{E}$; $\gamma_o \equiv \mathrm{Im}\,\mathcal{E}$; C and D *constants*.

More interesting appears, however, the assumption

$$o = \mathcal{H}, \tag{21}$$

since in this case our approach does surely possess a "Lie-admissible" structure(6) (due to the fact that the time-evolution operator with $\mathcal{H}$ is not unitary). In such a case one would simply get

$$\langle \psi | \psi_t \rangle = \overline{K} \cdot \exp[iE_o t + \gamma_o t] \tag{22}$$

with $\overline{K} \equiv \langle \psi | \psi \rangle$. But in this case the whole approach ought to be carefully rephrased in "Lie-admissible" terms (otherwise, e.g., *all* states would seem to be decaying).

PART II: ON FOUR-POSITION OPERATORS IN Q.F.T.

7. THE KLEIN-GORDON CASE: THREE-POSITION OPERATORS

The usual position-operators, being Hermitian, are known to possess real eigenvalues: i.e., they yield a *point-like* localization.

J. M. Jauch showed, however, that a point-like localization would be in contrast with "unimodularity". In the relativistic case, moreover, phenomena so as the pair production forbid a localization with precision better than one Compton wave-length. The eigenvalues of a realistic position-operator $\hat{\underline{\underline{z}}}$ are therefore expected to represent space *regions*, rather than points. This can be obtained only making recourse to non-Hermitian position-operators $\hat{\underline{\underline{z}}}$ (a priori, one can make recourse either to non-normal operators with commuting components, or to normal operators with non-commuting components([7])). Following the spirit of Refs. ([7]), we are going to show that the mean values of the *Hermitian part* of $\hat{\underline{\underline{z}}}$ will yield a mean (point-like) position([8]), while the mean values of the *anti-Hermitian part* of $\hat{\underline{\underline{z}}}$ will yield the sizes of the localization region([9]).

Let us consider e.g. the case of relativistic spin-zero particles, in natural units and with the metric (+ - - -). The position operator, $i\nabla_{\underline{\underline{p}}}$, is known to be actually non-Hermitian, and may be in itself a good candidate for an *extended-type position* operator. To show this, we want to split([8]) it into its Hermitian and anti-Hermitian parts.

Consider, then, a vector space V of complex differentiable functions on a 3-dimensional phase-space equipped with an inner product defined by $[p_o \equiv \sqrt{\underline{\underline{p}}^2 + m_o^2}\,]$:

$$(\psi,\phi) = \int \frac{d^3\underline{\underline{p}}}{p_o}\, \Psi^*(\underline{\underline{p}})\Phi(\underline{\underline{p}}). \tag{23}$$

Let the functions in V further satisfy a condition

$$\lim_{R\to\infty} \int_{S_R} \frac{dS}{p_o}\, \Psi^*(\underline{\underline{p}})\Psi(\underline{\underline{p}}) = 0, \tag{24}$$

where the integral is taken over the surface of a sphere of radius R. If $\mathcal{D} : V \to V$ is a differential operator of degree one, condition (24) allows a definition of the transpose $\mathcal{D}^T$ by

$$(\mathcal{D}^T\psi,\phi) = (\psi,\mathcal{D}\phi) \quad \text{for all} \quad \phi, \quad \psi \in V, \tag{25}$$

where $\mathcal{D}$ is changed into $\mathcal{D}^T$, or vice-versa, by means of integration by parts.

This allows further to introduce a *dual representation* $(\mathcal{D}_1,\mathcal{D}_2)$ of a *single* operator $\mathcal{D}_1^T + \mathcal{D}_2$ by

$$(\mathcal{D}_1\psi,\phi) + (\psi,\mathcal{D}_2\phi) = (\psi,(\mathcal{D}_1^T + \mathcal{D}_2)\phi). \tag{26}$$

With such a *dual* representation it is easy to split any operator into its Hermitian and anti-Hermitian (or skew-Hermitian) parts

$$(\psi,\mathcal{D}\phi) = \tfrac{1}{2}\,[(\psi,\mathcal{D}\phi) + (\mathcal{D}^*\psi,\phi)] + \tfrac{1}{2}\,[(\psi,\mathcal{D}\phi) - (\mathcal{D}^*\psi,\phi)]. \tag{27}$$

Here the pair

$$\tfrac{1}{2}(\mathcal{D}^*,\mathcal{D}) \equiv \overset{\leftrightarrow}{\mathcal{D}}_h \tag{28a}$$

corresponding to $\tfrac{1}{2}(\mathcal{D} + \mathcal{D}^{*T})$, represents the Hermitian part, while

$$\tfrac{1}{2}(-\mathcal{D}^*,\mathcal{D}) \equiv \overset{\leftrightarrow}{\mathcal{D}}_a \tag{28b}$$

represents the anti-Hermitian part.

Let us apply what precedes to the case of the Klein-Gordon position-operator $\underline{\underline{q}} = i\underline{\nabla}_{\underline{\underline{p}}}$. When

$$\mathcal{D} = i\,\frac{\partial}{\partial p_j} \tag{29}$$

we have [9,10]

$$\frac{1}{2}(\mathcal{D}^*,\mathcal{D}) = \frac{1}{2}(-\,i\,\frac{\partial}{\partial p_j}\,,\; i\,\frac{\partial}{\partial p_j}) \equiv \frac{i}{2}\,\frac{\overleftrightarrow{\partial}_{(-)}}{\partial p_j} \equiv \frac{i}{2}\,\frac{\overleftrightarrow{\partial}}{\partial p_j}\,, \tag{30a}$$

$$\frac{1}{2}\,(-\,\mathcal{D}^*,\mathcal{D}) = \frac{1}{2}(i\,\frac{\partial}{\partial p_j},\, i\,\frac{\partial}{\partial p_j}) \equiv \frac{i}{2}\,\frac{\overleftrightarrow{\partial}_{(+)}}{\partial p_j}\,. \tag{30b}$$

And the corresponding *single* operators turn out to be

$$\frac{1}{2}(\mathcal{D} + \mathcal{D}^{*T}) = i\,\frac{\partial}{\partial p_j} - \frac{i}{2}\,\frac{p_j}{\underline{\underline{p}}^2 + m_o^2} \tag{31a}$$

and

$$\frac{1}{2}(\mathcal{D} - \mathcal{D}^{*T}) = \frac{i}{2}\,\frac{p_j}{\underline{\underline{p}}^2 + m_o^2} \tag{31b}$$

It is netoworthy(10,9) that operator (31a) is nothing but the usual Newton-Wigner operator, while (31b) has been interpreted(7,9) as yielding the sizes of the localization-region (an ellipsoid) by means of its average values over the considered *wave-packet*.

Let us underline that the previous treatment justifies from the mathematical point of view the formalism used in Refs.(8,10): We want to report it briefly here, due to its immediate legibility (its significance being now mathematically clarified by the preceding approach). In Ref.(8) we split the operator $\hat{\underline{\underline{z}}}$ as follows:

$$\hat{\underline{\underline{z}}} \equiv i\nabla_{\underline{\underline{p}}} = \frac{i}{2}\,\frac{\overleftrightarrow{\partial}}{\partial \underline{\underline{p}}} + \frac{1}{2}\,\frac{\overleftrightarrow{\partial}_{(+)}}{\partial \underline{\underline{p}}}\,, \tag{32}$$

where

$$\psi^*\,\frac{\overleftrightarrow{\partial}_{(+)}}{\partial \underline{\underline{p}}}\,\phi \equiv \psi^*\,\frac{\partial\phi}{\partial \underline{\underline{p}}} + \phi\,\frac{\partial\psi^*}{\partial \underline{\underline{p}}}\,,$$

and where we always referred to a suitable space of *wave-packets*(10,9). Its Hermitian part(9.10)

$$\hat{x} \equiv \frac{i}{2}\,\frac{\overleftrightarrow{\partial}}{\partial \underline{\underline{p}}}\,, \tag{33}$$

which was expected to yield an (ordinary) point-like localization, was derived also by writing explicity

$$(\psi, \underline{\underline{x}}\phi) = i\int \frac{d^3\underline{\underline{p}}}{p_o} \Psi(p)\underline{\nabla}_{\underline{\underline{p}}}\Phi(\underline{\underline{p}})$$

and imposing Hermiticity, i.e. the reality of the diagonal elements. The calculation yielded

$$\mathrm{Re}(\phi, \underline{\underline{x}}\phi) = \frac{i}{2}\int \frac{d^3\underline{\underline{p}}}{p_o} \Phi^*(\underline{\underline{p}}) \frac{\overleftrightarrow{\partial}}{\partial\underline{\underline{p}}} \Phi(\underline{\underline{p}}),$$

just suggesting to adopt the Lorentz-invariant quantity (33) as Hermitian position operator. Then, integrating by parts (and due to the vanishing of the surface integral) we verified that (23) is equivalent to the ordinary Newton-Wigner operator N - W:

$$\frac{i}{2}\frac{\overleftrightarrow{\partial}}{\partial\underline{\underline{p}}} \equiv i\underline{\nabla}_{\underline{\underline{p}}} - \frac{i}{2}\frac{\underline{\underline{p}}}{\underline{\underline{p}}^2 + m_o^2} \equiv \mathrm{N - W}. \tag{34}$$

We were left with the anti-Hermitian part

$$\hat{y} \equiv \frac{1}{2}\frac{\overleftrightarrow{\partial}^{(+)}}{\partial\,\underline{\underline{p}}} \tag{35}$$

whose *average values* over the considered state (wave-packet) were regarded as yielding[7,9] the sizes of an ellipsoidal localization-region.

After this digression (eqs.(32) ÷ (35)), let us go back to our present formalism (represented by eqs.(23) ÷ (31)).

In general, the extended-type position operator $\hat{\underline{\underline{z}}}$ will give

$$\langle\psi|\underline{\underline{z}}|\psi\rangle = (\underline{\alpha} + \Delta\underline{\alpha}) + i(\underline{\beta} + \Delta\underline{\beta}), \tag{36}$$

where $\Delta\underline{\alpha}$ and $\Delta\underline{\beta}$ are the mean-errors encountered when measuring the point-like position and the sizes of the localization-region, respectively. It is interesting to evaluate the commutators

$[i, j = 1,2,3]$:

$$\left[\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial p^i} , \frac{1}{2} \frac{\overleftrightarrow{\partial}^{(+)}}{\partial p^j} \right] = \frac{i}{2p_o^2} \left(\delta_{ij} - \frac{2p_i p_j}{p_o^2}\right), \tag{37}$$

wherefrom the noticeable "uncertainty correlations" follow:

$$\Delta\alpha_i \cdot \Delta\beta_j \geq \frac{1}{4} \left| \left\langle \frac{1}{p_o^2} \left(\delta_{ij} - \frac{2p_i p_j}{p_o^2}\right) \right\rangle_\psi \right| . \tag{38}$$

8. FOUR-POSITION OPERATORS

It is tempting to propose as four-position operator the quantity $\hat{z}^\mu = \hat{x}^\mu + i\hat{y}^\mu$, whose Hermitian (Lorentz-covariant) part can be written:

$$\hat{x}^\mu \equiv - \frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial p^\mu} , \tag{39}$$

to be associated with its corresponding "operator" in four-momentum space:

$$\hat{p}^\mu \equiv + \frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial x_\mu} . \tag{40}$$

Let us recall the proportionality between the 4-momentum operator and the 4-current density operator in the chronotopical space, and underline then the canonical correspondence (in the 4-position and 4-momentum spaces, respectively) between the "operators" (cf. Sect. 7)

$$
\begin{cases}
\text{(41a)} \quad m_o \hat{\rho} \equiv \hat{p}_o = \frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial t} ; & \text{(41c)} \quad \hat{t} = -\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial p_o} ; \\
\text{(41b)} \quad m_o \hat{\underline{j}} \equiv \underline{p} = -\frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial \underline{r}} & \text{(41d)} \quad \hat{x} = \frac{i}{2} \frac{\overleftrightarrow{\partial}}{\partial \underline{p}} ,
\end{cases}
$$

where the four-position "operator" (41c,d) can be regarded as a 4-current density operator in the energy-impulse space(9). Analogous considerations can be carried on for the anti-Hermitian parts(9).

9. ON THE TIME-OPERATOR

Let us fix our attention only on the operator for time in the case of (non-relativistic) quantum mechanics. Time, as well as 3-position, sometimes is a parameter, but sometimes is an observable to be represented by an operator. We have shown elsewhere that in Q.M. the "operator" (41c) - cf. Sect. 7 - can be replaced with the "operator"

$$\hat{t} \equiv -i \frac{\partial}{\partial E} \tag{42}$$

provided that a suitable, subsidiary boundary-condition is imposed on the considered wave-packets(10).

In Q.M., however, the wave-packet space is a space of functions defined only over the interval $0 \leq E < \infty$, and not over the whole E-axis. As a consequence, $\hat{t}$ is Hermitian (and symmetric) but *not* self-adjoint, and does not allow the identity resolution. In Q.M., therefore, one has to use non-self adjoint operators(11) even for the observable time. However, even if $\hat{t}$ does not admit true eigenfunctions, nevertheless one succeeds in calculating the *average values* of $\hat{t}$ over wave-packets. And this is enough to evaluate the packet time-coordinate, the flight-times, the interaction-durations, the (mean) life-times of metastable states, and so on (8-10,12).

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