REPRESENTATION OF ANALYTIC FUNCTIONALS

BY VECTOR MEASURES

by Jorge Mujica

ABSTRACT

Let $\mathcal{H}(K)$ denote the space of holomorphic germs on a compact locally connected subset $K$ of a complex metrizable Schwartz space $E$, and let $\mathcal{P}^m(E)$ denote the space of continuous $m$-homogeneous polynomials on $E$. Given a continuous linear functional $T$ on $\mathcal{H}(K)$, we show the existence of a sequence $(\mu_m)$ of Borel vector measures on $K$, with values in the dual of $\mathcal{P}^m(E)$, such that

$$<f, T> = \sum_{m=0}^{\infty} \int_K \frac{1}{m!} \, \delta^m f(x) \, d\mu_m(x)$$

for all $f$ in $\mathcal{H}(K)$, where $\delta^m f(x) \in \mathcal{P}^m(E)$ denotes the $m$-th differential of $f$ at $x$. 

In this extension is given in Section 3, in Theorem 3.4. In Sections 1 and 2 we set up the
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INTRODUCTION. Let \( K \) be a compact locally connected set in the plane. A. Baernstein has shown in [2, p.31] that given an analytic functional on \( K \), i.e. a continuous linear functional \( T \) on \( \mathcal{H}(K) \), the space of holomorphic germs on \( K \), there exists a sequence \( (\mu_m) \) of complex Borel measures on \( K \) such that

\[
\langle f, T \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{K} f^{(m)}(x) \, dx
\]

for every \( f \in \mathcal{H}(K) \). In this paper we extend Baernstein's result to the case where \( K \) is a compact locally connected subset of a complex metrizable Schwartz space \( E \). In this extension, the complex measures \( \mu_m \) are replaced by vector measures with values in the dual of \( \mathcal{F}(mE) \), the space of continuous \( m \)-homogeneous polynomials on \( E \), and the derivatives \( f^{(m)}(x) \in \mathbb{C} \) are replaced by the differentials \( d^m f(x) \in \mathcal{F}(mE) \). This extension is given in Section 3, in Theorem 3.4. In Sections 1 and 2 we set up the
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for every $f \in \mathcal{H}(K)$. In this paper we extend Baernstein's result to the case where $K$ is a compact locally connected subset of a complex metrizable Schwartz space $E$. In this extension, the complex measures $\mu_m$ are replaced by vector measures with values in the dual of $\mathcal{P}(^mE)$, the space of continuous $m$-homogeneous polynomials on $E$, and the derivatives $f^{(m)}(x) \in \mathcal{C}$ are replaced by the differentials $\Delta^m f(x) \in \mathcal{P}(^mE)$. This extension is given in Section 3, in Theorem 3.4. In Sections 1 and 2 we set up the
necessary machinery. However, the main result in Section 1, namely Theorem 1.5, is also interesting in itself, and we give one further application of it in Section 4. A brief description of each section is the following.

In Section 1 we study the space $\mathcal{C}(X;F)$ of all continuous functions which are defined on a compact Hausdorff space $X$ and with values in a compactly-regular (LB)-space $F = \text{ind lim } F_j$. In Theorem 1.5 we prove that $\mathcal{C}(X;F)$, endowed with the topology of uniform convergence on $X$, is topologically isomorphic to the inductive limit of the Banach spaces $\mathcal{C}(X;F_j)$.

In Section 2 we study certain spaces of sequences of continuous functions. These spaces of sequences are a generalization of those already considered by Baernstein in [2]. Given a compact Hausdorff space $X$ and a sequence of compactly-regular (LB)-spaces $F_m = \text{ind lim } F_{mj}$, we define $S_j$ to be the Banach space of all sequences $\phi = (\phi_m)$ such that, for every $m$, $\phi_m \in \mathcal{C}(X;F_{mj})$ and

$$\|\phi\|_j = \sup_m j^{-m} \|\phi_m\|_{mj} < \infty,$$

and define $\mathcal{S}$ to be the inductive limit of the Banach spaces $S_j$. Then, using Theorem 1.5, we characterize the dual of $\mathcal{S}$ in terms of vector measures.
In Section 3 we give the announced extension of Baernstein's representation of $\mathcal{H}(K)$. To get the desired result, we embed $\mathcal{H}(K)$ as a topological subspace of a suitable space of sequences of continuous functions and use the results of Section 2.

Finally, in Section 4 we consider the following question, raised by Bierstedt and Meise in [4]. Given a Banach space $E$ and a compactly-regular (LB)-space $F = \text{ind lim } F_j$, is it true that the $\varepsilon$-product $E \varepsilon F$ is topologically isomorphic to the inductive limit of the $\varepsilon$-products $E \varepsilon F_j$? As an application of Theorem 1.5 we answer this question affirmatively in the case where $E = C(X)$, $X$ being a compact Hausdorff space.

I would like to thank J.B. Prolla for many useful discussions.

1. SPACES OF CONTINUOUS FUNCTIONS WITH VALUES IN AN INDUCTIVE LIMIT. Throughout this section $F$ denotes an (LB)-space, i.e. the inductive limit of an increasing sequence of Banach spaces $F_j$ such that each inclusion mapping $F_j \hookrightarrow F_{j+1}$ is continuous and $F = \bigcup F_j$. Without loss of generality we may assume that

$$\|x\|_{j+1} \leq \|x\|_j$$

for all $x \in F_j$. We devote this section to study some properties of the space of continuous functions $C(X;F)$, where $X$ denotes a compact Hausdorff space.
The usual topology on $\mathcal{E}(X;F)$ is the topology $\tau_u$ of uniform convergence on $X$, which is defined by the seminorms

$$f \in \mathcal{E}(X;F) \rightarrow \sup_{x \in K} q(f(x)) \in \mathbb{R}$$

where $q$ varies among the continuous seminorms on $F$.

Another natural topology on $\mathcal{E}(X;F)$ is the inductive topology $\tau_i$ with respect to the inclusion mappings $\mathcal{E}(X;F_j) \hookrightarrow \mathcal{E}(X;F)$ . It is clear that $\tau_u \leq \tau_i$ . We will show that whenever $\mathcal{E}(X;F) = \cup \mathcal{E}(X;F_j)$ , both topologies coincide.

First we will give sufficient conditions for $\mathcal{E}(X;F) = \cup \mathcal{E}(X;F_j)$ to happen. We recall the following definitions.

An inductive limit of subspaces $G = \text{ind lim } G_\alpha$ is said to be

(i) regular [7, p.123] if each bounded subset of $G$ is contained and bounded in some $G_\alpha$ ;

(ii) compactly-regular [3, p.100] if each compact subset of $G$ is contained and compact in some $G_\alpha$ ;

(iii) Cauchy-regular [9, Def.1.5] if given a bounded subset $B$ of $G$, there exists $\alpha$ such that $B$ is contained and bounded in $G_\alpha$ , and furthermore, $G$ and $G_\alpha$ induce the same Cauchy nets in $B$.

We remark that the term "Cauchy-regular" coincides,
in the case of (LB)-spaces, with each of the terms "boundedly retractive" and "strongly boundedly retractive", introduced by Bierstedt and Meise in [3, p.100].

The following result is clear.

1.1 PROPOSITION. If the inductive limit \( F = \text{ind lim } F_j \) is compactly-regular, or in particular Cauchy-regular, then \( \mathcal{C}(X;F) = \bigcup \mathcal{C}(X;F_j) \).

To prove that whenever \( \mathcal{C}(X;F) = \bigcup \mathcal{C}(X;F_j) \) the topologies \( \tau_u \) and \( \tau_i \) coincide, we will show that both topologies yield the same dual and the same equicontinuous subsets in the dual. We will use the following result of Singer's [14, p.398, Th.], which characterizes the dual of \( \mathcal{C}(X;G) \) in terms of vector measures, when \( G \) is a Banach space.

1.2 SINGER'S THEOREM. Let \( X \) be a compact Hausdorff space and let \( G \) be a Banach space. Then there is a one-to-one correspondence between the continuous linear functionals \( T \) on \( \mathcal{C}(X;G) \) and the regular Borel measures \( \mu \) on \( X \), with values in \( G' \), of bounded variation. This correspondence is given by the formula
< φ, T > = \int_X φ \, dμ

for all φ ∈ C(X; G), and the total variation ∥μ∥ of μ equals the norm ∥T∥ of T.

1.3 LEMMA. If C(X; F) = \bigcup C(X; F_j), then the topologies τ_u and τ_i on C(X; F) yield the same dual.

PROOF. Since τ_u ≤ τ_i, it is clear that
C'_u(X; F) ⊆ C'_i(X; F). We will show that the opposite inclusion also holds. Let T ∈ C'_i(X; F), let T_j ∈ C'_i(X; F_j) denote the restriction of T to C(X; F_j) and let \mathcal{B} denote the σ-algebra of all Borel subsets of X. By Singer's Theorem 1.2, for each j there exists a unique set function μ_j : \mathcal{B} → F'_j which is countably-additive, regular and of bounded variation, such that

< φ, T_j > = \int_X φ \, dμ_j

for all φ ∈ C(X; F_j). In view of the uniqueness of each μ_j, the following diagram is commutative whenever j < k.
It follows that for each $B \in \mathcal{B}$ the sequence $(\mu_j(B))$ defines an element $\mu(B) \in \text{proj lim } F'_j$. Since $F'$ is topologically isomorphic to $\text{proj lim } F'_j$ for the weak topologies [12, Ch.IV, Prop.4.5], it follows that $\mu : \mathcal{B} \to F'$ is a set function which is countably-additive and regular, and the following diagram is commutative for each $j$.

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\mu} & F' \\
\downarrow & & \downarrow \\
\mu_j & & F'_j
\end{array}
\]

We claim that the vector measure $\mu$ has total variation $\leq 1$ with respect to a certain continuous seminorm $q$ on $F$. To see this we will construct a convex, balanced $0$-neighborhood $W$ in $F$ such that

\[
(*) \quad \sum_{i=1}^{n} | \langle w_i, \mu(B_i) \rangle | \leq 1
\]

for each finite partition $\{ B_1, \ldots, B_n \}$ of $X$ with $B_i \in \mathcal{B}$, and each $\{ w_1, \ldots, w_n \} \subset W$. After constructing such a $W$, the Minkowski functional $q$ of $W$ will do the job.

$\mu_j$ being of bounded variation, there exists a $0$-neighborhood $V_j$ in $F_j$ such that
\[ (*) \sum_{i=1}^{n} | \langle v_i, \nu_j(B_i) \rangle | \leq 2^{-j} \]

for each finite partition \( \{ B_1, \ldots, B_n \} \) of \( X \) with \( B_i \in \mathcal{B} \), and each \( \{ v_1, \ldots, v_n \} \subseteq V_j \). Define \( W_j = V_1 + \ldots + V_j \). Then \( W_j \) is a convex, balanced 0-neighborhood in \( F_j \) and it follows from \((***)\) that

\[ (***) \sum_{i=1}^{n} | \langle w_i, \nu_j(B_i) \rangle | \leq \sum_{r=1}^{j} 2^{-r} \leq 1 \]

for each finite partition \( \{ B_1, \ldots, B_n \} \) of \( X \) with \( B_i \in \mathcal{B} \), and each \( \{ w_1, \ldots, w_n \} \subseteq W_j \). Define \( W = \bigcup W_j \). Since \( W_j \subseteq W_{j+1} \) for every \( j \), it follows that \( W \) is a convex, balanced 0-neighborhood in \( F \), and \((*)\) holds, as asserted.

Since \( \mu \) has total variation \( \leq 1 \) with respect to \( q \), it follows that the integral \( \int_X \phi d\mu \) defines a continuous linear functional \( S \) on \( \mathcal{C}(X;F_q) \) with \( \| S \| \leq 1 \), \( F_q \) being the vector space \( F \), seminormed by \( q \). But clearly \( S \) coincides with \( T_j \) on \( \mathcal{C}(X;F_j) \) for every \( j \). Hence \( S \) coincides with \( T \) on \( \mathcal{C}(X;F) \). Thus

\[ \langle \phi, T \rangle = \int_X \phi d\mu \]

and

\[ \left| \langle \phi, T \rangle \right| \leq \sup_{x \in X} q(f(x)) \]

for all \( \phi \in \mathcal{C}(X;F) \). Thus \( T \) is \( \tau_u \)-continuous, concluding the proof.
An examination of the proof of Lemma 1.3 shows that we can prove even more:

1.4 LEMMA. If \( \mathcal{C}(X;F) = \bigcup \mathcal{C}(X;F_j) \), then a collection of linear forms on \( \mathcal{C}(X;F) \) is \( \tau_u \)-equicontinuous if and only if it is \( \tau_i \)-equicontinuous.

Now we can prove

1.5 THEOREM. If \( \mathcal{C}(X;F) = \bigcup \mathcal{C}(X;F_j) \), or in particular if the inductive limit \( F = \text{ind lim } F_j \) is compactly-regular or Cauchy-regular, then the topologies \( \tau_u \) and \( \tau_i \) coincide.

PROOF. Theorem 1.5 follows from Lemmas 1.3 and 1.4, for any locally convex space has the topology of uniform convergence on the equicontinuous subsets of its dual space [12, Ch.IV, Th.1.5, Cor.3].

1.6 REMARK. If \( \mathcal{C}(X;F) = \bigcup \mathcal{C}(X;F_j) \), then \( \mathcal{C}(X;F) \) will be always endowed with the topology \( \tau_u \), or equivalently \( \tau_i \). We will then identify the continuous linear forms on \( \mathcal{C}(X;F) \) with the \( F' \)-valued regular Borel measures \( \mu \) on \( X \) which are of bounded variation with respect to some continuous seminorm on \( F \), writing indifferently \( \langle \phi, \mu \rangle \) or \( \int \phi \, d\mu \) for any \( \phi \in \mathcal{C}(X;F) \).
We conclude this section with a characterization of the bounded subsets of \( \mathcal{C}(X;F) \).

1.7 PROPOSITION. If the inductive limit 
\[ F = \text{ind lim } F_j \]
is Cauchy-regular, then 
\[ \mathcal{C}(X;F) = \text{ind lim } \mathcal{C}(X;F_j) \]
is a regular inductive limit.

PROOF. According to a result of Grothendieck [8, p.78, Th.9], each bounded subset of \( \mathcal{C}(X;F) \) is contained in the closure of a bounded subset of some \( \mathcal{C}(X;F_j) \). Thus to prove Proposition 1.7 we will show that the \( \mathcal{C}(X;F) \)-closure of the closed unit ball \( \mathcal{B}_j \) of \( \mathcal{C}(X;F_j) \) is contained in the closed unit ball \( \mathcal{B}_k \) of \( \mathcal{C}(X;F_k) \) for some \( k \geq j \).

If \( B_j \) denotes the closed unit ball of \( F_j \), then, \( F = \text{ind lim } F_j \) being Cauchy-regular, there exist \( r \) and \( k \), with \( j \leq r \leq k \), such that \( F \) and \( F_r \) induce the same Cauchy nets in \( B_j \), and \( F \) and \( F_k \) induce the same Cauchy nets in \( B_r \). Let \( \phi \) belong to the \( \mathcal{C}(X;F) \)-closure of \( \mathcal{B}_j \). Then there exists a net \( (\phi) \) in \( \mathcal{B}_j \) which converges to \( \phi \) in \( \mathcal{C}(X;F) \). It follows that \( \phi(x) \) converges to \( \phi(x) \) in \( F \) for each \( x \in X \), and hence that \( \phi(x) \) converges to \( \phi(x) \) in \( F_r \) for each \( x \in X \). Since

\[ \|\phi(x)\|_r \leq \|\phi(x)\|_j \leq \|\phi\|_j \leq 1 \]
for every $x \in X$, it follows that $\|\phi(x)\|_r \leq 1$ for every $x \in X$ and therefore $\phi(X) \subseteq \mathcal{B}_r$. Then $\phi \in C(X;F)$ implies that $\phi \in C(X;F_k)$ and it follows that the $C(X;F)$-closure of $\mathcal{G}_j$ is contained in $\mathcal{G}_k$, concluding the proof.

2. SPACES OF SEQUENCES OF CONTINUOUS FUNCTIONS. Let $X$ be a compact Hausdorff space and let $(F_m)$ be a sequence of (LB)-spaces, $F_m = \text{ind lim } F_{mj}$. Let $\mathcal{S}_j$ be the Banach space of all sequences $\phi = (\phi_m)$ such that, for every $m$, $\phi_m \in C(X;F_{mj})$ and

$$\|\phi\|_j = \sup_m j^{-m} \|\phi_m\|_{mj} < \infty.$$  

Then $\mathcal{S}_j \hookrightarrow \mathcal{S}_{j+1}$ continuously for every $j$. We define $\mathcal{S} = \bigcup \mathcal{S}_j$ and endow $\mathcal{S}$ with the inductive topology with respect to the inclusion mappings $\mathcal{S}_j \hookrightarrow \mathcal{S}$.

We give a lemma on inductive limits whose straightforward proof we omit.

2.1 LEMMA. Let $G = \text{ind lim } G_\alpha$ be an inductive limit of subspaces with $G = \bigcup G_\alpha$. Let $\pi : G \to G$ be a projection such that $\pi(G_\alpha) \subseteq G_\alpha$ and $\pi|_{G_\alpha} : G_\alpha \to G_\alpha$ is continuous for every $\alpha$. Let $M = \pi(G)$ and let $M_\alpha = \pi(G_\alpha)$ for every $\alpha$, endowed with the induced topologies of $G$ and $G_\alpha$, respectively. Then

(a) $M_\alpha = M \cap G_\alpha$, $M = \bigcup M_\alpha$.

(b) The identity mapping $M \to \text{ind lim } M_\alpha$ is a homeomorphism.
2.2 LEMMA. Consider the linear mappings

$$\pi_m : \phi = (\phi_m) \in \mathcal{S} \rightarrow (0, \ldots, 0, \phi_m, 0, \ldots) \in \mathcal{S}$$

$$\sigma_m : \phi \in \mathcal{C}(X; F_m) \rightarrow (0, \ldots, 0, \phi, 0, \ldots) \in \mathcal{S}$$

Then:

(a) $\pi_m$ is a continuous projection.

(b) $\phi = \sum \pi_m(\phi)$ for every $\phi \in \mathcal{S}$.

(c) $\sigma_m$ is a homeomorphism between $\mathcal{C}(X; F_m)$ and $\pi_m(\mathcal{S})$.

PROOF. (a) Clearly $\pi_m(\mathcal{S}_j) \subset \mathcal{S}_j$ and $\pi_m | \mathcal{S}_j$ is a continuous projection on $\mathcal{S}_j$.

(b) If $\phi = (\phi_m) \in \mathcal{S}_j$, then

$$\| \phi - \sum_{m=0}^{N} \pi_m(\phi) \|_{j+1} = \sup_{m \geq N+1} (j+1)^{-m} \| \phi_m \|_{m,j+1}$$

$$\leq \sup_{m \geq N+1} (j/(j+1))^m j^{-m} \| \phi_m \|_{m,j}$$

$$\leq (j/(j+1))^{N+1} \| \phi \|_j$$

Hence, as $N \rightarrow \infty$, $\sum_{m=0}^{N} \pi_m(\phi)$ converges to $\phi$ in $\mathcal{S}_{j+1}$, hence in $\mathcal{S}$.

(c) Certainly $\sigma_m | \mathcal{C}(X; F_m)$ is a homeomorphism between $\mathcal{C}(X; F_m)$ and $\pi_m(\mathcal{S}_j)$, and it follows from Lemma 2.1 that $\sigma_m$ is a homeomorphism between $\mathcal{C}(X; F_m)$ and $\pi_m(\mathcal{S})$.

Next we characterize the dual of $\mathcal{S}$ in terms of vector measures.
2.3 PROPOSITION. If \( \mathcal{G}(X;F_m) = \bigcup \mathcal{G}(X;F_{m_j}) \) for every \( m \), then given \( T \in \mathfrak{S}' \) there exists a unique sequence of vector measures \( \mu_m \) such that

(a) \( \mu_m \in \mathcal{G}'(X;F_m) \);

(b) \( \langle \phi, T \rangle = \sum \int_X \phi \, d\mu_m \) for every \( \phi = (\phi_m) \in \mathfrak{S} \);

(c) if \( \|\mu_m\|_{m_j} \) denotes the norm of \( \mu_m \) as a member of \( \mathcal{G}'(X;F_{m_j}) \), then \( \lim_{m \to \infty} \|\mu_m\|_{m_j}^{1/m} = 0 \) for every \( j \).

Conversely, given a sequence of vector measures \( \mu_m \) satisfying (a) and (c), then (b) defines \( T \in \mathfrak{S}' \).

PROOF. If \( \sigma_m \) is the mapping defined in Lemma 2.2, then we define \( \nu_m \in \mathcal{G}(X;F_m) \) by \( \nu_m = T \sigma_m \). Then by Lemma 2.2 for any \( \phi = (\phi_m) \in \mathfrak{S} \) we get

\[ \langle \phi, T \rangle = \sum \langle \pi_m(\phi), T \rangle = \sum \langle \phi_m, \nu_m \rangle = \sum \int_X \phi \, d\nu_m \]

Fix \( j \). We will show that \( \|\nu_m\|_{m_j}^{1/m} \to 0 \) as \( m \to \infty \).

There exists \( \phi_m \in \mathcal{G}(X;F_{m_j}) \) such that

\[ \|\phi_m\|_{m_j} = 1 \quad \text{and} \quad \langle \phi_m, \nu_m \rangle \geq (1/2) \|\nu_m\|_{m_j} \]

Let \( \phi_k = (k^m \phi_m) \). Then, whenever \( k \geq j \), \( \phi_k \in \mathfrak{S}_k \) and

\[ \langle \phi_k, T \rangle = \sum k^m \langle \phi_m, \nu_m \rangle \geq (1/2) \sum k^m \|\nu_m\|_{m_j} \]

This proves the first part. The second part follows from a similar argument.
Thus \( \sum_{m} k^{m} \| \mu_{m} \|_{mj} < \infty \) for every \( k \geq j \) and therefore
\[
\| \mu_{m} \|_{mj}^{1/m} \to 0 \quad \text{as} \quad m \to \infty.
\]

The uniqueness of the sequence \((\mu_{m})\) is clear.

To prove the converse, let \((\mu_{m})\) be a sequence of vector measures satisfying (a) and (c). Then, given \( \phi = (\phi_{m}) \in S_{j} \), we have:

\[
\sum_{m} |\langle \phi_{m}, \mu_{m} \rangle| \leq \sum_{m} \| \phi_{m} \|_{mj} \| \mu_{m} \|_{mj} \leq \| \phi \| \sum_{m} m^{j} \| \mu_{m} \|_{mj}
\]

and the last written series converges for \( \| \mu_{m} \|_{mj}^{1/m} \to 0 \) as \( m \to \infty \). Thus (b) defines a linear form \( T \) on \( S \) whose restriction to each \( S_{j} \) is continuous.

We conclude this section with a characterization of the bounded subsets of \( S \).

2.4 PROPOSITION. Let the \((LB)\)-spaces \( F_{m} = \text{ind lim } F_{mj} \) have the following property: \( F_{m} \) and \( F_{m,j+1} \) induce the same Cauchy nets in the closed unit ball of \( F_{mj} \) for every \( j \) and every \( m \). Then the inductive limit \( S = \text{ind lim } S_{j} \) is regular.

PROOF. The proof is similar to that of Proposition 1.7. Again by [8, p.78, Th.9] it suffices to show that the \( S \)-closure of the closed unit ball \( B_{j} \) of \( S_{j} \) is contained in the closed unit ball \( B_{j+2} \) of \( S_{j+2} \). Let \( \phi \) belong
to the $S$-closure of $B_j$ and let $(\phi_\alpha)$ be a net in $B_j$ which converges to $\phi$ in $S$. Then $\phi_{am}(x)$ converges to $\phi_m(x)$ in $F_m$ for every $x \in X$ and every $m$. Since 
\[ \|\phi_\alpha\|_j \leq 1 \]
it follows that 
\[ \|\phi_{am}(x)\|_{mj} \leq j^m \]
for every $x \in X$ and every $m$. Therefore, it follows from the hypothesis that $\phi_{am}(x)$ converges to $\phi_m(x)$ in $F_{m,j+1}$ for every $x \in X$ and every $m$. Thus $\phi_m(x) \in F_{m,j+1}$ and 
\[ \|\phi_m(x)\|_{m,j+1} \leq j^m \]
for every $x \in X$ and every $m$. Then we show as in the proof of Proposition 1.7 that 
\[ \phi_m \in \mathcal{C}(X;F_m,j+2) \]
for every $m$. It follows that $\phi$ belongs to $S_{j+2}$, concluding the proof.

2.5 COROLLARY. Let the (LB)-spaces $F_m = \text{ind lim } F_{mj}$ have the following property: $F_m$ and $F_{m,j+1}$ induce the same Cauchy nets in the closed unit ball of $F_{mj}$ for every $j$ and every $m$. Then, given $\mathcal{X} \subseteq S$, the following conditions are equivalent:

(a) $\mathcal{X}$ is bounded in $S$.

(b) $\mathcal{X}$ is contained and bounded in some $S_j$.

(c) There exists $j$ such that, for every $\phi = (\phi_m) \in \mathcal{X}$, 
$\phi_m \in \mathcal{C}(X;F_{mj})$ and $\|\phi_m\|_{mj} \leq j^m$, for every $m$.

3. THE SPACE OF HOLOMORPHIC GERMS. Throughout this section $\mathcal{H}(K)$ denotes the space of all complex-valued
holomorphic germs on a compact subset \( K \) of a complex locally convex space \( E \). \( E \) will be assumed metrizable and Schwartz in our main result, namely Theorem 3.4. We refer to Grothendieck [8, p.117, Def.5] for the definition of Schwartz spaces. \( \mathcal{H}(K) \) is endowed with the inductive topology coming from \( \mathcal{H}(K) = \text{ind lim} \mathcal{H}^\infty(V) \), where \( \mathcal{H}^\infty(V) \) denotes the Banach space of all complex-valued, bounded holomorphic functions on \( V \), \( V \) varying among all open neighborhoods of \( K \). \( \mathcal{P}^{(m)}(E) \) denotes the space of all complex-valued continuous \( m \)-homogeneous polynomials on \( E \). Our main references for \( \mathcal{H}(K) \) and \( \mathcal{P}^{(m)}(E) \) are [9], [5] and [1].

We begin with an intrinsic characterization of the bounded subsets of \( \mathcal{H}(K) \).

3.1 THEOREM. Let \( E \) be metrizable and let \( K \subset E \) be compact and locally connected. Then, for any \( \mathcal{X} \subset \mathcal{H}(K) \), the following conditions are equivalent:

(a) \( \mathcal{X} \) is bounded in \( \mathcal{H}(K) \).

(b) \( \mathcal{X} \) is contained and bounded in \( \mathcal{H}^\infty(V) \), for some open neighborhood \( V \) of \( K \).

(c) There exists a continuous seminorm \( \alpha \) on \( E \) and a constant \( \rho > 0 \) such that

\[
\left\| \frac{1}{m!} \frac{d^m f(x)}{d x^m} \right\|_\alpha \leq \rho^{-m}
\]

for every \( f \in \mathcal{X} \), every \( x \in K \) and every \( m \).
PROOF. The equivalence (a)$\iff$(b) is nothing but
[9, Th.3.1]. That (b) implies (c) follows readily from
the Cauchy inequalities. Thus we only have to prove
that (c) implies (b). The proof we will give is
essentially due to A. Baernstein, who proved that (c)
implies (b) for $E = C$ in [2, p.31].

For each $x \in K$ we choose $U_x \subseteq E$ open such that
$x \in U_x \subseteq B_\alpha(x;\rho)$ and $K \cap U_x$ is connected, where

$$B_\alpha(x;\rho) = \{ t \in E : \alpha(t-x) < \rho \}$$

Next we choose $r_x$ such that $B_\alpha(x;2r_x) \subseteq U_x$ and define

$$V = \bigcup_{x \in K} B_\alpha(x;r_x)$$

Given $f \in \mathcal{C}$ and $x \in K$ we define $f_x$ on $B_\alpha(x;\rho)$ by

$$f_x(t) = \sum_{m} \frac{1}{m!} \alpha^m f(x)(t-x)$$

Clearly $f_x$ is holomorphic on $B_\alpha(x;\rho)$. We claim that
$f_x(t) = f_y(t)$ for every $t \in B_\alpha(x;r_x) \cap B_\alpha(y;r_y)$.

We may assume $r_x \leq r_y$. If the intersection
$B_\alpha(x;r_x) \cap B_\alpha(y;r_y)$ is nonvoid then $\alpha(x-y) \leq 2r_y$,
so that $x \in B_\alpha(y;2r_y) \subseteq U_y$. Thus $x$ and $y$ both belong
to the connected set $K \cap U_y$. Let $U$ be an open
neighborhood of $K$ such that $f$ is holomorphic on $U$. 
Then \( \hat{\kappa} \cap U_y = U \cap B_\alpha(y; \rho) \). If \( W \) is the connected component of \( U \cap B_\alpha(y; \rho) \) which contains \( \hat{\kappa} \cap U_y \), then \( W \) is open and \( x \in W \). Write \( \frac{1}{m!} \hat{a}^m f(y) = B_m \) for every \( m \). Then

\[
f(t) = \sum_{m=0}^{\infty} \hat{B}_m(t-y)
\]

3.2 REMARK. Theorem 3.1 was stated incorrectly by [10, §7, Prop.3].

for every \( t \in W \) and therefore, by [10, §7, Prop.3] we have that

\[
\frac{1}{k!} \hat{d}^k f(x) = \sum_{m=k}^{\infty} \frac{1}{k!} \hat{d}^k \hat{B}_m(x-y)^{m-k}
\]

3.3 REMARK. Theorem 3.1 has already been proved properly the locally connected ones, but his proof is much more involved.

Consequently, for any \( t \in B_\alpha(x; x) \cap B_\alpha(y; y) \), using [10, §5, Prop.1] we get

\[
f_x(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \hat{d}^k f(x)(t-x)^k
\]

We will characterize the dual of \( X_k(t) \) in terms of the vector measures defined by the empty subset of a complete separable Schwartz space. In this situation, using the vector space \( \mathbb{R} \) semi-normed by \( \| \cdot \| \) and \( \| \cdot \|_{\infty} \), we may define a function \( f^* \) holomorphic on \( V \) by

\[
f_y(t) = \sum_{m=0}^{\infty} \hat{B}_m(t-y)^m
\]

proving our claim. Thus we may define a function \( f^* \) holomorphic on \( V \) by

\[
\text{Thus the (l) spaces } \mathbb{R}^{(l)} \text{ and } \mathbb{R}^{(\infty)} \text{ induce the same limits in the closed unit ball of}
\]

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\]
\[ f^*(t) = \sum_{m} \frac{1}{m} \sigma^m f(x)(t-x) \]

for any \( t \in V \) and any \( x \in K \) such that \( t \in B_a(x;r_x) \). Clearly \( f \) is the germ of \( f^* \) on \( K \) and \( |f^*(t)| \leq 2 \) for every \( t \in V \), concluding the proof.

3.2 REMARK. Theorem 3.1 was stated incorrectly by S.B. Chae for an arbitrary compact subset of a Banach space \( E \) [6, p.116, Prop.3.2], but R. Aron found a simple counterexample in \( E = \mathbb{C} \).

3.3 REMARK. Theorem 3.1 has already been proved by R.L. Soraggi [15, Teor.5.1] for a large class of compact sets which includes properly the locally connected ones, but his proof is much more involved.

We will characterize the dual of \( \mathcal{K}(K) \) in terms of vector measures when \( K \) is a compact locally connected subset of a complex metrizable Schwartz space. In this situation, using [1, Lemma 4], we can find an increasing sequence of seminorms \( \sigma_j \) defining the topology of \( E \) such that, letting \( E_j \) denote the vector space \( E \) seminormed by \( \sigma_j \), then \( \sigma_{(nE)} = \text{ind } \lim \sigma_{(nE_j)} \) and \( \sigma_{(nE_{j+1})} \) induce the same Cauchy nets in the closed unit ball of \( \sigma_{(nE_j)} \). Thus the (LB)-spaces \( \sigma_{(nE)} = \text{ind } \lim \sigma_{(nE_j)} \) are all Cauchy-regular, and even the stronger hypothesis in Proposition 2.4 or its Corollary 2.5 is fulfilled.
With this notation we have:

3.4 THEOREM. Let $K$ be a compact locally connected subset of a complex metrizable Schwartz space $E$. Then, given a continuous linear functional $T$ on $H(K)$, there exists a sequence of vector measures $\mu_m$ such that

(a) $\mu_m \in C'(K; \mathcal{F}(^mE))$;

(b) $\langle f, T \rangle = \sum_m \frac{1}{m!} \mathcal{d}^m f \, d\mu_m$ for every $f \in H(K)$;

(c) If $\|\mu_m\|_{mj}$ denotes the norm of $\mu_m$ as a member of $C'(K; \mathcal{F}(^mE_j))$, then, for every $j$, $\lim_{m \to \infty} \|\mu_m\|_{m1}^{1/m} = 0$.

Conversely, given a sequence of vector measures $\mu_m$ satisfying (a) and (c), then (b) defines $T \in H'(K)$.

PROOF. Let $S = \text{ind lim } S_j$ be the space of sequences of continuous functions defined by the compact set $K$ and the (LB)-spaces $\mathcal{F}(^mE) = \text{ind lim } \mathcal{F}(^mE_j)$. We define an injective linear mapping $L : H(K) \to S$ by $L(f) = \phi = (\phi_m)$, where

$$\phi_m = \frac{1}{m!} \mathcal{d}^m f \in C(K; \mathcal{F}(^mE))$$

for every $m$. Using the Cauchy inequalities we see readily that $L$ is well-defined and continuous. We claim that $L$ is also open. To see this observe that:

(i) every closed bounded subset of $H(K)$ is compact: this follows for $H(K)$ is a Silva space,
according to [5, Th.7] or [1, Th.4];

(ii) \( S \) is a (DF)-space: this follows from [8, p.78, Th.9];

(iii) if \( X \) is a bounded subset of \( S \) then \( L^{-1}(X) \) is a bounded subset of \( \mathcal{H}(K) \); this follows from Corollary 2.5 and Theorem 3.1.

Thus \( L \) is open according to Baernstein's open mapping theorem [2, p.29, Lemma], proving our claim.

Thus \( L \) is a topological isomorphism between \( \mathcal{H}(K) \) and a subspace of \( S \), and Theorem 3.4 follows from the Hahn-Banach theorem and Proposition 2.3.

4. INDUCTIVE LIMITS AND THE \( \varepsilon \)-PRODUCT. If \( G \) and \( H \) are locally convex spaces, then \( G \otimes H \) denotes the \( \varepsilon \)-product of \( G \) and \( H \) [13], i.e. the space \( \mathcal{L}_\varepsilon(G';H) \) of all continuous linear mappings of \( G' \) into \( H \), with the topology of uniform convergence on the equicontinuous subsets of \( G' \), where \( G' \) denotes the dual \( G' \) of \( G \), endowed with the topology of uniform convergence on all compact subsets of \( G \) which are convex and balanced.

Let \( E \) be a Banach space and let \( F = \text{ind lim } F_j \) be a compactly-regular (LB)-space. Then each inclusion mapping \( \mathcal{L}_\varepsilon(E'_c;F_j) \hookrightarrow \mathcal{L}_\varepsilon(E'_c;F) \) is continuous, and using [13, p.41, Prop.8] one can prove that \( \mathcal{L}(E'_c;F) = \bigcup \mathcal{L}(E'_c;F_j) \). Thus we may identify \( E \otimes F \) with \( \text{ind lim } E \otimes F_j \) algebraically and the identity mapping
ind lim E_F \rightarrow E_F \text{ is continuous. It is not known whether this mapping is a homeomorphism. } Bierstedt and Meise have shown that this is the case whenever } F \text{ is a nuclear space } [4, \text{ p.209, Bemerkung 13}.] \text{ As an application of Theorem 1.5 we obtain another partial answer to this question.}

4.1 PROPOSITION. If } X \text{ is a compact Hausdorff space and } F = \text{ind lim } F_j \text{ is a compactly-regular (LB)-space, then the identity mapping \text{ind lim } E_F \rightarrow E_F \text{ is a homeomorphism.}

PROOF. By } [11, \text{ Th.8.6}] \text{ the linear mapping}

T \in \mathcal{L}(C_c'(X) ; F_j) \rightarrow f \in C(X ; F_j)

given by

f(x) = T(\delta_x) \quad (x \in X),

where } \delta_x \in C_c'(X) \text{ is the evaluation at } x, \text{ is a topological isomorphism between } C(X)_F \text{ and } C(X;F_j) \text{ for every } j. \text{ Likewise for } C(X)_F \text{ and } C_u(X;F). \text{ And since by Theorem 1.5 the identity mapping ind lim } C(X;F_j) \rightarrow C_u(X;F) \text{ is a homeomorphism, Proposition 4.1 follows.
4.2 REMARK. To conclude that $\mathcal{C}(X)\epsilon F$ is topologically isomorphic to $\mathcal{C}_u(X;F)$ it is required in [11, Th.8.6] that $F$ be quasi-complete. But certainly the closed, convex, balanced hull of each compact subset of $F$ is compact, and [11, Th.8.6] still holds under this weaker hypothesis.
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