ON SEPARATELY HOLOMORPHIC AND SILVA HOLOMORPHIC MAPPINGS

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ABSTRACT

It is examined relations of the type $\mathcal{H}(U \times V; G) \cong \mathcal{H}(U; (\mathcal{H}(V;G), \tau_0))$. Counter-examples and some positive results are given.

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E, F and G denote complex locally convex spaces. The subsets $U \subset E$ and $V \subset F$ indicate non-void open subsets. $\mathcal{B}_E$ denote the family of all bounded closed absolutely convex subsets of E. If $B \in \mathcal{B}_E$ let E_B denote the vector subspace of E generated by B and normed by the Minkowsky functional $\|\cdot\|_B$ associated to B.

We recall that a subset K of E is a strict compact set it there is B in $\mathcal{B}_E$ such that B is contained and compact in E_B.

$\mathcal{H}_f(U;F)$, $\mathcal{H}(U;F)$ and $\mathcal{H}_S(U;F)$ denote respectively the vector spaces of all F-valued finitely holomorphic mappings, all F-valued holomorphic mappings and all F-valued Silva holomorphic mappings. It is clear that $\mathcal{H}(U;F) \subset \mathcal{H}_S(U;F) \subset \mathcal{H}_f(U;F)$. If $g \in \mathcal{H}_f(U;F)$, $u \in U$ and $n \in \mathbb{N}$, $(n!)^{-1}\hat{\delta}^n f(u)$ denotes the n-homogeneous polynomial from E into F which is the n-th Taylor series coefficient of g at u.

For fundamental properties of the spaces $\mathcal{H}_f(U;F)$, $\mathcal{H}(U;F)$ and $\mathcal{H}_S(U;F)$ we give some references: Barroso [1], Dineen [2], Matos [3], Matos--Nachbin [4], Nachbin [5], Noverraz [6], Paques [7], Pisanelli [8] Silva [9].

DEFINITION 1 - A mapping g from U x V into G is separately finitely holomorphic (respectively: separately holomorphic, separately Silva holomorphic) in U x V if for every $u \in U$ and $v \in V$ the mappings

$$g_u: x \in V \longmapsto g_u(x) = g(u,x) \in G$$

and

$$g_v: y \in U \longmapsto g_v(y) = g(y,v) \in G$$

are such that $g_u \in \mathcal{H}_f(V;G)$ (respectively: $\mathcal{H}(V;G)$, $\mathcal{H}_S(V;G)$) and $g_v \in \mathcal{H}_f(U;G)$

(respectively: $\mathcal{H}(U;G)$, $\mathcal{H}_S(U;G)$).

$\mathcal{H}_f(U,V;G)$ (respectively: $\mathcal{H}(U,V;G)$, $\mathcal{H}_S(U,V;G)$) denotes the vector space of all separately finitely holomorphic (respectively: separately holomorphic, separately Silva holomorphic) mappings from $U \times V$ into G. It is clear that the following inclusions are true:

$$\begin{array}{ccccc} \mathcal{H}(U \times V;G) & \subset & \mathcal{H}_S(U \times V;G) & \subset & \mathcal{H}_f(U \times V;G) \\ \cap & & \cap & & \cap \\ \mathcal{H}(U,V;G) & \subset & \mathcal{H}_S(U,V;G) & \subset & \mathcal{H}_f(U,V;G) \end{array}$$

The following theorem is a consequence of a classical Hartogs' Theorem (see [10] for a proof of this theorem)

THEOREM 2 - (Hartogs) - $\mathcal{H}_f(U,V;G) = \mathcal{H}_f(U \times V;G)$.

Versions of Hartogs' Theorem for holomorphic and Silva holomorphic mappings may be found in Alexander [11], Bochnak-Siciak [12], Dineen [2], Lazet [13], Matos [14], Matos [3], Noverraz [15], Pisanelli [8].

Next example shows that the Hartogs' Theorem is false for cartesian products of a normed and a Banach space.

EXAMPLE 3 - Let E be the vector space c_{00} of all sequences of complex numbers with finite support. We consider on E the ℓ^1-norm. Let F be a Banach space. It is known that there is a sequence (g_m) of elements of $\mathcal{H}(F;\mathbb{C})$ which is pointwise bounded but not locally bounded (this is known for F finite dimensional and, by composing with projections, for any normed space F). Thus we can find a sequence (y^m) in F converging to a point y of F and (h_m) in $\mathcal{H}(F;\mathbb{C})$ pointwise bounded such that $h_m(y^m) > m^2$ for every $m=1,2,\ldots$ Now we define

$$g: \begin{array}{ccc} E \times F & \longmapsto & \mathbb{C} \\ ((x_m),t) & \longmapsto & \sum x_m h_m(t) \end{array}$$

$g \in \mathcal{H}(E,F;\mathbb{C})$. Now if we take $z^m = (0,\ldots,0,1/m,0,\ldots)$, $1/m$ in the m-th position, the sequence $((z^m,y^m))$ converges in $E \times F$. But $g(z^m,y^m) > m$ for $m=1,2,3,\ldots$ Thus $g \notin \mathcal{H}(E \times F;\mathbb{C})$.

In this paper our function spaces will be considered with the following natural locally convex topologies. τ_{0f} the topology of the uniform convergence on the finite dimensional compact sets, τ_0 the compact-open topology, τ_{0e} the topology of the uniform convergence on the strict compact sets.

We consider the following mappings:

(1) $\psi : g \in \mathcal{H}_f(U,V;G) \longmapsto \psi(g) = \psi_g \in (G^V)^U$
where $\psi_g(u)(v) = g(u,v)$ for every $u \in U$ and $v \in V$.

(2) $\varphi : g \in \mathcal{H}_f(U,V;G) \longmapsto \varphi(g) = \varphi_g \in (G^U)^V$
where $\varphi_g(v)(u) = g(u,v)$ for every $u \in U$ and $v \in V$.

It is easy to see that each of these mappings is 1-1 and linear. We want to find out conditions on E, F and G which imply equalities of the type:

$$\psi(\mathcal{H}_S(U,V;G)) = \mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0f}))$$
$$\psi(\mathcal{H}_S(U,V;G)) = \mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0e}))$$
$$\varphi(\mathcal{H}_S(U,V;G)) = \mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0f}))$$
$$\varphi(\mathcal{H}_S(U,V;G)) = \mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0e}))$$

and the analogous equalities for the spaces of separately holomorphic mappings. In these cases we also want to know if ψ and φ are homeomorphisms for the natural topologies.

PROPOSITION 4 - (a) If $g \in \mathcal{H}_S(U,V;G)$ then $\psi_g \in \mathcal{H}_f(U;(\mathcal{H}_S(V;G),\tau_{0f}))$ and $\varphi_g \in \mathcal{H}_f(V;(\mathcal{H}_S(U;G),\tau_{0f}))$

(b) If $g \in \mathcal{H}(U,V;G)$ then $\psi_g \in \mathcal{H}_f(U;(\mathcal{H}(V;G),\tau_{0f}))$ and $\varphi_g \in \mathcal{H}_f(V;(\mathcal{H}(U;G),\tau_{0f}))$.

PROOF - We may suppose without loss of generality that G is seminormed by $\|\cdot\|$.

Let $g \in \mathcal{H}_S(U,V;G)$ (respectively $\mathcal{H}(U,V;G)$). It is clear that $\psi_g(u) = g_u$ is in $\mathcal{H}_S(V;G)$ (respectively $\mathcal{H}(V;G)$) for every $u \in U$. By Theorem 2 g is in $\mathcal{H}_f(U \times V;G)$. We consider $u_0 \in U$, $x \in E$ and L a finite dimensional compact subset of V. The Taylor series expansion of g and the Cauchy inequalities imply

$$\sup_{t \in L} \| \alpha^{-1}(g(u_o + \alpha x,t) - g(u_o,t)) - \hat{\delta}^1 g(u_o,t)(x,0) \| \leq$$

$$\leq |\alpha| \cdot \sup_{t \in L} \| \sum_{n=2}^{\infty} \alpha^{n-2}(n!)^{-1} \hat{\delta}^n g(u_o,t)(x,0) \| \leq$$

$$\leq |\alpha| \sum_{n=2}^{\infty} |\alpha|^{n-2} \sup_{t \in L} \left| \int_{|\lambda|=\rho} \lambda^{-n-1}(g((u_o,t) + \lambda(x,0)) \, d\lambda \right| = \circledast$$

where $\rho > 0$ is such that $(u_o,t) + \lambda(x,0) \in U \times V$ for every $t \in L$ and $|\lambda| \leq \rho$. Thus for $|\alpha| \leq 2^{-1}\rho$ we have:

$$\circledast \leq |\alpha| \sum_{n=2}^{\infty} 2^{-n+2} \rho^{n-2} \rho^{-n} \sup_{\substack{t \in L \\ |\lambda|=\rho}} \| g((u_o,t) + \lambda(x,0)) \| = |\alpha| . M$$

where M is a positive real number. Hence

$$(1) \qquad \lim_{\alpha \to 0} \sup_{t \in L} \| \alpha^{-1}(g(u_o + \alpha x,t) - g(u_o,t)) - \hat{\delta}^1 g(u_o,t)(x,0) \| = 0$$

The mapping $\hat{\delta}^1 g(u_o,.)(x,0)$ is Silva holomorphic (respectively holomorphic) in V since g is separately Silva holomorphic (respectively separately holomorphic) in U x V. By (1) we get ψ_g finitely holomorphic from U into the space $(\mathcal{H}_S(V;G), \tau_{Of})$. The proof for φ_g is analogous with the obvious changes.

We recall the definition of a holomorphically barreled locally convex space (see Barroso-Matos-Nachbin [16], [17] and Nachbin [18]).

DEFINITION 5 - The space E is holomorphically barreled if for every U, G and $\mathcal{X} \subset \mathcal{H}(U;G)$ τ_{Of}-bounded, $\mathcal{X}$ is equicontinuous on U.

THEOREM 6 - (a) If E is holomorphically barreled, then $\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G), \tau_{Of}))$ for $g \in \mathcal{H}(U,V;G)$ and ψ is a homeomorphism between the spaces $(\mathcal{H}(U,V;G),\tau_{0f})$ and $(\mathcal{H}(U;(\mathcal{H}(V;G)),\tau_{0f}),\tau_{0f})$.

(b) If F is holomorphically barreled, then $\varphi_g \in \mathcal{H}(V;(\mathcal{H}(U;G),\tau_{0f})$ for each $g \in \mathcal{H}(U,V;G)$ and φ is a homeomorphism between the spaces $(\mathcal{H}(U,V;G),\tau_{0f})$ and $(\mathcal{H}(V;\mathcal{H}(U;G)),\tau_{0f}),\tau_{0f})$.

PROOF - If $g \in \mathcal{H}(U,V;G)$ $\psi_g \in \mathcal{H}_f(U;(\mathcal{H}(V;G),\tau_{0f})$ by Proposition 6. In order to prove that ψ_g is holomorphic it is enough to prove that it is amply bounded (i. e. for every continuous seminorm p on $(\mathcal{H}(V;G),\tau_{0f})$ ψ_g is locally bounded as a mapping from U into $\mathcal{H}(V;G)$ seminormed by p). Without loss of generality we may suppose G seminormed by $\|\cdot\|$. Let L be a finite dimensional compact subset of V and let p_L be the seminorm on $\mathcal{H}(V;G)$ given by

$$p_L(g) = \sup_{t \in L} \|g(t)\|.$$

We observe that if K is a finite dimensional compact subset of U then

$$\sup_{\substack{x \in K \\ t \in L}} \|\psi_g(x)(t)\| = \sup_{\substack{x \in K \\ t \in L}} \|g(x,t)\| = M < +\infty$$

Hence $\mathcal{X} = \{ g_t \,;\, t \in L\} \subset \mathcal{H}(U;G)$ is τ_{0f}-bounded. Since we suppose E to be holomorphically barreled, $\mathcal{X}$ is equicontinuous, therefore amply bounded, i. e. for every $x_o \in U$ there is a neighborhood V_o of x_o such that

$$\sup_{\substack{x \in V_o \\ t \in L^o}} \|g_t(x)\| = N < +\infty$$

But this means that

$$\sup_{x \in V_o} p_L(\psi_g(x)) = N < +\infty$$

Thus ψ_g is holomorphic in U. Now it is very easy to show that ψ is a homeomorphism. The proof of the second part is analogous with the obvious modifications.

THEOREM 7 - If E is metrizable and F is a Fréchet space and we suppose that

$\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G), \tau_{0f}))$ for every $g \in \mathcal{H}(U,V;G)$, then it follows that $\mathcal{H}(U \times V;G) = \mathcal{H}(U,V;G)$.

PROOF - If $g \in \mathcal{H}(U,V;G)$ then $g \in \mathcal{H}_f(U \times V;G)$ by Theorem 2. The hypothesis implies that $\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G), \tau_{0f}))$. Thus for every compact subset K of U $\psi_g(K) \subset \mathcal{H}(V;G)$ is τ_{0f}-compact, hence τ_{0f}-bounded. Since a Fréchet space is holomorphically barreled (see [16] and [17]) $\psi_g(K)$ is equicontinuous, hence τ_0-bounded. This implies that g is bounded over the compact subsets of U x V. By a result of [16] (see also [17]) it follows that g is holomorphic since E x F is metrizable.

REMARK 8 - Theorem 7 and Example 3 show that we can not take E arbitrary and F holomorphically barreled in Theorem 6 part (a) and still get the same conclusion.

THEOREM 9 - (a) If E is quasi-complete, then $\psi_g \in \mathcal{H}_S(U;(\mathcal{H}_S(V;G), \tau_{0f}))$ for every g in $\mathcal{H}_S(U,V;G)$. The mapping ψ is a homeomorphism between the spaces $(\mathcal{H}_S(U,V;G), \tau_{0f})$ and $(\mathcal{H}_S(U;(\mathcal{H}_S(V;G), \tau_{0f})), \tau_{0f})$.
(b) If F is quasi-complete, then $\varphi_g \in \mathcal{H}_S(V;(\mathcal{H}_S(U;G), \tau_{0f}))$ for every g in the space $\mathcal{H}_S(U,V;G)$. The mapping φ is a homeomorphism between the spaces $(\mathcal{H}_S(U,V;G), \tau_{0f})$ and $(\mathcal{H}_S(V;(\mathcal{H}_S(U;G), \tau_{0f})), \tau_{0f})$.

PROOF - If $g \in \mathcal{H}_S(U,V;G)$ then $\psi_g \in \mathcal{H}_f(U;(\mathcal{H}_S(V;G), \tau_{0f}))$ by Proposition 4, part (a). Let $B \in \mathcal{B}_E$. Then E_B is a Banach space since E is quasi-complete. In order to show that ψ_g is Silva-holomorphic we must prove that $\psi_g \mid U \cap E_B$ is holomorphic for the normed topology. Thus we must prove that this mapping is amply bounded in $U \cap E_B$ for the normed topology. But this can be shown as in the proof of the preceding theorem since E_B being a Banach space is holomorphically barreled. The proof that ψ is a homeomorphism is trivial. The proof of part (b) is analogous with the obvious modifications.

THEOREM 10 - (a) If E and F are holomorphically barreled, every $g \in \mathcal{H}(U,V;G)$

is bounded over the compact subsets of $U \times V$ and $\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G),\tau_0))$, $\varphi_g \in \mathcal{H}(V;(\mathcal{H}(U;G),\tau_0))$. Moreover ψ is a homeomorphism between the spaces $(\mathcal{H}(U,V;G),\tau_0)$ and $(\mathcal{H}(U;(\mathcal{H}(V;G),\tau_0)),\tau_0)$ and φ is a homeomorphism between $(\mathcal{H}(U,V;G),\tau_0)$ and $(\mathcal{H}(V;(\mathcal{H}(U;G),\tau_0)),\tau_0)$.

(b) If E and F are quasi-complete then every $g \in \mathcal{H}_S(U,V;G)$ is bounded over the strict compact subsets of $U \times V$ (hence g is in $\mathcal{H}_S(U \times V;G)$) and ψ_g is in $\mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0e}))$, $\varphi_g \in \mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0e}))$. Moreover ψ is a homeomorphism between $(\mathcal{H}_S(U,V;G),\tau_{0e})$ and $(\mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0e})),\tau_{0e})$ and φ is a homeomorpism between $(\mathcal{H}_S(U,V;G),\tau_{0e})$ and the space $(\mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0e})),\tau_{0e})$.

PROOF - With no loss of generality we maysuppose G seminormed.

(a) Let $K \subset U$ and $L \subset V$ compact sets. Since E is holomorphically barreled $\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G),\tau_{0f}))$ for $g \in \mathcal{H}(U,V;G)$. This implies that g is bounded over $K \times M$ for every finite dimensional compact subset M of V. Hence the set $\mathcal{X} = \{ g_x; x \in K \} \subset \mathcal{H}(V;G)$ is τ_{0f}-bounded. Since F is holomorphically barreled $\mathcal{X}$ is equicontinuous, thus τ_0-bounded. It follows that g is bounded over $K \times L$. In order to prove that $\psi_g \in \mathcal{H}(U;(\mathcal{H}(V;G),\tau_0))$ we observe that $\mathcal{Y} = \{ g_t ; t \in L \} \subset \mathcal{H}(U;G)$ is τ_{0f}-bounded, hence equicontinuous in U since E is holomorphically barreled. Thus for each $x_0 \in U$ there is a neighborhood V_0 of x_0 such that

$$\sup \left\{ \| \psi_g(x)(t) \| ; t \in L, x \in V_0 \right\} < +\infty$$

This means that ψ_g is amply bounded from U into $(\mathcal{H}(V;G),\tau_0)$ and ψ_g is in $\mathcal{H}(U;(\mathcal{H}(V;G),\tau_0))$. The proof that ψ is a homeomorphism is now easy. The proofs of the second part of (a) is analogous with the obvious modifications.

(b) The proof of this part is reduced to an application of part (a) by the fact that restrictions of g to $E_B \times F_D \cong (E \times F)_{B \times D}$, when $B \in \mathcal{B}_E$ and $D \in \mathcal{B}_F$, satisfy the hypothesis of part (a).

REMARK 11 - The conclusion of Theorem 10, part (a), does not hold if one of the spaces is not holomorphically barreled. Example 3 gives the counter-example (see also Remark 8).

It is easy to see that the enverse mappings ψ^{-1} and φ^{-1} of ψ and φ respectively are such that:

$$\psi^{-1}(\mathcal{H}(U;(\mathcal{H}(V;G),\tau_0))) \subset \psi^{-1}(\mathcal{H}(U;(\mathcal{H}(V;G),\tau_{0f}))) \subset \mathcal{H}(U,V;G)$$

$$\psi^{-1}(\mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0e}))) \subset \psi^{-1}(\mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0f}))) \subset \mathcal{H}_S(U,V;G)$$

$$\varphi^{-1}(\mathcal{H}(V;(\mathcal{H}(U;G),\tau_0))) \subset \varphi^{-1}(\mathcal{H}(V;(\mathcal{H}(U;G),\tau_{0f}))) \subset \mathcal{H}(U,V;G)$$

$$\varphi^{-1}(\mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0e}))) \subset \varphi^{-1}(\mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0f}))) \subset \mathcal{H}(U,V;G)$$

Thus combining the preceding results with the known versions of Hartogs' Theorem (see references following Theorem 2) we can write:

THEOREM 12 - If (i) E and F are Fréchet spaces, (ii) E and F are Silva spaces, (iii) E x F is a Baire space with one of them metrizable, then

(a) ψ is a homeomorpism between $(\mathcal{H}(U \times V;G),\tau_0)$ and the space $(\mathcal{H}(U;(\mathcal{H}(V;G),\tau_0)),\tau_0)$.

(b) φ is a homeomorphism between $(\mathcal{H}(U \times V;G),\tau_0)$ and the space $(\mathcal{H}(V;(\mathcal{H}(U;G),\tau_0)),\tau_0)$.

If E and F are quasi-complete then:

(a) ψ is a homeomorphism between $(\mathcal{H}_S(U \times V;G),\tau_{0_e})$ and the space $(\mathcal{H}_S(U;(\mathcal{H}_S(V;G),\tau_{0e})),\tau_{0e})$.

(b) φ is a homeomorphism between $(\mathcal{H}_S(U \times V;G),\tau_{0e})$ and the space $(\mathcal{H}_S(V;(\mathcal{H}_S(U;G),\tau_{0e})),\tau_{0e})$.

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