ABSTRACT

All Nachbin spaces $CV_\infty(X)$ of continuous scalar-valued functions have the approximation property.

§1. INTRODUCTION

Throughout this paper X is a Hausdorff space such that $C_b(X;\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) separates the points of X, and E is a non-zero locally convex space. Our aim is to prove that certain function spaces $L \subset C(X;E)$ have the approximation property as soon as E has the approximation property. We show this for the class of all Nachbin spaces $CV_\infty(X;E)$. Such spaces include $C(X;E)$ with the compact-open topology; $C_b(X;E)$ with the strict topology; $C_0(X;E)$ with the uniform topology. When $E = \mathbb{K}$, Bierstedt [1], using the technique of ε-products, had proved that $CV_\infty(X;\mathbb{K})$ has the approximation property, under the hypothesis that X is a completely regular $k_{\mathbb{R}}$-space, and that the family V of weights is such that given a compact subset $K \subset X$, one can find a weight $v \in V$ such that $v(x) \geq 1$ for all $x \in K$.

The technique we use here was suggested by the paper [5] of Gierz, who proved the analogue of Theorem 1 below for the case of X compact and bundles of Banach spaces. This technique of "localization" of the approximation property was used by Bierstedt, in the case of the partition by antisymmetric sets (Bierstedt [2]), but the main idea of representing the space of operators of L as another Nachbin space of cross sections is due to Gierz. However our presentation is much simpler, in particular we do not use the concept of a locally $C(X)$-convex $C(X)$-module. In the Introduction to his paper, Gierz said that his method could be applied to the vector fibrations in the sense of [8], and this led to our effort at simplifying his proof and adapting it to our context.

§2. THE APPROXIMATION PROPERTY FOR NACHBIN SPACES.

A *vector fibration* over a Hausdorff topological space X is a pair $(X,(F_x)_{x\in X})$, where each F_x is a vector space over the field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *cross-section* is then any element f of the Cartesian product of the spaces F_x, i.e., $f=(f(x))_{x\in X}$.

A *weight on* X is a function v on X such that v(x) is a seminorm over F_x for each $x \in X$. A *Nachbin space* LV_∞ is a vector space L of cross-sections f such that the mapping

$$x \in X \longmapsto v(x)\,[f(x)]$$

is upper semicontinuous and null at infinity on X for each weight v belonging to a *directed set* V of weights (directed means that, given $v_1, v_2 \in V$, there is some $v \in V$ and $\lambda > 0$ such that $v_i(x) \leq \lambda v(x)$ $(i = 1,2)$ for all $x \in X$); the space L is then equipped with the topology defined by the directed set of seminorms

$$f \longmapsto \| f \|_v = \sup\{v(x)[f(x)] \; ; \; x \in X\} ,$$

and it is denoted by LV_∞.

Since only the subspace $L(x) = \{f(x) \; ; \; f \in L\} \subset F_x$ is relevant, we may assume that $L(x) = F_x$ for each $x \in X$.

The cartesian product of the spaces F_x has the structure of a $C(X ; \mathbb{K})$-module, where $C(X ; \mathbb{K})$ denotes the ring of all continuous $\mathbb{K}$-valued functions on X, if we define the product ϕf for each $\phi \in C(X ; \mathbb{K})$ and each cross-section f by

$$(\phi f)(x) = \phi(x) f(x)$$

for all $x \in X$. If $W \subset L$ is a vector subspace and $B \subset C(X ; \mathbb{K})$ is a subalgebra, we say that W is a B-module, if $BW = \{\phi f ; \phi \in B, f \in W\} \subset W$.

We recall that a locally convex space E has the *approximation property* if the identity map e on E can be approximated, uniformly on every totally bounded set in E, by continuous linear maps of finite rank. This is equivalent to say that $E' \otimes E$ is dense in $\mathcal{L}_c(E)$, the space $\mathcal{L}(E)$ with the topology of uniform convergence on totally bounded sets of E. Let cs(E) be the set of all continuous seminorms on E. For each seminorm $p \in cs(E)$, let E_p denote the space E seminormed by p. If, for each $p \in cs(E)$, the space E_p has the approximation property, then E has the approximation property.

THEOREM 1. *Suppose that, for each $x \in X$, the space F_x equipped with the topology defined by the family of seminorms $\{v(x);\ v \in V\}$ has the approximation property. Let $B \subset C_b(X\ ;\ \mathbb{K})$ be a self-adjoint and separating subalgebra. Then any Nachbin space LV_∞ which is a B-module has the approximation property.*

The idea of the proof is to represent the space $\mathcal{L}(W)$, where $W = LV_\infty$, as a Nachbin space of cross-sections over X, each fiber being $\mathcal{L}(W\ ;\ F_x)$, and then apply the solution of the Bernstein-Nachbin approximation problem in the separating and self-adjoint bounded case. Before proving theorem 1 let us state some corollaries.

COROLLARY 1. *Let X be a Hausdorff space, and for each $x \in X$ let F_x be a normed space with the approximation property. Let $B \subset C_b(X\ ;\ \mathbb{K})$ be a self-adjoint and separating subalgebra.*

Let L be a vector space of cross-sections pertaining to $(X\ ;\ (F_x)_{x \in X})$ such that

(1) for every $f \in L$, the map $x \longmapsto \|f(x)\|$ is upper semicontinuous and null at infinity;

(2) L is a B-module;

(3) $L(x) = F_x$ for each $x \in X$.

Then L equipped with norm $\|f\| = \sup\{\|f(x)\|\ ;\ x \in X\}$ has the approximation property.

PROOF. Consider the weight v on X defined by $v(x)$ = norm of F_x, for each $x \in X$. Then LV_∞ is just L equipped with the norm

$$\| f \| = \sup \{ \| f(x) \| \ ; \ x \in X \}.$$

REMARK. From Corollary 1 it follows that all "continuous sums", in the sense of Godement [6] or [7], of Banach spaces with the approximation property have the approximation property, if the "base space" X is compact and if such a "continuous sum" is a $C_b(X ; \mathbb{K})$-module. In particular, all "continuous sums" of Hilbert spaces and of C*-algebras, in the sense of Dixmier and Douady [3] have the approximation property, if X is compact. Indeed, a "continuous sum" in the sense of [3] is a $C(X ; \mathbb{K})$-module.

COROLLARY 2. *Let* X *be a Hausdorff space such that* $C_b(X ; \mathbb{K})$ *is separating; let* V *be a directed set of real-valued, non-negative, upper semicontinuous functions on* X*; and let* E *be a locally convex space with the approximation property. Then* $CV_\infty(X ; E)$ *has the approximation property.*

PROOF. By definition, $CV_\infty(X;E) = \{f \in C(X ; E);\ v f$ vanishes at infinity, for all $v \in V\}$, equipped with the topology defined by the family of seminorms

$$\| f \|_{v,p} = \sup\{v(x)\ p(f(x)) \ ; \ x \in X\}$$

where $v \in V$ and $p \in cs(E)$.

Let L_v denote $CV_\infty(X ; E)$ equipped with the topology defined by the above seminorms when $v \in V$ is kept fixed. Then, for each $x \in X$, either $L_v(x) = 0$ or $L_v(x) = E$ equipped with the topology defined by the seminorms $\{v(x)p \ ; \ p \in cs(E)\}$. Hence in both cases, $L_v(x)$ has the approximation property. It remains to notice that all Nachbin spaces are $C_b(X ; \mathbb{K})$-modules. Therefore L_v has the approximation property. Since $v \in V$ was arbitrary, $CV_\infty(X ; E)$ has the approximation property.

COROLLARY 3. *Let* X *and* E *be as in Corollary 2. Then*

(a) $C(X\ ;\ E)$ *with the compact-open topology has the approximation property.*

(b) $C_0(X\ ;\ E)$ *with the uniform topology has the approximation property.*

REMARK. In (a) above, it is sufficient to assume that $C(X\ ;\ \mathbb{K})$ is separating.

COROLLARY 4. (Fontenot [4]) *Let* X *be a locally compact Hausdorff space, and let* E *be a locally convex space with the approximation property. Then* $C_b(X\ ;\ E)$ *with the strict topology* β *has the approximation property.*

PROOF. Apply Corollary 2, with $V = \{v \in C_0(X\ ;\ \mathbb{R})\ ;\ v \geq 0\}$.

COROLLARY 5. *All Nachbin spaces of continuous scalar-valued functions have the approximation property.*

PROOF. In Corollary 2, take $E = \mathbb{K}$.

§3. PROOF OF THEOREM 1

Let $W = LV_\infty$ and let $A \subset W$ be a totally bounded set.

Let $v_0 \in V$ and $\varepsilon > 0$ be given.

For each $T \in \mathcal{L}(W)$ consider the map

$$\varepsilon_x \circ T : W \longrightarrow F_x$$

for $x \in X$, where $\varepsilon_x : W \longrightarrow F_x$ is the evaluation map, i.e., $\varepsilon_x(f) = f(x)$, for all $f \in W$.

STEP 1. $\varepsilon_x \circ T \in \mathcal{L}(W\ ;\ F_x)$.

PROOF. Just notice that $\varepsilon_x \in \mathcal{L}(W\ ;\ F_x)$, since

$$v(x)\ [f(x)] \leq \|f\|_v\ , \quad \text{for any} \quad v \in V.$$

For each $T \in \mathcal{L}(W)$, consider the cross-section $\hat{T} = (\varepsilon_x \circ T)_{x \in X}$; and for each $v \in V$ consider the weight $\hat{v}$ on X defined by

$$\hat{v}(x)[U(x)] = \sup\{v(x)[(U(x))(f)]\ ;\ f \in A\}$$

for every $U(x) \in \mathcal{L}(W\ ;\ F_x)$. Then

$$\hat{v}(x)[\hat{T}(x)] = \hat{v}(x)[\varepsilon_x \circ T] = \sup\{v(x)[(T\ f)(x)]\ ;\ f \in A\}$$

for any $T \in \mathcal{L}(W)$.

STEP 2. *The map* $x \longmapsto \hat{v}(x)[\hat{T}(x)]$ *is upper semicontinuous and vanishes at infinity on* X, *for each* $T \in \mathcal{L}(W)$.

PROOF. Let $x_o \in X$ and assume

$$\hat{v}(x_o)[\hat{T}(x_o)] < h\ .$$

Choose h" and h' such that

(1) $\hat{v}(x_o)[\hat{T}(x_o)] < h' < h'' < h.$

Let $\delta = 2(h'' - h')$. Then $\delta > 0$. Since $T(A)$ is totally bounded, there exist $f_1, f_2, \ldots, f_m \in A$ such that, given $f \in A$, there is $i \in \{1,2,\ldots,m\}$ such that

(2) $\|T\ f - T\ f_i\|_v < \delta/4$

Since $x \longmapsto v(x)[(T\ f_i)(x)]$ is upper semicontinuous , there are $V_1, V_2, \ldots, V_m$ neighborhoods of x_o such that

(3) $v(x)[(T\ f_i)(x)] < v(x_o)[(T\ f_i)(x_o)] + \delta/4$

for all $x \in V_i$ $(i = 1,2,\ldots,m)$.

Let $U = V_1 \cap V_2 \cap \ldots \cap V_m$. Then U is a neighborhood of x_o in X. Let $x \in U$ and let $f \in A$. Choose $i \in \{1,2,\ldots,m\}$ such that (2) is true. Then

$$v(x)[(T\,f)(x)] \leq v(x)[(T\,f)(x) - (T\,f_i)(x)] + v(x)[(T\,f_i)(x)]$$

$$< \|T\,f - T\,f_i\|_v + v(x_o)[(T\,f_i)(x_o)] + \delta/4$$

$$< \delta/2 + v(x_o)[(T\,f_i)(x_o)]$$

$$= h'' - h' + v(x_o)[(T\,f_i)(x_o)].$$

On the other hand, by (1), we have

$$v(x_o)[(T\,f_i)(x_o)] \leq \hat{v}(x_o)[\hat{T}(x_o)] < h'.$$

Hence $v(x)[(T\,f)(x)] < h''$ for all $f \in A$, and $x \in U$.

Therefore $\hat{v}(x)[\hat{T}(x)] \leq h'' < h$, for all $x \in U$.

Let us now prove that the mapping $x \longmapsto \hat{v}(x)[\hat{T}(x)]$ vanishes at infinity.

Let $\delta > 0$ be given and define

$$K_\delta = \{x \in X;\ \hat{v}(x)[\hat{T}(x)] \geq \delta\}.$$

Since $K_\delta = \emptyset$, if $\sup\{\|T\,f\|_v\ ;\ f \in A\} < \delta$, we may assume $\sup\{\|T\,f\|_v\ ;\ f \in A\} \geq \delta$.

Since T(A) is totally bounded, there are $f_1,\ldots,f_m \in A$ such that, given $f \in A$, there is $i \in \{1,\ldots,m\}$ such that

(4) $\quad \| T f - T f_i \|_v < \delta/4.$

Let $K = \bigcup_{i=1}^{m} \{t \in X \; ; \; v(t)[(T f_i)(t)] \geq \delta/2\}.$

Then K is compact, since each of the functions $x \longmapsto v(x)[(T f_i)(x)]$ vanishes at infinity. Let now $x \in K_\delta$ and choose $f \in A$ such that

(5) $\quad v(x)[(T f)(x)] > \frac{3\delta}{4}.$

Choose $f_i \in A$ satisfying (4). Then

(6) $\quad v(x)[(T f)(x)] < v(x)[(T f_i)(x)] + \delta/4.$

Therefore $\delta/2 < v(x)[(T f_i)(x)]$ and so $x \in K$, i.e., $K_\delta \subset K$. Since K_δ is closed, this ends the proof.

The above two steps show that the image $\mathcal{L} = \{\hat{T} \; ; \; T \in \mathcal{L}(W)\}$ of $\mathcal{L}(W)$ under the map $T \longmapsto \hat{T}$ is a Nachbin space $\mathcal{L}\mathcal{V}_\infty$ of cross sections over X, pertaining to the vector fibration $(X \; ; \; (\mathcal{L}(W \; ; \; F_x))_{x \in X})$, if we take as family $\mathcal{V}$ of weights the family $\mathcal{V} = \{\hat{v} \; ; \; v \in V\}$

STEP 3. *For every* $T \in \mathcal{L}(W)$,

$$\sup_{f \in A} \| T f \|_v \leq \sup_{x \in X} \hat{v}(x)[\hat{T}(x)].$$

PROOF. Let $f \in A$. Then

$$\| T f \|_v = \sup_{x \in X} v(x)[(T f)(x)]$$

$$= \sup_{x \in X} v(x) \left[(\varepsilon_x \circ T)(f) \right]$$

$$= \sup_{x \in X} \hat{v}(x) \left[\hat{T}(x) \right] = \| \hat{T} \|_{\hat{v}} .$$

Let now $\mathcal{F} = \{\hat{T} \; ; \; T \in W' \otimes W\}$.

Our aim is to prove that we can find $T \in W' \otimes W$ such that

$$\sup_{f \in A} \| T f - f \|_{v_0} < \varepsilon$$

Hence, by Step 3, it is enough to prove that $\| \hat{T} - \hat{I} \|_{\hat{v}_0} < \varepsilon$, where $\hat{I} = (\varepsilon_x)_{x \in X}$.

By the bounded case of the Bernstein-Nachbin approximation problem (Theorem 11, [8], pg. 314) it is enough to prove that

STEP 4. *$\mathcal{F}$ is a B-module.*

STEP 5. *For each $x \in X$, $\mathcal{F}(x)$ is dense in $\mathcal{L}(W \; ; \; F_x)$, equipped with the topology defined by the seminorms $\{\hat{v}(x) \; ; \; v \in V\}$.*

PROOF. To prove that $\mathcal{F}$ is a B-module, it is enough to prove that

$$(7) \qquad (M_\phi \circ T)\hat{} = \phi \, \hat{T}$$

for all $\hat{T} \in \mathcal{F}$, i.e., for all $T \in W' \otimes W$ and for all $\phi \in B$; and $M_\phi : W \to W$ is defined by $M_\phi(f) = \phi f$, for all $f \in W$.

Now to prove (7), one has to prove that

$$(8) \qquad (M_\phi \circ T)\hat{}\,(x) = (\phi \, \hat{T})(x)$$

for all $x \in X$. However,

$$(M_\phi \circ T)^\wedge (x) = \varepsilon_x \circ (M_\phi \circ T), \text{ and}$$

$$(\phi \hat{T})(x) = \phi(x)\, \hat{T}(x)$$

$$= \phi(x)(\varepsilon_x \circ T).$$

And, for all $f \in W$ one has

$$[\varepsilon_x \circ (M_\phi \circ T)](f) = \varepsilon_x((M_\phi \circ T)(f))$$

$$= \varepsilon_x (\phi(T f))$$

$$= \phi(x)(T f)(x)$$

$$= \phi(x)(\varepsilon_x \circ T)(f).$$

This ends the proof of step 4.

To prove step 5, we first notice that, since each F_x equipped with the topology defined by $\{v(x) ; v \in V\}$ has the approximation property, then $W' \otimes F_x$ is dense in $\mathcal{L}_c(W ; F_x)$, a fortiori in $\mathcal{L}(W ; F_x)$ with the topology of the seminorms $\{\hat{v}(x) ; v \in V\}$.

Hence, all we have to prove is that $\mathcal{F}(x)$ contains $W' \otimes F_x$, for each $x \in X$.

Let then $T \in W' \otimes F_x$ be a continuous linear operator of finite rank, say

$$T = \sum_{i=1}^{n} \phi_i \otimes v_i$$

where $\phi_i \in W'$ and $v_i \in F_x$. Since $W(x) = F_x$, choose $f_i \in W$ such that

$$f_i(x) = v_i$$

for $i = 1,2,\ldots,n$.

Define $U = \sum_{i=1}^{n} \phi_i \otimes f_i$.

Then $U \in W' \otimes W$; so $\hat{U} \in \mathcal{F}$. Now

$$\hat{U}(x) = \varepsilon_x \circ \left(\sum_{i=1}^{n} \phi_i \otimes f_i \right)$$

and therefore

$$\hat{U}(x)(f) = \sum_{i=1}^{n} \phi_i(f)\, f_i(x) = \sum_{i=1}^{n} \phi_i(f) v_i = T(f)$$

for all $f \in W$.

REFERENCES

[1] Bierstedt, K-D., Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. I, J. reine angew. Math. 259 (1973), 186-210.

[2] Bierstedt, K-D., The approximation property for weighted function spaces, Bonner Math. Schriften 81 (1975), 3-25.

[3] Dixmier, J., and A. Douady, Champs continus d'espaces hilbertiens et de C*-algèbres, Bull. Soc. Math. France 91 (1963), 227-284.

[4] Fontenot, R.A., Strict topologies for vector-valued functions, Can. J. Math. 26 (1974), 841-853.

[5] Gierz, G., Representation of spaces of compact operators and applications to the approximation property, Preprint Nr. 335, Techn. Hochschule Darmstadt, Feb. 1977.

[6] Godement, R., Théorie générale des sommes continues d'espaces de Banach, C.R. Acad. Sci. Paris 228 (1949), 1321-1323.

[7] Godement, R., Sur la théorie des répresentations unitaires, Ann. of Math. 53 (1951), 68-124.

[8] Nachbin, L., S. Machado, and J. B. Prolla, Weighted approximation, vector fibrations and algebras of operators, J. Math. pures et appl. 50 (1971), 299-323.

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