All Nachbin spaces $CV_\infty(X)$ of continuous scalar-valued functions have the approximation property.
§1. INTRODUCTION

Throughout this paper X is a Hausdorff space such that \( C_b(X;\mathbb{K}) \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)) separates the points of X, and E is a non-zero locally convex space. Our aim is to prove that certain function spaces \( L \subseteq C(X;E) \) have the approximation property as soon as E has the approximation property. We show this for the class of all Nachbin spaces \( C_{\infty}(X;E) \). Such spaces include \( C(X;E) \) with the compact-open topology; \( C_b(X;E) \) with the strict topology; \( C_0(X;E) \) with the uniform topology. When \( E = \mathbb{K} \), Bierstedt [1], using the technique of \( \varepsilon \)-products, had proved that \( C_{\infty}(X;\mathbb{K}) \) has the approximation property, under the hypothesis that X is a completely regular \( k_{\mathbb{R}} \)-space, and that the family \( V \) of weights is such that given a compact subset \( K \subseteq X \), one can find a weight \( v \in V \) such that \( v(x) \geq 1 \) for all \( x \in K \).

The technique we use here was suggested by the paper [5] of Gierz, who proved the analogue of Theorem 1 below for the case of X compact and bundles of Banach spaces. This technique of "localization" of the approximation property was used by Bierstedt, in the case of the partition by antisymmetric sets (Bierstedt [2]), but the main idea of representing the space of operators of \( L \) as another Nachbin space of cross sections is due to Gierz. However our presentation is much simpler, in particular we do not use the concept of a locally \( C(X) \)-convex \( C(X) \)-module. In the Introduction to his paper, Gierz said that his method could be applied to the vector fibrations in the sense of [8], and this led to our effort at simplifying his proof and adapting it to our context.

§2. THE APPROXIMATION PROPERTY FOR NACHBIN SPACES.

A vector fibration over a Hausdorff topological space X is a pair \( (X,(F_x)_{x \in X}) \), where each \( F_x \) is a vector space over the field \( \mathbb{K} \) (where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)). A cross-section is then any element \( f \) of the Cartesian product of the spaces \( F_x \), i.e., \( f = (f(x))_{x \in X} \).
A weight on $X$ is a function $v$ on $X$ such that $v(x)$ is a seminorm over $F_x$ for each $x \in X$. A Nachbin space $L_{V_{\infty}}$ is a vector space $L$ of cross-sections $f$ such that the mapping

$$x \in X \mapsto v(x)[f(x)]$$

is upper semicontinuous and null at infinity on $X$ for each weight $v$ belonging to a directed set $V$ of weights (directed means that, given $v_1, v_2 \in V$, there is some $v \in V$ and $\lambda > 0$ such that $v_i(x) \leq \lambda v(x)$ $(i = 1, 2)$ for all $x \in X$); the space $L$ is then equipped with the topology defined by the directed set of seminorms

$$f \mapsto \| f \|_v = \sup\{v(x)[f(x)] : x \in X\},$$

and it is denoted by $L_{V_{\infty}}$.

Since only the subspace $L(x) = \{f(x) : f \in L\} \subset F_x$ is relevant, we may assume that $L(x) = F_x$ for each $x \in X$.

The cartesian product of the spaces $F_x$ has the structure of a $C(X \mid X)$-module, where $C(X \mid X)$ denotes the ring of all continuous $X$-valued functions on $X$, if we define the product $\phi f$ for each $\phi \in C(X \mid X)$ and each cross-section $f$ by

$$(\phi f)(x) = \phi(x) f(x)$$

for all $x \in X$. If $W \subset L$ is a vector subspace and $B \subset C(X \mid X)$ is a subalgebra, we say that $W$ is a $B$-module, if $BW = \{\phi f : \phi \in B, f \in W\} \subset W$.

We recall that a locally convex space $E$ has the approximation property if the identity map $e$ on $E$ can be approximated, uniformly on every totally bounded set in $E$, by continuous linear maps of finite rank. This is equivalent to say that $E' \otimes E$ is dense in $L_c(E)$, the space $L(E)$ with the topology of uniform convergence on totally bounded sets of $E$. Let $cs(E)$ be the set of all continuous seminorms on $E$. For each seminorm $p \in cs(E)$, let $E_p$ denote the
space $E$ seminormed by $p$. If, for each $p \in cs(E)$, the space $E_p$ has the approximation property, then $E$ has the approximation property.

**Theorem 1.** Suppose that, for each $x \in X$, the space $F_x$ equipped with the topology defined by the family of seminorms $\{v(x); v \in V\}$ has the approximation property. Let $B \subseteq C_b(X; \mathbb{K})$ be a self-adjoint and separating subalgebra. Then any Nachbin space $LV_\infty$ which is a $B$-module has the approximation property.

The idea of the proof is to represent the space $L(W)$, where $W = LV_\infty$, as a Nachbin space of cross-sections over $X$, each fiber being $L(W; F_x)$, and then apply the solution of the Bernstein-Nachbin approximation problem in the separating and self-adjoint bounded case. Before proving theorem 1 let us state some corollaries.

**Corollary 1.** Let $X$ be a Hausdorff space, and for each $x \in X$, let $F_x$ be a normed space with the approximation property. Let $B \subseteq C_b(X; \mathbb{K})$ be a self-adjoint and separating subalgebra.

Let $L$ be a vector space of cross-sections pertaining to $(X; (F_x)_{x \in X})$ such that

1. For every $f \in L$, the map $x \mapsto \|f(x)\|$ is upper semicontinuous and null at infinity;
2. $L$ is a $B$-module;
3. $L(x) = F_x$ for each $x \in X$.

Then $L$ equipped with norm $\|f\| = \sup \{\|f(x)\|; x \in X\}$ has the approximation property.

**Proof.** Consider the weight $v$ on $X$ defined by $v(x) = \text{norm of } F_x$, for each $x \in X$. Then $LV_\infty$ is just $L$ equipped with the norm.
\[ \| f \| = \sup \{ \| f(x) \| ; x \in X \}. \]

**REMARK.** From Corollary 1 it follows that all "continuous sums", in the sense of Godement [6] or [7], of Banach spaces with the approximation property have the approximation property, if the "base space" \( X \) is compact and if such a "continuous sum" is a \( C_b(X; \mathbb{R}) \)-module. In particular, all "continuous sums" of Hilbert spaces and of \( C^* \)-algebras, in the sense of Dixmier and Douady [3] have the approximation property, if \( X \) is compact. Indeed, a "continuous sum" in the sense of [3] is a \( C(X; \mathbb{R}) \)-module.

**COROLLARY 2.** Let \( X \) be a Hausdorff space such that \( C_b(X; \mathbb{R}) \) is separating; let \( V \) be a directed set of real-valued, non-negative, upper semicontinuous functions on \( X \); and let \( E \) be a locally convex space with the approximation property. Then \( CV_\infty(X; E) \) has the approximation property.

**PROOF.** By definition, \( CV_\infty(X; E) = \{ f \in C(X; E); v \cdot f \) vanishes at infinity, for all \( v \in V \} \), equipped with the topology defined by the family of seminorms

\[ \| f \|_{v,p} = \sup \{ v(x) \cdot p(f(x)) ; x \in X \} \]

where \( v \in V \) and \( p \in \text{cs}(E) \).

Let \( L_v \) denote \( CV_\infty(X; E) \) equipped with the topology defined by the above seminorms when \( v \in V \) is kept fixed. Then, for each \( x \in X \), either \( L_v(x) = 0 \) or \( L_v(x) = E \) equipped with the topology defined by the seminorms \( \{ v(x)\cdot p ; p \in \text{cs}(E) \} \). Hence in both cases, \( L_v(x) \) has the approximation property. It remains to notice that all Nachbin spaces are \( C_b(X; \mathbb{R}) \)-modules. Therefore \( L_v \) has the approximation property. Since \( v \in V \) was arbitrary, \( CV_\infty(X; E) \) has the approximation property.

**COROLLARY 3.** Let \( X \) and \( E \) be as in Corollary 2. Then
(a) \( C(X ; E) \) with the compact-open topology has the approximation property.

(b) \( C_0(X ; E) \) with the uniform topology has the approximation property.

REMARK. In (a) above, it is sufficient to assume that \( C(X ; \mathbb{K}) \) is separating.

COROLLARY 4. (Fontenot [4]) Let \( X \) be a locally compact Hausdorff space, and let \( E \) be a locally convex space with the approximation property. Then \( C_b(X ; E) \) with the strict topology \( \beta \) has the approximation property.

PROOF. Apply Corollary 2, with \( V = \{ v \in C_0(X ; \mathbb{R}) ; v \geq 0 \} \).

COROLLARY 5. All Nachbin spaces of continuous scalar-valued functions have the approximation property.

PROOF. In Corollary 2, take \( E = \mathbb{K} \).

§3. PROOF OF THEOREM 1

Let \( W = LV_\infty \) and let \( A \subseteq W \) be a totally bounded set.

Let \( v_0 \in V \) and \( \varepsilon > 0 \) be given.

For each \( T \in \mathcal{L}(W) \) consider the map

\[ \varepsilon_x \circ T : W \longrightarrow F_x \]

for \( x \in X \), where \( \varepsilon_x : W \longrightarrow F_x \) is the evaluation map, i.e.,

\[ \varepsilon_x(f) = f(x), \text{ for all } f \in W. \]

STEP 1. \( \varepsilon_x \circ T \in \mathcal{L}(W ; F_x) \).

PROOF. Just notice that \( \varepsilon_x \in \mathcal{L}(W ; F_x) \), since
\[ v(x) \leq \| f \|_V, \text{ for any } v \in V. \]

For each \( T \in \mathcal{L}(W) \), consider the cross-section \( \hat{T} = (e_x \circ T) \in X \) and for each \( v \in V \) consider the weight \( \hat{\nu} \) on \( X \) defined by

\[ \hat{\nu}(x)[U(x)] = \sup \{ v(x)[(U(x))(f)] : f \in A \} \]

for every \( U(x) \in \mathcal{L}(W ; F_X) \). Then

\[ \hat{\nu}(x)[\hat{T}(x)] = \hat{\nu}(x)[e_x \circ T] = \sup \{ v(x)[(T f)(x)] : f \in A \} \]

for any \( T \in \mathcal{L}(W) \).

**STEP 2.** The map \( x \mapsto \hat{\nu}(x)[\hat{T}(x)] \) is upper semicontinuous and vanishes at infinity on \( X \), for each \( T \in \mathcal{L}(W) \).

**PROOF.** Let \( x_0 \in X \) and assume

\[ \hat{\nu}(x_0)[\hat{T}(x_0)] < h. \]

Choose \( h'' \) and \( h' \) such that

1. \( \hat{\nu}(x_0)[\hat{T}(x_0)] < h' < h'' < h. \)

Let \( \delta = 2(h'' - h') \). Then \( \delta > 0 \). Since \( T(A) \) is totally bounded, there exist \( f_1, f_2, \ldots, f_m \in A \) such that, given \( f \in A \), there is \( i \in \{1,2,\ldots,m\} \) such that

2. \( \| T f - T f_i \|_V < \delta/4 \)

Since \( x \mapsto v(x)[(T f_i)(x)] \) is upper semicontinuous, there are \( V_1, V_2, \ldots, V_m \) neighborhoods of \( x_0 \) such that

3. \( v(x)[(T f_i)(x)] < v(x_0)[(T f_i)(x_0)] + \delta/4 \)
for all $x \in V_i$ ($i = 1, 2, \ldots, m$).

Let $U = V_1 \cap V_2 \cap \ldots \cap V_m$. Then $U$ is a neighborhood of $x_0$ in $X$. Let $x \in U$ and let $f \in A$. Choose $i \in \{1, 2, \ldots, m\}$ such that (2) is true. Then

$$v(x)[(T f)(x)] \leq v(x)[(T f)(x) - (T f_i)(x)] + v(x)[(T f_i)(x)]$$

$$< \|T f - T f_i\|_v + v(x_0)[(T f_i)(x_0)] + \delta/4$$

$$< \delta/2 + v(x_0)[(T f_i)(x_0)]$$

$$= h'' - h' + v(x_0)[(T f_i)(x_0)].$$

On the other hand, by (1), we have

$$v(x_0)[(T f_i)(x_0)] \leq \hat{v}(x_0)[\hat{T}(x_0)] < h'.$$

Hence $v(x)[(T f)(x)] < h''$ for all $f \in A$, and $x \in U$.

Therefore $\hat{v}(x)[\hat{T}(x)] \leq h'' < h$, for all $x \in U$.

Let us now prove that the mapping $x \mapsto \hat{v}(x)[\hat{T}(x)]$ vanishes at infinity.

Let $\delta > 0$ be given and define

$$K_\delta = \{x \in X; \hat{v}(x)[\hat{T}(x)] \geq \delta\}.$$

Since $K_\delta = \emptyset$, if $\sup\{\|T f\|_v; f \in A\} < \delta$, we may assume $\sup\{\|T f\|_v; f \in A\} \geq \delta$.

Since $T(A)$ is totally bounded, there are $f_1, \ldots, f_m \in A$ such that, given $f \in A$, there is $i \in \{1, \ldots, m\}$ such that
(4) \[ \| T f - T f_i \|_v < \delta / 4. \]

Let \( K = \bigcup_{i=1}^{m} \{ t \in X ; v(t)[(T f_i)(t)] > \delta / 2 \}. \)

Then \( K \) is compact, since each of the functions \( x \mapsto v(x)[(T f_i)(x)] \) vanishes at infinity. Let now \( x \in K_{\delta} \) and choose \( f \in A \) such that

(5) \[ v(x)[(T f)(x)] > \frac{3\delta}{4}. \]

Choose \( f_i \in A \) satisfying (4). Then

(6) \[ v(x)[(T f)(x)] < v(x)[(T f_i)(x)] + \delta / 4. \]

Therefore \( \delta / 2 < v(x)[(T f_i)(x)] \) and so \( x \in K_{\delta} \), i.e., \( K_{\delta} \subseteq K \). Since \( K_{\delta} \) is closed, this ends the proof.

The above two steps show that the image \( L = \{ \tilde{T} ; T \in L(W) \} \) of \( L(W) \) under the map \( T \mapsto \tilde{T} \) is a Nachbin space \( L \mathcal{U}_{\infty} \) of cross sections over \( X \), pertaining to the vector fibration \( (X ; (L(W ; F_x))_{x \in X}) \), if we take as family \( \mathcal{U} \) of weights the family \( \mathcal{V} = \{ \tilde{v} ; v \in V \} \)

STEP 3. For every \( T \in L(W) \),

\[ \sup_{f \in A} \| T f \|_v \leq \sup_{x \in X} \tilde{v}(x)[\tilde{T}(x)]. \]

PROOF. Let \( f \in A \). Then

\[ \| T f \|_v = \sup_{x \in X} v(x)[(T f)(x)] \]
\[
\sup_{x \in X} \varphi(x) \left( (\varepsilon_x \circ T)(f) \right) 
\]

\[
= \sup_{x \in X} \hat{\varphi}(x) \left( \hat{T}(x) \right) = \| \hat{T} \|_{\hat{\varphi}}.
\]

Let now \( \mathcal{F}' = \{ \hat{T} : T \in W' \otimes W \} \).

Our aim is to prove that we can find \( T \in W' \otimes W \) such that

\[
\sup_{f \in A} \| T f - f \|_{\varphi_0} < \varepsilon.
\]

Hence, by Step 3, it is enough to prove that

\[
\| \hat{T} - \hat{I} \|_{\hat{\varphi}_0} < \varepsilon,
\]

where \( \hat{I} = (\varepsilon_x)_{x \in X} \).

By the bounded case of the Bernstein-Nachbin approximation problem (Theorem 11, [8], pg. 314) it is enough to prove that

STEP 4. \( \mathcal{F} \) is a B-module.

STEP 5. For each \( x \in X \), \( \mathcal{F}(x) \) is dense in \( \mathcal{L}(W ; F_{x}) \), equipped with the topology defined by the seminorms \( \{ \hat{\varphi}(x) ; \varphi \in \mathcal{V} \} \).

PROOF. To prove that \( \mathcal{F} \) is a B-module, it is enough to prove that

\[
(M_{\phi} \circ T)^{\wedge} = \phi \hat{T}
\]

for all \( \hat{T} \in \mathcal{F} \), i.e., for all \( T \in W' \otimes W \) and for all \( \phi \in B \); and \( M_{\phi} : W + W \) is defined by \( M_{\phi}(f) = \phi f \), for all \( f \in W \).

Now to prove (7), one has to prove that

\[
(M_{\phi} \circ T)^{\wedge}(x) = (\phi \hat{T})(x)
\]

for all \( x \in X \). However,
\((M_\phi \circ T)^\sim (x) = \varepsilon_x \circ (M_\phi \circ T)\), and

\((\varphi^\sim T)(x) = \varphi(x) \; \hat{T}(x) = \varphi(x) (\varepsilon_x \circ T)\).

And, for all \(f \in W\) one has

\[
[\varepsilon_x \circ (M_\phi \circ T)](f) = \varepsilon_x ((M_\phi \circ T)(f))
= \varepsilon_x (\varphi(T \; f))
= \varphi(x) (T \; f)(x)
= \varphi(x) (\varepsilon_x \circ T)(f).
\]

This ends the proof of step 4.

To prove step 5, we first notice that, since each \(F_x\) equipped with the topology defined by \(\{v(x) \; ; \; v \in V\}\) has the approximation property, then \(W' \otimes F_x\) is dense in \(\mathcal{L}_c(W \; ; \; F_x)\), a fortiori in \(\mathcal{L}(W \; ; \; F_x)\) with the topology of the seminorms \(\{\hat{v}(x) \; ; \; v \in V\}\).

Hence, all we have to prove is that \(\mathcal{F}(x)\) contains \(W' \otimes F_{x'}\) for each \(x \in X\).

Let then \(T \in W' \otimes F_x\) be a continuous linear operator of finite rank, say

\[T = \sum_{i=1}^{n} \phi_i \otimes v_i\]

where \(\phi_i \in W'\) and \(v_i \in F_x\). Since \(W(x) = F_x\), choose \(f_i \in W\) such that
\[ f_i(x) = v_i \]

for \( i = 1, 2, \ldots, n. \)

Define \( U = \sum_{i=1}^{n} \phi_i \otimes f_i. \)

Then \( U \in W' \otimes W; \) so \( \hat{U} \in \mathcal{F}. \) Now

\[ \hat{U}(x) = \varepsilon_x \circ (\sum_{i=1}^{n} \phi_i \otimes f_i) \]

and therefore

\[ \hat{U}(x)(f) = \sum_{i=1}^{n} \phi_i(f) f_i(x) = \sum_{i=1}^{n} \phi_i(f) v_i = T(f) \]

for all \( f \in W. \)

REFERENCES


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