An introduction to regularity structures

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These slides can be downloaded from my home page

Plan of the course

In this course I want to discuss mainly the analytical aspects of the theory of regularity structures.

The plan for the four lectures is roughly:

- I. Reconstruction Theorem
- II. Models and modelled distributions
- III. Schauder estimates for germs
- IV. Multilevel Schauder estimates for modelled distributions
- V. Products and equations

Lecture notes and papers in collaboration with F. Caravenna and L. Broux, see my web page.

Chapter 1: The Reconstruction Theorem

A theory, a theorem

This talk is based on a paper (appeared in 2021 in the EMS Surveys in Mathematics)

- *Hairer's Reconstruction Theorem without Regularity Structures*
  by F. Caravenna and L.Z.

In this paper we have extracted a single result (the Reconstruction Theorem) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

A later paper by Pavel Zorin-Kranich, to appear in Revista Matematica Iberoamericana, has introduced introduced further simplifications and improvements to our results.

Prelude: The Sewing Lemma

You have seen in Sebastian's lectures last week that the main tool in rough analysis is the

Theorem (Sewing Lemma)

Let $0 < \alpha \leq 1 < \beta$. *There exists a unique map* $\mathcal{I} : \mathcal{C}_2^{\alpha,\beta}([0,T] : \mathbb{R}^d) \to \mathcal{C}^\alpha([0,T] : \mathbb{R}^d)$ *s.t.*

$$(\mathcal{I}\Xi)_0 = 0, \qquad |\mathcal{I}\Xi_t - \mathcal{I}\Xi_s - \Xi_{s,t}| \lesssim |t-s|^\beta, \qquad s,t \in [0,T].$$

We recall that $\mathcal{C}_2^{\alpha,\beta}$ denotes the space of continuous $\Xi : \{(s,t) : 0 \leq s \leq t \leq T\} \to \mathbb{R}^d$ s.t.

$$\sup_{0 \leq s < t \leq T} \frac{|\Xi_{s,t}|}{|t-s|^\alpha} + \sup_{0 \leq s < u < t \leq T} \frac{|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}|}{|t-s|^\beta} < +\infty.$$

This theorem was proved around 2003 indipendently by Gubinelli and Feyel-de la Pradelle.

It is restricted to functions depending on a one-dimensional parameter.

It took ten years to find a version of this result in higher dimension... This is Martin's Reconstruction Theorem.

Distributions

This talk will concern the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions or generalised functions.

We consider the space $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$.

A distribution on $\mathbb{R}^d$ is a linear functional $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k|\le r} \|\partial^k \varphi\|_\infty, \qquad \forall \varphi \in C_0^\infty(K)$$

where throughout the lectures $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le C\, g$.

When r can be chosen uniformly over K we say that T has order r.

Distributions

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\,\varphi(x)\,\mathrm{d}x, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics, which actually motivated the whole theory of distributions, is the Dirac measure δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \qquad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1+\cdots+k_d}\, T(\partial^k \varphi).$$

Products of distributions

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ then it is possible to define canonically the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \qquad \forall\, \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathcal{D}'(\mathbb{R}^d)$, in general there is no canonical way of defining $T \cdot T'$.

One may use some form of regularisation of T, T' or both. Then, the result could heavily depend on the regularisation and thus be neither unique nor canonical.

For example, one can not define the square $(\delta_x)^2$ of the Dirac function.

The main question of reconstruction

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$. If for all $\psi \in \mathcal{D}$ the map

$$\mathbb{R}^d \ni x \mapsto F_x(\psi)$$

is measurable, then we call $(F_x)_{x\in\mathbb{R}^d}$ a germ.

Problem:
Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is locally well approximated by $(F_x)_{x\in\mathbb{R}^d}$?

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Note that for $j \in \mathbb{N}^d$, $w \in \mathbb{R}^d$, we use the notation

$$|j| := \sum_{k=1}^{d} j_k, \qquad w^j := \prod_{k=1}^{d} w_k^{j_k}, \qquad j! := \prod_{k=1}^{d} j_k!$$

with the convention $0^0 := 1$.

Taylor expansions

For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x) \, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d \, .$$

Then the classical Taylor theorem says that there exists a function $R(x,y)$ such that

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

uniformly for every x, y on compact sets of $\mathbb{R}^d$.

We say that the distribution f is locally well approximated by the germ $(F_x)_{x\in\mathbb{R}^d}$.

Scaling

Let us introduce now the following fundamental tool:

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Then the local approximation property

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

implies for any $\varphi \in \mathcal{D}$, uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

$$\begin{aligned}\left|(f-F_y)(\varphi_y^\lambda)\right| &= \left|\int_{\mathbb{R}^d} R(y,w)\,\varphi_y^\lambda(w)\,\mathrm{d}w\right| \\ &\lesssim \frac{1}{\lambda^d}\int_{B_y(\lambda)} |w-y|^\gamma\,\mathrm{d}w \lesssim \lambda^\gamma\end{aligned}$$

Taylor expansions

Another simple formula in this context is

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y - z| + \lambda)^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0, 1]$.

We call this property coherence, see below.

This comes from a simple estimate of $F_z(w) - F_y(w)$.

Coherence of Taylor expansions

Let us note that we can Taylor expand also the derivatives of f: for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z), \qquad |R^k(y,z)| \lesssim |y-z|^{\gamma-|k|}.$$

Then we can write

$$\begin{aligned} F_y(w) &= \sum_{|k|<\gamma} \partial^k f(y) \, \frac{(w-y)^k}{k!} \\ &= \sum_{|k|<\gamma} \left(\sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k|<\gamma} R^k(y,z) \, \frac{(w-y)^k}{k!}. \end{aligned}$$

Coherence of Taylor expansions

Therefore

$$F_z(w) - F_y(w) = -\sum_{|k|<\gamma} R^k(y,z)\,\frac{(w-y)^k}{k!}.$$

In particular

$$\begin{aligned}|F_z(w) - F_y(w)| &\le \sum_{|k|<\gamma} |R^k(y,z)|\,\frac{|w-y|^k}{k!}\\ &\lesssim \sum_{|k|<\gamma} |y-z|^{\gamma-|k|}\,|w-y|^k\\ &\lesssim (|y-z|+|w-y|)^\gamma\end{aligned}$$

since $a^t b^s \le (a+b)^t(a+b)^s$ for $a,b,t,s \ge 0$.

Coherence of Taylor expansions

Now recall that

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (F_z(w) - F_y(w))\ \varphi_y^\lambda(w)\,\mathrm{d}w\right| &\lesssim \frac{1}{\lambda^d}\int_{B_y(\lambda)} (|y-z| + |w-y|)^\gamma\,\mathrm{d}w \\ &\lesssim (|y-z| + \lambda)^\gamma.\end{aligned}$$

We have obtained for the germ $(F_y)_{y\in\mathbb{R}^d}$ and for any $\varphi \in \mathcal{D}$, $y, z \in \mathbb{R}^d$

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z| + \lambda)^\gamma.$$

Coherence

Let us set from now on

$$\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{N}.$$

In particular for the germ related to a Taylor expansion we have for $\lambda \in \{\varepsilon_n : n \in \mathbb{N}\}$

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim (|y-z| + \varepsilon_n)^\gamma, \qquad \left|(f - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

We say that a germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathcal{D}'$ is (α, γ)-coherent for $\alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$, if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha (|y-z| + \varepsilon_n)^{\gamma-\alpha}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

Hairer's Reconstruction Theorem (without regularity structures)

Theorem (Hairer 14, Caravenna-Z. 20)

Consider a (α,γ)*-coherent germ with* $\gamma > 0$*, namely we suppose that there exists* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *such that* $\int \varphi \neq 0$ *and*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y *in compact sets of* $\mathbb{R}^d$ *and* $n \in \mathbb{N}$ *(coherence condition). Then there exists a unique* $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ *with a fixed* $r > -\alpha$.

Comments

- This result was stated and proved by Martin in [Hai14] for a subclass of germs related to regularity structures. He used wavelets.
- Later Otto-Weber proposed an approach based on a semigroup. This corresponds to a special choice of the test functions φ, ψ.
- Our statement is more general and requires no knowledge of regularity structures.
- This result can be seen as a generalisation of the Sewing Lemma in rough paths (Gubinelli, Feyel-de La Pradelle).
- The construction is completely local: constants and even the exponent α can depend on the compact set.
- We also cover the case $\gamma \leq 0$ (see below).
- Pavel Zorin-Kranich recently showed how to simplify, shorten and (slightly) improve our proof.

Proof for $\gamma > 0$: Uniqueness

Suppose we have two distributions $f, g \in \mathcal{D}'$ which satisfy, uniformly for $x \in K$ for any compact $K \subset \mathbb{R}^d$,

$$\lim_{n\to+\infty} |(f - F_x)(\varphi_x^{\varepsilon_n})| = \lim_{n\to+\infty} |(g - F_x)(\varphi_x^{\varepsilon_n})| = 0\,. \tag{1}$$

We may assume that $c := \int \varphi = 1$ (otherwise just replace φ by $c^{-1}\,\varphi$).

We set $T := f - g$, we fix a test function $\psi \in \mathcal{D}$. We recall the definition of the convolution

$$\psi * \varphi(w) = \int_{\mathbb{R}^d} \psi(y)\,\varphi(w-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \psi(w-y)\,\varphi(y)\,\mathrm{d}y\,,$$

for $w \in \mathbb{R}^d$. This implies

$$T(\psi * \varphi) = \int_{\mathbb{R}^d} \psi(y)\,T(\varphi(\cdot - y))\,\mathrm{d}y = \int_{\mathbb{R}^d} T(\psi(\cdot - y))\,\varphi(y)\,\mathrm{d}y\,. \tag{2}$$

Proof for $\gamma > 0$: Uniqueness

It follows that

$$T(\psi) = \lim_{n\to+\infty} T(\psi * \varphi_0^{\varepsilon_n}).$$

Moreover

$$T(\psi * \varphi_0^{\varepsilon_n}) = \int_{\mathbb{R}^d} T(\varphi_0^{\varepsilon_n}(\cdot - y))\, \psi(y)\, \mathrm{d}y = \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y\,,$$

$$|T(\psi * \varphi_0^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\, \psi(y)\, \mathrm{d}y \right| \leq \|\psi\|_{L^1} \sup_{y\in \mathrm{supp}(\psi)} \left| T(\varphi_y^{\varepsilon_n}) \right| .$$

It remains to show that $\lim_{n\to+\infty} \sup_{y\in\mathrm{supp}(\psi)} \left|T(\varphi_y^{\varepsilon_n})\right| = 0$. Now

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \leq |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

which vanishes as $n \to +\infty$ uniformly for $y \in \mathrm{supp}(\psi)$, by the reconstruction bound (1).

Proof for $\gamma > 0$: Existence

We fix a test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$ which makes the germ F coherent.

We can find in an elementary way a related $\hat{\varphi} \in \mathcal{D}(B(0,1))$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(y)\, \mathrm{d}y = 1\,, \quad \int_{\mathbb{R}^d} y^k\, \hat{\varphi}(y)\, \mathrm{d}y = 0\,, \quad \forall\, k \in \mathbb{N}_0^d :\; 1 \leq |k| \leq r-1\,,$$

for a given $r > -\alpha$. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \qquad \text{and} \qquad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2,$$

where by $\hat{\varphi}^{\frac{1}{2}}, \hat{\varphi}^2$ we mean $\hat{\varphi}^{\lambda}(z) = \lambda^{-d}\hat{\varphi}(\lambda^{-1}z)$ for $\lambda = \frac{1}{2}, 2$, respectively.

This peculiar choice of ρ ensures that the difference $\rho^{\frac{1}{2}} - \rho$ is a convolution:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi}\,.$$

It follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}\,.$$

Proof for $\gamma > 0$: Existence

Finally we define

$$f_n(z) := F_z(\rho_z^{\varepsilon_n}), \qquad f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\,\psi(z)\,\mathrm{d}z\,, \qquad z \in \mathbb{R}^d,\ \psi \in \mathcal{D}\,.$$

Then we want to prove that $f_n(\psi) \to f(\psi)$ and $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$ for all $\psi \in \mathcal{D}$, namely that

$$\mathcal{R}F = \lim_{n\to+\infty} f_n \qquad \text{in } \mathcal{D}'.$$

We study the function

$$f_{x,n}(z) := f_n(z) - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n}), \qquad x, z \in \mathbb{R}^d. \tag{3}$$

Proof for $\gamma > 0$: Existence

We write $f_{x,n}$ as a telescoping sum:

$$f_{x,k+1}(z) - f_{x,k}(z) = (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \tag{4}$$

$$= (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}_z^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y \tag{5}$$

$$= \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y}_{g''_k(z)}\,, \tag{6}$$

where again we use (2). By coherence we have

$$|g''_k(z)| \le \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z|\le\varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}\, \varepsilon_k^{\gamma-\alpha} = \varepsilon_k^{\gamma}\,,$$

$$\left|\int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi(z)\, \mathrm{d}z\right| \le \sup_{y\in\bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\, \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^{\alpha} \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}\,.$$

Proof for $\gamma > 0$: Existence

By the properties of $\check\varphi$ we can write

$$(\check\varphi^\varepsilon * \psi)(y) = \int_{\mathbb{R}^d} \check\varphi^\varepsilon(y-z)\left\{\psi(z) - p_y(z)\right\} \mathrm{d}z\,,$$

where $p_y(z) := \sum_{|k|\le r-1} \frac{\partial^k\psi(y)}{k!}(z-y)^k$ is the Taylor polynomial of ψ of order $r-1$ based at y; since $|\psi(z) - p_y(z)| \lesssim \|\psi\|_{C^r}\,|z-y|^r$, we obtain

$$\|\check\varphi^{\varepsilon_k} * \psi\|_{L^1} \lesssim \int_{\mathbb{R}^d} |\check\varphi^{\varepsilon_k}(y-z)|\,|z-y|^r\,\mathrm{d}z \lesssim \varepsilon_k^r\,.$$

We obtain

$$\left|\int_{\mathbb{R}^d} g'_{x,k}(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\alpha+r}\,, \qquad \left|\int_{\mathbb{R}^d} g''_k(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\gamma}\,.$$

Now we have by assumptions $\gamma > 0$ and $\alpha + r > 0$.

Proof for $\gamma > 0$: Existence

In particular, as $n \to +\infty$,

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]$$

converges to a distribution of order r. Now that $F_x(\rho_\cdot^{\varepsilon_n})$ converges to F_x in $\mathcal{D}'$. We obtain $f_n = f_{x,n} + F_x(\rho_\cdot^{\varepsilon_n})$ converges to a distribution $\mathcal{R}F$ in $\mathcal{D}'$. We also obtain for all ℓ

$$\mathcal{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi) \right] ,$$

and the latter formula yields similarly the reconstruction bound $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$.

The Reconstruction Theorem for $\gamma \leq 0$.

Theorem (Hairer 14, Caravenna-Z. 20)

Let $F : \mathbb{R}^d \to \mathcal{D}'(\mathbb{R}^d)$ *be a* (α, γ)*-coherent germ, with* $\alpha \leq \gamma \leq 0$*, namely there exists a* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$ *s.t.*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha}, \qquad n \in \mathbb{N},\ x, y \in \mathbb{R}^d,$$

(coherence condition). Then there exists a non-unique $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^{\gamma} & \text{if } \gamma < 0 \\ \left(1 + |\log \varepsilon_n|\right) & \text{if } \gamma = 0 \end{cases}.$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ *with a fixed* $r > -\alpha$.

Proof for $\gamma \leq 0$

In the proof with $\gamma > 0$, we wrote, see (6) and (3),

$$f_{x,n} := f_n - F_x(\rho_\cdot^{\varepsilon_n}) = f_{x,0} + \sum_{k=0}^{n-1} \left[g'_{x,k} + g''_k \right], \qquad |g'_{x,n}| \lesssim \varepsilon_n^{\alpha+r}, \qquad |g''_n| \leq \varepsilon_n^{\gamma}.$$

Now we can choose r such that $\alpha + r > 0$, but $\gamma \leq 0$ is fixed.

The solution is to define a different approximation sequence, eliminating the term g''_n whose convergence depends on $\gamma > 0$, and the proof follows with the same estimates. Namely

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g''_k, \qquad \bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} g'_{x,k}(\psi).$$

Then with the same arguments $\bar{f}_n(\psi) \to \bar{f}(\psi)$ and $|(\bar{f} - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$.

Homogeneity

The coherence assumption only concerns $F_z - F_y$, never F_y alone.

Under coherence alone, the reconstruction $\mathcal{R}F$ exists in $\mathcal{D}'$ but we have little more information.

Another crucial notion for germs is homogeneity (with exponent $\bar{\alpha}$)

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(B(0,1))$ with $\|\psi\|_{C^r} \leq 1$, for some fixed $r > -\bar{\alpha}$.

Negative Hölder (Besov) spaces

Given $\bar{\alpha} \in]-\infty, 0[$, we define $\mathcal{C}^{\bar{\alpha}} = \mathcal{C}^{\bar{\alpha}}(\mathbb{R}^d)$ as the space of distributions $T \in \mathcal{D}'$ such that for all $\psi \in \mathcal{D} \setminus \{0\}$

$$\frac{|T(\psi_x^\varepsilon)|}{\|\psi\|_{C^{r_{\bar{\alpha}}}}} \lesssim \varepsilon^{\bar{\alpha}}$$

$$\text{uniformly for } x \text{ in compact sets and } \varepsilon \in (0,1]\,,$$

where we define $r_{\bar{\alpha}}$ as the smallest integer $r \in \mathbb{N}$ such that $r > -\bar{\alpha}$.

Theorem

The reconstruction $\mathcal{R}F$ of a (α, γ)-coherent germ F with homogeneity exponent $\bar{\alpha}$ is in $\mathcal{C}^{\bar{\alpha}}$ (and the map $F \mapsto \mathcal{R}F \in \mathcal{C}^{\bar{\alpha}}$ is linear continuous).

Sewing versus reconstruction

In dimension $d = 1$, the Sewing Lemma and the Reconstruction are almost equivalent.

For a continuous $\Xi : \{(s,t) : 0 \le s \le t \le T\} \to \mathbb{R}$ which vanishes on the diagonal we can define the germ $F_t(\cdot) := \partial_s \Xi_{\cdot,t}$.

Let $z > y > x$ and $\varphi := \mathbb{1}_{(-1,0)}$, so that $\varphi_y^{y-x} = \frac{1}{y-x}\mathbb{1}_{(x,y)}$. Then

$$(F_z - F_y)(\varphi_y^{y-x}) = \frac{1}{y-x}\int_x^y \left(\partial_s \Xi_{s,z} - \partial_s \Xi_{s,y}\right) \mathrm{d}s$$
$$= -\frac{1}{y-x}\left(\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}\right).$$

Then

$$\left|(F_z - F_y)(\varphi_y^{y-x})\right| \lesssim |y-x|^{-1}(|z-y| + |y-x|)^{\beta-1+1} \iff |\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}| \lesssim |z-x|^{\beta}$$

namely $(-1, \beta-1)$-coherence of F is equivalent to $\delta\Xi \in \mathcal{C}_3^{\beta}$.

Sewing versus reconstruction

In particular, we can interpret the conditions

$$\underbrace{\sup_{0\le s<t\le T} \frac{|\Xi_{s,t}|}{|t-s|^{\alpha}} < +\infty}_{\text{homogeneity}} \qquad \underbrace{\sup_{0\le s<u<t\le T} \frac{|\Xi_{s,t}-\Xi_{s,u}-\Xi_{u,t}|}{|t-s|^{\beta}} < +\infty}_{\text{coherence}}.$$

As for reconstruction, also Sewing is possible under mere coherence

- coherence implies existence of $\mathcal{I}\Xi$
- homogeneity implies that $\mathcal{I}\Xi \in \mathcal{C}^{\alpha}$.

Moreover for $\beta \le 1$ we still have a version of the Sewing Lemma, as for Reconstruction with $\gamma = \beta - 1 \le 0$ (see Broux/Z.).

Singular product

Let $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $F_y(w) = \sum_{|k|<\alpha} \partial^k f(y) \frac{(w-y)^k}{k!}$.

Let also $g \in \mathcal{C}^\beta$ with $\beta \leq 0$. We define the germ $P = (P_x := g \cdot F_x)_{x \in \mathbb{R}^d}$, that is

$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi F_x), \qquad \varphi \in \mathcal{D}\,.$$

Theorem

If $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, with $\alpha > 0$ and $\beta \leq 0$, then the germ $P = (P_x)_{x \in \mathbb{R}^d}$ is $(\beta, \alpha + \beta)$-coherent, namely

$$\left|(P_z - P_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\beta (|y - z| + \varepsilon_n)^\alpha.$$

If $\alpha + \beta > 0$, the unique distribution $\mathcal{R}P$ can be used to construct a canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

If $\alpha + \beta \leq 0$, the (non-unique) distribution $\mathcal{R}P$ can be used to construct a non-canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

Recent developments

- *Reconstruction Theorem for Germs of Distributions on Smooth Manifolds*
  by Paolo Rinaldi and Federico Sclavi
- *On a Microlocal Version of Young's Product Theorem*
  by Claudio Dappiaggi, Paolo Rinaldi and Federico Sclavi
- *Besov Reconstruction*
  by Lucas Broux and David Lee
- *Reconstruction theorem in quasinormed spaces*
  by Pavel Zorin-Kranich
- *A stochastic reconstruction theorem*
  by Hannes Kern
- *The Sewing lemma for* $0 < \gamma \leq 1$
  by Lucas Broux and L.Z.

What we did yesterday

We defined the notion of coherent germs: $(F_x)_{x\in\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ such that

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\alpha}(|y-z| + \varepsilon_n)^{\gamma-\alpha},$$

where for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^{\varepsilon_n}(w) := \frac{1}{\varepsilon_n^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Here $\gamma, \alpha \in \mathbb{R}$ and $\alpha \leq \gamma$.

We stated the Reconstruction Theorem: there exists $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$$

(with a log-correction for $\gamma = 0$) and $\mathcal{R}F$ is unique if $\gamma > 0$.

An important special case of reconstruction

Let F be a (α, γ)-coherent germ with $\gamma > 0$.

We know that the (unique) reconstruction $\mathcal{R}F$ satisfies

$$\mathcal{R}F(\psi) = \lim_{n \to +\infty} \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z, \qquad \forall\, \psi \in \mathcal{D}.$$

Let us suppose now that $(x, y) \mapsto F_x(y)$ is continuous.

Then by dominated convergence we obtain

$$\mathcal{R}F(\psi) = \int_{\mathbb{R}^d} F_z(z)\, \psi(z)\, \mathrm{d}z, \qquad \forall\, \psi \in \mathcal{D},$$

namely the reconstruction $\mathcal{R}F$ is equal to the function $z \mapsto F_z(z)$.

This includes the Taylor polynomial example where $F_x(x) = f(x)$.

Non-uniqueness for $\gamma \le 0$

Let F be a (α,γ)-coherent germ with $\alpha \le \gamma < 0$.

Suppose that $T \in \mathcal{D}'$ is a reconstruction of F, namely

$$|(T - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

uniformly for x in compact sets etc.

Then for any $D \in \mathcal{C}^\gamma$, the distribution $T + D$ is also a reconstruction of F.

Viceversa, if T' is a reconstruction of F, then

$$|(T - T')(\psi_x^{\varepsilon_n})| \le |(T - F_x)(\psi_x^{\varepsilon_n})| + |(T' - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

so that $T - T' \in \mathcal{C}^\gamma$.

Therefore, for $\gamma < 0$, the reconstruction of F is unique up to an element of $\mathcal{C}^\gamma$.

Again on singular products

Let us go back to the singular product between $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $g \in \mathcal{C}^\beta$ with $\beta \leq 0$.

We defined a germ P which is $(\alpha, \alpha + \beta)$-coherent.

If $\alpha + \beta > 0$ then the product $f\, g = \mathcal{R}P$ is canonical (we can call it the Young product).

If $\alpha + \beta < 0$ then the reconstruction $\mathcal{R}P$ is unique up to an element of $\mathcal{C}^{\alpha+\beta}$.

Chapter 2: Models and modelled distributions

More on germs

The reconstruction theorem can be applied to coherent germs, which form a large (vector) space.

However this space is too large. When we want to solve SPDEs, we are going to use a much smaller space to set up a fixed point.

We are going to study germs which can be written as suitable linear combinations of a fixed finite family of germs.

An example in one-dimension

You saw in Theorem 55 of Riedel3.pdf that given

- $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$
- $\mathbf{X} = (X, \mathbb{X})$ a α-rough path
- $(Y, Y') \in \mathcal{D}_X^\alpha([0,T])$ a controlled path

then setting

$$\Xi_{u,v} := Y_u\, \delta X_{u,v} + Y'_u\, \mathbb{X}_{u,v}$$

one obtains $\delta\Xi \in C_3^{3\alpha}$ and one can apply the Sewing Lemma to define the rough integral

$$I_t = \int_0^t Y_u\, \mathrm{d}\mathbf{X}_u,$$

which is the unique continuous function $I : [0,T] \to \mathbb{R}$ s.t.

$$I_0 = 0, \quad |I_t - I_s - \Xi_{s,t}| \lesssim |t-s|^{3\alpha}.$$

For the reconstruction theorem, we want analogs of $\mathbf{X}$ and Y to build coherent germs.

Pre-models

Definition

A *pre-model* is a pair (Π, Γ) s.t.

1. $\Pi = (\Pi^i)_{i \in I}$ is a family of germs $\Pi^i = (\Pi^i_x)_{x \in \mathbb{R}^d}$ labelled by a finite index set I,
2. $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto (\Gamma^{ij}_{xy})_{i,j \in I}$ is a matrix-valued function such that

$$\Pi^j_y = \sum_{i \in I} \Pi^i_x \, \Gamma^{ij}_{xy} \,, \qquad j \in I, \; x, y \in \mathbb{R}^d,$$

3. there exist $(\alpha_i)_{i \in I} \subset \mathbb{R}$ and a $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ such that

$$|\Pi^i_x(\varphi^{\varepsilon_n}_x)| \lesssim \varepsilon_n^{\alpha_i},$$

   uniformly over x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$.

We denote $\bar{\alpha} := \min(\alpha_i, i \in I)$.

An example

For a fixed $\gamma > 0$, the family of classical monomials

$$\Pi^j_y(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y, w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \leq \gamma\},$$

with $\alpha_i = |i|$, any $\varphi \in \mathcal{D}$ and

$$\Gamma^{ij}_{xy} = \mathbb{1}_{(i \leq j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i, j \in I,$$

forms a pre-model.

Modelled distributions

Definition

Let (Π, Γ) be a pre-model, and let $\gamma > \max(\alpha_i, i \in I)$.
If $f : \mathbb{R}^d \to \mathbb{R}^I$ is measurable and satisfies for all $i \in I$

$$\left| f_x^i \right| \lesssim 1, \qquad \left| f_x^i - \sum_{j \in I} \Gamma_{xy}^{ij} f_y^j \right| \lesssim |x - y|^{\gamma - \alpha_i},$$

uniformly for x, y in compact subsets of $\mathbb{R}^d$, then we call f a *distribution modelled* by (Π, Γ), or simply a *modelled distribution*, and we write $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$.

Given a pre-model (Π, Γ) and a modelled distribution $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$, we define the germ

$$\langle \Pi, f \rangle_x := \sum_{i \in I} \Pi_x^i f_x^i, \qquad x \in \mathbb{R}^d.$$

Coherence of $\langle \Pi, f \rangle$

We want to show that $\langle \Pi, f \rangle$ is $(\bar\alpha, \gamma)$-coherent, where $\bar\alpha := \min(\alpha_i, i \in I)$. Using the reexpansion property $\Pi^j_z = \sum_{i\in I} \Pi^i_y \, \Gamma^{ij}_{yz}$ we have

$$\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y = \sum_{j\in I} \Pi^j_z f^j_z - \sum_{i\in I} \Pi^i_y f^i_y = -\sum_{i\in I} \Pi^i_y \left(f^i_y - \sum_{j\in I} \Gamma^{ij}_{yz} f^j_z \right).$$

Therefore

$$(\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi^\varepsilon_y) = -\sum_{i\in I} \Pi^i_y(\varphi^\varepsilon_y) \left(f^i_y - \sum_{j\in I} \Gamma^{ij}_{yz} f^j_z \right),$$

namely

$$\left| (\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi^\varepsilon_y) \right| \lesssim \sum_{i\in I} \varepsilon^{\alpha_i} |z-y|^{\gamma-\alpha_i} \lesssim \varepsilon^{\bar\alpha} (\varepsilon + |z-y|)^{\gamma-\bar\alpha},$$

uniformly for y, z in compact sets.

Homogeneity of $\langle \Pi, f \rangle$

Moreover

$$\left|\langle \Pi, f\rangle_y(\varphi_y^\varepsilon)\right| \le \sum_{i\in I} f_y^i \, |\Pi_y^i(\varphi_y^\varepsilon)| \lesssim \sum_{i\in I} \varepsilon^{\alpha_i} \lesssim \varepsilon^{\bar\alpha},$$

uniformly over y in compact subsets of $\mathbb{R}^d$. In other words we have proved that

Theorem

If (Π, Γ) is a pre-model and $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$, then $\langle \Pi, f\rangle$ is a $(\bar\alpha, \gamma)$-coherent germs with uniform homogeneity bound with exponent $\bar\alpha$.

Note that here $\alpha = \bar\alpha$.

Hölder functions as modelled distributions

We have see that the classical polynomial family

$$\Pi_y^i(w) = \frac{(w-y)^i}{i!}, \quad \Gamma_{xy}^{ij} = \mathbb{1}_{(i\le j)}\,\frac{(x-y)^{j-i}}{(j-i)!}, \quad i,j\in\mathbb{N}^d,$$

forms a pre-model. It is an interesting exercise to check that modelled distributions with respect to this pre-model are actually classical Hölder functions.

Let us consider for simplicity the case $\gamma \notin \mathbb{N}$. Now, a modelled distribution $f \in \mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ satisfies by definition

$$\left| f_x^i - \sum_{j\ge i,\,|j|<\gamma} \frac{(x-y)^{j-i}}{(j-i)!}\, f_y^j \right| \lesssim |x-y|^{\gamma-|i|}, \qquad \forall\,|i|<\gamma\,.$$

This is in fact a Taylor expansion of f^i at order $\lfloor \gamma - |i| \rfloor$ with a remainder of order $\gamma - |i|$, and this implies that f^i is of class $C^{\gamma-|i|}$ and

$$f^j = \partial_{j-i} f^i, \qquad \forall\, j \ge i.$$

Hölder functions as modelled distributions

In particular, for $i = 0$ we see that f^0 is of class C^γ and satisfies

$$f^0(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

Then f^0 is a reconstruction of $\langle \Pi, f \rangle$, and since $\gamma > 0$ it is the unique reconstruction. In other words we have seen that

$$f^0 = \mathcal{R}\langle \Pi, f \rangle \in C^\gamma, \qquad f^i = \partial_i f^0, \quad \forall |i| < \gamma.$$

The fact that f^0 is the reconstruction of $\langle \Pi, f \rangle$ is also a consequence of $\mathcal{R}\langle \Pi, f \rangle = \{x \mapsto \langle \Pi, f \rangle_x(x)\} = \{x \mapsto f_x^0\}$.

Semi-norms

Back to the general case, for a fixed pre-model (Π, Γ) we can interpret, by analogy with the case of Hölder functions of the previous section, the space $\mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ of all distributions modelled by (Π, Γ) as the collection of *generalised derivatives* of $u := \mathcal{R}\langle \Pi, f\rangle$ with respect to the pre-model (Π, Γ).

We can define a system of seminorms for $f \in \mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$

$$[f]_{\mathcal{D}^{\gamma}_{(\Pi,\Gamma)},K} = \sup_{i\in I} \sup_{x,y\in K,\, x\neq y} \frac{\left| f^i_x - \sum_{j\in I} \Gamma^{ij}_{xy} f^j_y \right|}{|x-y|^{\gamma-\alpha_i}},$$

where K is a compact subset of $\mathbb{R}^d$.

This is rather original with respect to the standard situation in ODEs or PDEs, where one sets an equation in a fixed Banach space. Here the Banach (Fréchet) space depends on an external parameter, the pre-model (Π, Γ). For SDEs and SPDEs, the pre-model (or rough path) (Π, Γ) is actually *random*.

Models

Definition

A *model* is a pre-model (Π, Γ), such that moreover

1. $\Gamma^{ii}_{xy} = 1$ for all $i \in I$,
2. $\Gamma^{ij}_{xy} = 0$ if $\alpha_i \geq \alpha_j$ and $i \neq j$,
3. $|\Gamma^{ij}_{xy}| \lesssim |x-y|^{\alpha_j - \alpha_i}$ if $\alpha_i < \alpha_j$.

For a fixed $\gamma > 0$, the family of classical monomials

$$\Pi^j_y(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y, w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \leq \gamma\},$$

$$\Gamma^{ij}_{xy} = \mathbb{1}_{(i \leq j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i, j \in I,$$

with $\alpha_i = |i|$, forms a model.

Lemma

Let (Π,Γ) *be a model. Fix an exponent* $\gamma > \max(\alpha_i : i \in I)$ *and set* $\bar\alpha := \min(\alpha_i : i \in I)$*. Then*

1. *The space* $\mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ *is not reduced to the null vector.*
2. *For any* $\gamma' > \bar\alpha$*, the restricted family* $(\Pi',\Gamma') := (\Pi^i,\Gamma^{ij})_{i,j\in I'}$ *labelled by* $I' := \{i \in I : \alpha_i < \gamma'\}$ *is a model. If* $\gamma > \gamma'$*, the projection*

$$f = (f^i)_{i\in I} \mapsto f' = (f^i)_{i\in I'}$$

maps $\mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ *to* $\mathcal{D}^{\gamma'}_{(\Pi',\Gamma')}$.

Proof.

For the first assertion, we consider an element Π^i_x of minimal homogeneity $\bar\alpha = \min_I \alpha$. In this case we see that $\Gamma^{ij}_{xy} = \delta_{ij}$ for all $j \in I$, where δ is the Kronecker symbol, and the function $f^j_x = \delta_{ij}$ is a modelled distribution. $\square$

Go back to page 80.

Chapter 3: The Schauder estimates for germs

A theory, a theorem

This lecture and the next are based on work with L. Broux and F. Caravenna (see the Lecture Notes and a forthcoming paper). We discuss one of the most important operations on coherent germs: the convolution with a regularising integration kernel.

The temptative title for this paper is

- *Hairer's multilevel Schauder estimates without Regularity Structures*

In this paper we have extracted a single result (the multilevel Schauder estimates) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

Definition (Regularising kernel)

Fix a dimension $d \in \mathbb{N}$ and an exponent $\beta > 0$. A measurable function $\mathsf{K} : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is called a *β-regularizing kernel up to degree $m \in \mathbb{N}$* if the following conditions hold:

- the function $x \mapsto \mathsf{K}(x)$ is of class C^m on $\mathbb{R}^d \setminus \{0\}$;
- the following upper bound holds:

$$\forall\, k \in \mathbb{N}^d \;\text{ with }\; |k| \leq m : \qquad |\partial^k \mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta+|k|}} \mathbb{1}_{\{|x|\leq 1\}} \qquad (7)$$

$$\text{uniformly for } x \text{ in compact sets}\,.$$

By the way, let us introduce the notations

$$\mathcal{G}^{\alpha,\gamma} := \{(H_x)_{x\in\mathbb{R}^d} \,:\, H \text{ is } (\alpha,\gamma)\text{-coherent}\}$$
$$\mathcal{G}^{\bar{\alpha};\alpha,\gamma} := \{(H_x)_{x\in\mathbb{R}^d} \in \mathcal{G}^{\alpha,\gamma} \,:\, H \text{ has homogeneity bound with exponent } \bar{\alpha}\}$$

Classical Schauder Estimates

Theorem

Let $\gamma \in \mathbb{R}$ *and* $\beta > 0$.
Let K *be a* β*-regularising kernel up to degree* $m > \gamma + \beta$.
Suppose that $\gamma \neq 0$ *and* $\gamma + \beta \notin \mathbb{N}$.
Then, the convolution by K *defines a continuous linear map from* $\mathcal{C}^\gamma$ *to* $\mathcal{C}^{\gamma+\beta}$.

We want to lift this result to coherent germs, in a way which is compatible with the reconstruction.

Convolution with coherent germs

Fix two real numbers α, γ such that

$$\alpha \leq \gamma, \qquad \gamma \neq 0.$$

We define r_α as the smallest integer larger than $-\alpha$, namely

$$r_\alpha := \min\{k \in \mathbb{N} : \ k > -\alpha\}\,.$$

Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a (α, γ)-coherent germ. We now want to *lift the convolution with* K *on the space of coherent germs*, i.e. to find a coherent germ $H = (H_x)_{x \in \mathbb{R}^d}$ with the property

$$\mathcal{R}H = \mathsf{K} * \mathcal{R}F\,.$$

A simple solution is the constant germ $H_x \equiv \mathsf{K} * \mathcal{R}F$, which is trivially coherent, but this does not allow to construct a fixed-point theory for PDEs.

Convolution with coherent germs

The naive guess $H_x = \mathsf{K} * F_x$ needs not give a coherent germ, therefore we need to enrich it. To this purpose, we look for H_x of the following special form:

$$\forall x \in \mathbb{R}^d : \qquad H_x = \mathsf{K} * F_x + R_x \qquad \text{where } R_x(\cdot) \text{ is a } \textit{polynomial}\,.$$

Remarkably, this is possible with the following explicit solution:

$$H_x := \mathsf{K} * F_x + \underbrace{\sum_{|\ell|<\gamma+\beta} (\mathcal{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell}_{R_x(\cdot)},$$

where we denote for $x \in \mathbb{R}^d$, $\ell \in \mathbb{N}^d$ the classical monomials

$$\mathbb{X}_x^\ell : \mathbb{R}^d \to \mathbb{R}, \qquad \mathbb{X}_x^\ell(w) := \frac{(w-x)^\ell}{\ell!}$$

and where we agree that

$$R_x(\cdot) \equiv 0 \qquad \text{if} \qquad \gamma + \beta \leq 0.$$

Schauder estimates on coherent germs

Theorem

Fix $\alpha, \gamma, \beta \in \mathbb{R}$ *such that*

$$\alpha \le \gamma, \qquad \gamma \ne 0, \qquad \beta > 0\,,$$

where we further assume for simplicity that $\{\alpha + \beta,\ \gamma + \beta\} \cap \mathbb{N} = \emptyset$. *Consider*

- $F = (F_x)_{x \in \mathbb{R}^d} \in \mathcal{G}^{\alpha,\gamma}$ *is a* (α, γ)*-coherent germ;*
- K *is a* β*-regularizing kernel up to degree* $m > \gamma + \beta + r_\alpha$.

Then

1. *the germ* $H = (H_x)_{x \in \mathbb{R}^d}$ *is well-defined.*
2. H *is* $((\alpha + \beta) \wedge 0, \gamma + \beta)$*-coherent, namely* $H \in \mathcal{G}^{(\alpha+\beta)\wedge 0, \gamma+\beta}$.
3. H *satisfies* $\mathcal{R}H = \mathsf{K} * \mathcal{R}F$.

Schauder estimates on coherent germs

In other words, setting $\mathcal{K}F := H$, with

$$H_x := \mathsf{K} * F_x + \sum_{|\ell|<\gamma+\beta} (\mathcal{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell \,,$$

we have a well-defined linear operator satisfying

$$\mathcal{K} : \mathcal{G}^{\alpha,\gamma} \to \mathcal{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathcal{R} \circ \mathcal{K} = \mathsf{K} * \mathcal{R}.$$

Let us define the new germ

$$J_x := F_x - \mathcal{R}F \,,$$

which allows to rewrite H as

$$\begin{aligned} H_x &= \mathsf{K} * F_x - \sum_{|\ell|<\gamma+\beta} J_x \left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell \\ &= \mathsf{K} * \mathcal{R}F + L_x, \qquad \text{where} \qquad L_x := \mathsf{K} * J_x - \sum_{|\ell|<\gamma+\beta} J_x \left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell \,. \end{aligned}$$

Sketch of the proof

The proof is based on two steps:

- L is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent,
- L has homogeneity bound with exponent $\gamma+\beta$.

In other words we show that $L \in \mathcal{G}^{\gamma+\beta;(\alpha+\beta)\wedge 0,\gamma+\beta}$.

(Recall that we did not assume homogeneity of F. Indeed, $H_x = \mathsf{K} * \mathcal{R}F + L_x$ is not homogeneous either, in general.)

Then 0 is a $(\gamma+\beta)$-reconstruction of L, i.e. $\mathsf{K} * \mathcal{R}F$ is a $(\gamma+\beta)$-reconstruction of H, namely

$$\mathcal{R} \circ \mathcal{K} = \mathsf{K} * \mathcal{R}.$$

Chapter 4: A digression

SDEs

Let B be a Brownian motion in $\mathbb{R}^d$, $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ sufficiently smooth, $Y_0 \in \mathbb{R}^k$.

The Itô integration theory gives well-posedness of the SDE

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\, \mathrm{d}B_s, \qquad t \geq 0.$$

More generally, for a class of processes $(X_t)_{t \geq 0}$ (semimartingales) and $(h_t)_{t \geq 0}$ (predictable locally bounded...) one has an integration theory

$$(h, X) \mapsto \int_0^t h_s\, \mathrm{d}X_s, \qquad t \geq 0$$

as a (local) martingale.

Itô's integration theory

This marvelous theory is based on measurability of the maps $X \mapsto (\int_0^t h_s \, \mathrm{d}X_s)_{t \geq 0}$ and $B \mapsto Y$.

In general the question of continuity of these maps is not even mentioned.

In fact, within the Itô theory such continuity fails.

Terry Lyons introduced rough paths with the aim of filling this gap.

An example

You saw in Theorem 55 of Riedel3.pdf that given

- $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$
- $\mathbf{X} = (X, \mathbb{X})$ a α-rough path
- $(Y, Y') \in \mathcal{D}_X^{\alpha}([0,T])$ a controlled path

then setting

$$\Xi_{u,v} := Y_u \, \delta X_{u,v} + Y'_u \, \mathbb{X}_{u,v}$$

one obtains $\delta\Xi \in C_3^{3\alpha}$ and one can apply the Sewing Lemma to define the rough integral

$$I_t = \int_0^t Y_u \, \mathrm{d}\mathbf{X}_u,$$

which is the unique continuous function $I : [0,T] \to \mathbb{R}$ s.t.

$$I_0 = 0, \quad |I_t - I_s - \Xi_{s,t}| \lesssim |t-s|^{3\alpha}.$$

For the reconstruction theorem, we want analogs of $\mathbf{X}$ and Y to build coherent germs.

Rough integration

With respect to the classical situation, we are replacing X with $\mathbf{X} = (X, \mathbb{X})$ and Y with (Y, Y').

Then one of the results of the theory is that the map

$$\left(\mathbf{X}, (Y, Y')\right) \mapsto \left(\int_0^t Y_u \, \mathrm{d}\mathbf{X}_u\right)_{t \geq 0}$$

is indeed continuous (with respect to natural distances).

This is related to the fact that the product

$$Y_u \; \mathrm{d}X_u$$

is ill-defined as a distribution if $\alpha \leq \frac{1}{2}$. Since $\frac{1}{2} - \frac{1}{2} \leq 0$, this is the setting where the reconstruction theorem gives a non-unique result.

Rough integration

By the way, I had left this point suspended. Let us define

$$A_{st} := Y_s\,(X_t - X_s).$$

We expect that an integral $(\int_0^t Y_u\,\mathrm{d}X_u)_{t\geq 0}$ should satisfy

$$\int_s^t Y_u\,\mathrm{d}X_u - Y_s\,(X_t - X_s) = \int_s^t (Y_u - Y_s)\,\mathrm{d}X_u = O(|t-s|^{2\alpha}).$$

Suppose we have one such integral $(I_t)_{t\geq 0}$, is it unique?

Let $g \in \mathcal{C}^{2\alpha}$ and set $\bar{I} := I + g$. Then

$$\bar{I}_t - \bar{I}_s - Y_s\,(X_t - X_s) = I_t - t_s - Y_s\,(X_t - X_s) + g_t - g_s = O(|t-s|^{2\alpha}).$$

Viceversa, we obtain that for any pair $(I, \bar{I})$ of such integrals $I - \bar{I} \in \mathcal{C}^{2\alpha}$.

Itô versus Stratonovich

A prominent example is given by stochastic integrals: given a Brownian motion $(B_t)_{t\geq 0}$ in $\mathbb{R}$, we can define the Itô integral

$$\int_0^t B_s \, \mathrm{d}B_s = \frac{1}{2}\left(B_t^2 - t\right)$$

or the Stratonovich integral

$$\int_0^t B_s \circ \, \mathrm{d}B_s = \frac{1}{2} B_t^2 .$$

The difference between the two of them is $g_t = -\frac{1}{2}t$ which is clearly $\mathcal{C}^{2\alpha}$ for all $\alpha \leq \frac{1}{2}$.

Controlled ODEs

We are interested also in studying solutions $Y : [0, T] \to \mathbb{R}^k$ to an ordinary differential equation *controlled* by a smooth function $X : [0, T] \to \mathbb{R}^d$

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)\, \dot{X}_s \, \mathrm{d}s, \tag{8}$$

where $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is sufficiently smooth. Let us rewrite (8), for $s < t$,

$$\begin{aligned} Y_t - Y_s &= \int_s^t \dot{Y}_r \, \mathrm{d}r = \int_s^t \sigma(Y_r)\, \dot{X}_r \, \mathrm{d}r = \\ &= \sigma(Y_s)(X_t - X_s) + \int_s^t (\sigma(Y_r) - \sigma(Y_s))\, \dot{X}_r \, \mathrm{d}r \\ &= \sigma(Y_s)(X_t - X_s) + R_{st}. \end{aligned}$$

If σ is at least continuous, then by uniform continuity of $r \mapsto \sigma(Y_r)$ we can see that

$$R_{st} = o(t - s).$$

Controlled ODEs

Suppose now that $X : [0, T] \to \mathbb{R}^d$ is of class C^α. We would like to give an analog of the controlled equation (8). For that, we define

$$\delta X_{st} := X_t - X_s, \qquad |\delta X_{st}| \lesssim |t-s|^\alpha, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Taking inspiration from the previous slide we set the following

Definition

Let $\alpha > 1/2$ and $X \in C^\alpha([0,T];\mathbb{R}^d)$. A solution to (8) is a $\mathrm{y} \in C^\alpha([0,T];\mathbb{R}^k)$ such that

$$\mathrm{y}^2_{st} := \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\,\delta X_{st}, \qquad |\mathrm{y}^2_{st}| \lesssim |t-s|^\zeta, \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{9}$$

namely $\mathrm{y}^2 \in C^\zeta_2$, for some $\zeta > 1$.

Theorem

Let $\alpha \in \left(\frac{1}{2}, 1\right)$. Then for every $X \in C^\alpha([0,T])$ and $\mathrm{y}_0 \in \mathbb{R}^d$ there exists a unique $\mathrm{y} : [0,T] \to \mathbb{R}^k$ satisfying (9). Moreover the map $(\mathrm{y}_0, X) \mapsto \mathrm{y}$ is continuous.

Controlled ODEs

If $\alpha \in \left]\frac{1}{3}, \frac{1}{2}\right]$, then we have to modify the argument. We suppose for the moment that $X \in C^1([0,T];\mathbb{R}^d)$. We rewrite, for $s < t$,

$$\begin{aligned}
Y_t - Y_s &= \int_s^t \dot{Y}_r \,\mathrm{d}r \\
&= \int_s^t \sigma(Y_r)\dot{X}_r \,\mathrm{d}r \\
&= \int_s^t \left(\sigma(Y_s) + \int_s^r \frac{\mathrm{d}}{\mathrm{d}v}(\sigma(Y_v))\,\mathrm{d}v\right)\dot{X}_r \,\mathrm{d}r \\
&= \sigma(Y_s)(X_t - X_s) + \int_s^t \left(\int_s^r \nabla\sigma(Y_v)\sigma(Y_v)\dot{X}_v \,\mathrm{d}v\right)\dot{X}_r \,\mathrm{d}r.
\end{aligned}$$

Controlled ODEs

We next expand, for $s < r$,

$$
\begin{aligned}
&\int_s^r \nabla\sigma(Y_v)\sigma(Y_v)\dot{X}_v \,\mathrm{d}v = \\
&= \int_s^r \left(\nabla\sigma(Y_s)\sigma(Y_s) + \int_s^v \frac{\mathrm{d}}{\mathrm{d}w}(\nabla\sigma(Y_w)\sigma(Y_w)) \,\mathrm{d}w \right) \dot{X}_v \,\mathrm{d}v \\
&= \nabla\sigma(Y_s)\sigma(Y_s)(X_r - X_s) + \int_s^r O(|v-s|)\dot{X}_v \,\mathrm{d}v \\
&= \nabla\sigma(Y_s)\sigma(Y_s)(X_r - X_s) + O(|r-s|^2).
\end{aligned}
$$

Controlled ODEs

Hence

$$
\begin{aligned}
Y_t - Y_s &= \\
&= \sigma(Y_s)(X_t - X_s) + \int_s^t \nabla\sigma(Y_s)\sigma(Y_s)(X_r - X_s) \otimes \dot{X}_r \, \mathrm{d}r + \int_s^t O(|r-s|^2)\dot{X}_r \, \mathrm{d}r \\
&= \sigma(Y_s)(X_t - X_s) + \sigma_2(Y_s) \int_s^t (X_r - X_s) \otimes \dot{X}_r \, \mathrm{d}r + O(|t-s|^3),
\end{aligned}
$$

where, for $x, y \in \mathbb{R}^d$, we define $x \otimes y \in \mathbb{R}^{d\times d}$ by

$$x \otimes y := (x_i y_j)_{1 \leqslant i,j \leqslant d} \,,$$

and where we introduce the notation $\sigma_2 : \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d \otimes \mathbb{R}^d$

$$\sigma_2(y) := \nabla\sigma(y)\sigma(y).$$

Controlled ODEs

Here we introduce the notations $\mathbb{X}^1 : [0,T]^2_{\leqslant} \to \mathbb{R}^d$, $\mathbb{X}^2 : [0,T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$

$$\mathbb{X}^1_{st} := X_t - X_s, \qquad \mathbb{X}^2_{st} := \int_s^t (X_r - X_s) \otimes \dot{X}_r \, \mathrm{d}r, \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{10}$$

We note now that for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \int_u^t (X_u - X_s) \otimes \dot{X}_r \, \mathrm{d}r = (X_u - X_s) \otimes (X_t - X_u) = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}.$$

Moreover

$$|\mathbb{X}^1_{st}| \lesssim |t-s|, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^2. \tag{11}$$

The controlled equation (29) can be rewritten therefore

$$Y_t - Y_s \quad = \sigma(Y_s)\mathbb{X}^1_{st} + \sigma_2(Y_s)\mathbb{X}^2_{st} + O(|t-s|^3), \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{12}$$

Controlled ODEs

Suppose now that $X : [0, T] \to \mathbb{R}^d$ is of class C^α with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. We define again

$$\mathbb{X}^1_{st} := X_t - X_s, \qquad |\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha,$$

but the definition of $\mathbb{X}^2$ in (10) does not make sense anymore.

It is possible to construct a robust theory for the controlled equation (29) with X of class C^α with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, provided we *choose* a function $\mathbb{X}^2 : [0, T]^2_{\leqslant} \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut}, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}.$$

Definition

Let $\alpha \in (1/3, 1/2]$, $d \in \mathbb{N}$ and $X \in C^\alpha([0,T];\mathbb{R}^d)$. A d-dimensional α-rough path over X is a pair $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ with $\mathbb{X}^1 : [0,T]^2_\leqslant \to \mathbb{R}^d$, $\mathbb{X}^2 : [0,T]^2_\leqslant \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfying for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

$$\mathbb{X}^1_{st} := X_t - X_s, \qquad \mathbb{X}^2_{st} - \mathbb{X}^2_{su} - \mathbb{X}^2_{ut} = \mathbb{X}^1_{su} \otimes \mathbb{X}^1_{ut},$$
$$|\mathbb{X}^1_{st}| \lesssim |t-s|^\alpha, \qquad |\mathbb{X}^2_{st}| \lesssim |t-s|^{2\alpha}.$$

Definition

Let $\alpha > 1/3$ and $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ a α-rough path. A solution to (8) is a $\mathrm{y} \in \mathcal{C}^\alpha([0,T];\mathbb{R}^k)$ such that for some $\zeta > 1$

$$|\mathrm{y}^3_{st}| \lesssim |t-s|^\zeta, \qquad \mathrm{y}^3_{st} := \delta \mathrm{y}_{st} - \sigma(\mathrm{y}_s)\mathbb{X}^1_{st} - \sigma_2(\mathrm{y}_s)\mathbb{X}^2_{st}. \tag{13}$$

Theorem

Let $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

- *For every* α*-rough path* $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ *and* $\mathrm{y}_0 \in \mathbb{R}^d$ *there exists a unique* $\mathrm{y} : [0, T] \to \mathbb{R}^d$ *satisfying* (13)*. Moreover the map* $(\mathrm{y}_0, \mathbb{X}) \mapsto \mathrm{y}$ *is continuous.*
- *Let* B *be a Brownian motion in* $\mathbb{R}^d$*. Define*

$$\mathbb{B}^1_{st} := B_t - B_s, \qquad \mathbb{B}^2_{st} := \int_s^t (B_r - B_s) \otimes \mathrm{d}B_r \quad \textit{(Itô integral).}$$

Then $\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2)$ *is a.s. a* α*-rough path and the corresponding solution to* (13) *is a.s. equal to the unique solution to the SDE*

$$\mathrm{y}_t = \mathrm{y}_0 + \int_0^t \sigma(\mathrm{y}_s)\, \mathrm{d}B_s.$$

Controlled PDEs

Regularity structures extend this approach to (stochastic) PDEs. For example for a (temporarily) smooth ξ on $\mathbb{R}^2$

$$-\Delta u = (\alpha + \beta u)\xi,$$

which we write in mild formulation

$$u = G * ((\alpha + \beta u)\xi) .$$

Lucas explained that this equation is ill-defined (which is worse than ill-posed) if $\xi \in \mathcal{C}^{-1-\kappa}$ with $\kappa > 0$, the a.s. regularity of white noise on $\mathbb{R}^d$.

Can we play the same game as for controlled SDEs?

Controlled PDEs

$$
\begin{aligned}
u &= u(x) + G * ((\alpha + \beta u)\xi) - u(x) \\
&= u(x) + G * ((\alpha + \beta u)\xi) - G * ((\alpha + \beta u)\xi)\,(x) \\
&= u(x) + G * ((\alpha + \beta(u(x) + u - u(x)))\xi) - G * ((\alpha + \beta(u(x) + u - u(x))\xi)\,(x) \\
&= u(x) + (\alpha + \beta u(x))\,(G * \xi - G * \xi(x)) + \\
&\qquad + \underbrace{\beta\, G * ((u - u(x))\xi)}_{f_x} - \underbrace{\beta\, G * ((u - u(x))\xi)\,(x)}_{f_x(x)}
\end{aligned}
$$

Controlled PDEs

In Lucas' lectures we saw that the set of indices for the model is here $I = \{\mathbf{1}, X_1, X_2, \bullet, \overset{\bullet}{|}, \overset{\bullet}{\underset{\bullet}{|}}, X_1\bullet, X_2\bullet\}$ with respective homogeneities $\{0, 1, 1, -1-\kappa, 1-\kappa, -2\kappa, -\kappa, -\kappa\}$.

$$u = u(x) + (\alpha + \beta u(x)) \underbrace{(G * \xi - G * \xi(x))}_{\Pi_x \overset{\bullet}{|}} + \underbrace{f_x - f_x(x)}_{R_x}$$

$$= u(x)\, \Pi_x \mathbf{1} + (\alpha + \beta u(x))\, \Pi_x \overset{\bullet}{|} + R_x.$$

Note that

$$R_x(y) = f_x(y) - f_x(x), \qquad f_x(y) := \beta\, G * ((u - u(x))\xi)\,(y).$$

Controlled PDEs

Let us set now $\Pi_x X_i(y) := y_i - x_i$, $i = 1, 2$. Then we want to continue the expansion

$$u = u(x)\,\Pi_x \mathbf{1} + (\alpha + \beta u(x))\,\Pi_x \mathsf{I} + C_1 \Pi_x X_1 + C_2 \Pi_x X_2 + \overline{R}_x$$

where $\overline{R}_x = R_x - C_1 \Pi_x X_1 - C_2 \Pi_x X_2$. We want $\overline{R}_x$ to be small around x. Since

$$\overline{R}_x(y) = f_x(y) - f_x(x) - C_1(y_1 - x_1) - C_2(y_2 - x_2),$$

in order to make this smaller than $|y - x|^{1+\kappa}$, we are forced to choose $C_i = \partial_i f_x(x)$ so that

$$\overline{R}_x(y) = f_x(y) - f_x(x) - \sum_{i=1}^{2} \partial_i f_x(x)\,(y_i - x_i).$$

Controlled PDEs

Finally

$$u = u(x)\,\Pi_x\mathbf{1} + (\alpha + \beta u(x))\,\Pi_x\upharpoonleft + \sum_{i=1}^{2} \partial_i f_x(x)\,\Pi_x X_i + \overline{R}_x.$$

Here we expect to have a model (Π, Γ) (the one discussed by Lucas) such that, setting

$$U(x) := u(x)\mathbf{1} + (\alpha + \beta u(x))\upharpoonleft + \sum_{i=1}^{2} \partial_i f_x(x)\,X_i$$

then $U \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$ for a $\kappa > 0$ small.

This U is such that $\langle \Pi, U\rangle_x(x) = u(x)$, namely $\mathcal{R}\langle \Pi, U\rangle = u$. Recall page 41.

Now we have

$$\mathcal{R}\langle \Pi, U\rangle = u = G * ((\alpha + \beta u)\xi).$$

Can we find a modelled distribution whose reconstruction gives the right-hand side as well?

Controlled PDEs

Let us recall that such that

$$\Pi_x \bullet = \xi, \quad \Pi_x \overset{\bullet}{\underset{\bullet}{|}} = (G * \xi - G * \xi(x))\, \xi, \quad \Pi_x(\bullet X_i) = \xi(\cdot_i - x_i).$$

Then we expect that (note the product rule $\bullet \overset{\bullet}{|} = \overset{\bullet}{\underset{\bullet}{|}}$)

$$\bullet(\alpha + \beta U)(x) := (\alpha + \beta u(x)) \bullet + \beta(\alpha + \beta u(x)) \overset{\bullet}{\underset{\bullet}{|}} + \beta\, \partial_i G * ((u - u(x))\xi)(x) \bullet X_i$$

defines a modelled distribution in $\mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$ with $\kappa > 0$ small and

$$\mathcal{R}\langle \Pi, V \rangle = (\alpha + \beta u)\xi.$$

Then we have

$$G * ((\alpha + \beta u)\xi) = G * \mathcal{R}\langle \Pi, \bullet(\alpha + \beta U)\rangle = \mathcal{R} \circ \mathcal{K}\langle \Pi, \bullet(\alpha + \beta U)\rangle$$

where $\mathcal{K}$ is the convolution operator on coherent germs we discussed on Tuesday.

Controlled PDEs

We also expect now to have another operator $\overline{\mathcal{K}} : \mathcal{D}^{\kappa}_{(\Pi,\Gamma)} \to \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$ such that

$$\mathcal{K}\langle \Pi, \bullet(\alpha + \beta U)\rangle = \langle \Pi, \overline{\mathcal{K}}\bullet(\alpha + \beta U)\rangle.$$

This operator exists (it is one of the topics of today's lecture), and is equal in our case to

$$\overline{\mathcal{K}}\bullet(\alpha + \beta U)(x) = \boldsymbol{G} * ((\alpha + \beta u)\xi)(x)\,\mathbf{1} + (\alpha + \beta u(x))\,\rotatebox[origin=c]{90}{$\multimap$} + \sum_{i=1}^{2} \beta\,\partial_i \boldsymbol{G} * ((u - u(x))\xi)(x)\,X_i.$$

Controlled PDEs

We would like to find u as the fixed point of a map

$$v \mapsto G * ((\alpha + \beta v)\xi)\,,$$

in such a way that this fixed point is a continuous functional of the noise ξ in a distributional norm.

This is in fact impossible. However we can lift the equation to a space of modelled distributions. Following the previous slide, we write for a $V \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$ of the form

$$V(x) = v(x)\mathbf{1} + V^{\overset{\bullet}{|}}(x)\,\overset{\bullet}{|} + \sum_{i=1}^{2} V^{X_i}(x)\,X_i, \qquad v = \mathcal{R}V,$$

we write

$$\bullet V(x) = v(x)\bullet + V^{\overset{\bullet}{|}}(x)\,\overset{\bullet}{\underset{\bullet}{|}} + \sum_{i=1}^{2} V^{X_i}(x)\bullet X_i.$$

Controlled PDEs

Then we expect that $\bullet V \in \mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$ and also $\bullet(\alpha + \beta V) \in \mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$.

We apply therefore $\overline{\mathcal{K}}\bullet(\alpha + \beta V) \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$, which has the form

$$\overline{\mathcal{K}}\bullet(\alpha + \beta V)(x) = G * ((\alpha + \beta v)\xi)(x)\,\mathbf{1} + (\alpha + \beta v(x))\,\stackrel{\bullet}{|} + \sum_{i=1}^{2} \beta\, \partial_i G * ((v - v(x))\xi)(x)\, X_i$$

Now the fixed point equation $U = \overline{\mathcal{K}}\bullet(\alpha + \beta U)$ is equivalent to the system

$$u = G * ((\alpha + \beta u)\xi), \qquad U^{\stackrel{\bullet}{|}} = (\alpha + \beta u(x)), \qquad U^{X_i}(x) = \beta\, \partial_i G * ((u - u(x))\xi)(x),$$

which shows that the equation is the same. What changes is the topology: the $\mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$-norm turns out to make the map $V \mapsto \overline{\mathcal{K}}\bullet(\alpha + \beta V)$ a contraction (for $\beta > 0$ small enough).

Theorem

The map $(\Pi, \Gamma) \mapsto u$ *is continuous.*

The topology of (Π, Γ) has to be made clear, but can be understood as the topology of each component Π^i_x in $\mathcal{C}^{\alpha_i}$.

The topology of u is $\mathcal{C}^\eta$ for some $\eta \in \mathbb{R}$.

Chapter 5: The Schauder estimates for modelled distributions

The operator $\overline{\mathcal{K}}$

Given a model (Π, Γ) and a β-regularising kernel K. We need an additional assumption

$$\forall\, i \in I, \quad \text{if } \alpha_i + \beta \in \mathbb{N} \text{ then } \Pi^i_x \left(\partial^k_x \mathsf{K}(x - \cdot)\right) = 0 \quad \forall\, k \in \mathbb{N}^d \text{ with } |k| = \alpha_i + \beta,\ x \in \mathbb{R}^d.$$

Theorem

If $\gamma \neq 0$ *and* $\gamma + \beta \notin \mathbb{N}$, *there exists a model* $(\overline{\Pi}, \overline{\Gamma})$ *and a linear continuous operator*

$$\overline{\mathcal{K}} : \mathcal{D}^{\gamma}_{(\Pi,\Gamma)} \to \mathcal{D}^{\gamma+\beta}_{(\overline{\Pi},\overline{\Gamma})}$$

such that

$$\mathsf{K} * \mathcal{R}\langle \Pi, V\rangle = \mathcal{R}\langle \overline{\Pi}, \overline{\mathcal{K}} V\rangle, \qquad \forall\, V \in \mathcal{D}^{\gamma}_{(\Pi,\Gamma)}.$$

Moreover $\mathcal{K}\langle \Pi, V\rangle = \langle \overline{\Pi}, \overline{\mathcal{K}} V\rangle$, where $\mathcal{K}$ is the convolution operator acting on coherent germs.

Fixed point

In fact in the application to the SPDE I wrote $\overline{\mathcal{K}} : \mathcal{D}^{\gamma}_{(\Pi,\Gamma)} \to \mathcal{D}^{\gamma+\beta}_{(\Pi,\Gamma)}$ instead of $\overline{\mathcal{K}} : \mathcal{D}^{\gamma}_{(\Pi,\Gamma)} \to \mathcal{D}^{\gamma+\beta}_{(\overline{\Pi},\overline{\Gamma})}$ because I needed a map acting on the same space.

In practice we prepare recursively the model in such a way that a truncation of $(\overline{\Pi}, \overline{\Gamma})$ is contained in (Π, Γ).

For example in the classical case of Hölder functions $\Delta^{-1} : \mathcal{C}^{\gamma} \to \mathcal{C}^{\gamma+2}$ (for $\gamma \notin \mathbb{N}$) and in this case

$$(\Pi_x)_{i\in I} = \left((\cdot - x)^k\right)_{k\in\mathbb{N}^d, |k|<\gamma}, \qquad (\overline{\Pi}_x)_{i\in I} = \left((\cdot - x)^k\right)_{k\in\mathbb{N}^d, |k|<\gamma+2}.$$

Example

Let us discuss again

$$u = G * ((\alpha + \beta u)\xi)\,.$$

Here we start from $\{\bullet\}$. Then at the fist iteration we obtain

$$\{\overset{\bullet}{|}, \mathbf{1}\}, \qquad \Pi_x \overset{\bullet}{|}(y) = G * \xi(y) - G * \xi(x),$$

so that our model should contain $\{\bullet, \overset{\bullet}{|}, \mathbf{1}\}$.

However in this way we do not recover $\overset{\bullet}{\underset{\bullet}{|}} = \bullet \overset{\bullet}{|}$. For that the product has to come into play:

$$\bullet \left(v(x)\mathbf{1} + V^{\overset{\bullet}{|}}(x)\, \overset{\bullet}{|} \right) = v(x) \bullet + V^{\overset{\bullet}{|}}(x)\, \overset{\bullet}{\underset{\bullet}{|}}.$$

Then we want to have $\{\bullet, \overset{\bullet}{|}, \overset{\bullet}{\underset{\bullet}{|}}, \mathbf{1}\}$ in our model.

Example

Now since the homogeneity of $\underset{\bullet}{\dot{\mid}}$ is -2κ, then the next integration symbol would have homogeneity $2-2\kappa > 1+2\kappa$ and therefore we do not consider it.

On the other hand we want to include $\{X_i\}_{i=1,2}$ in the model since we want to work in $\mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$. But then since

$$\bullet\left(v(x)\mathbf{1} + V^{\dot{\mid}}(x)\,\dot{\mid} + \sum_{i=1}^{2} V^{X_i}(x)\,X_i \right) = v(x)\bullet + V^{\dot{\mid}}(x)\,\underset{\bullet}{\dot{\mid}} + \sum_{i=1}^{2} V^{X_i}(x)\bullet X_i.$$

we have to include $\{\bullet X_i\}_{i=1,2}$ as well.

Example

The recursive construction defines uniquely all Π_x's but one: $\Pi_x \mathbf{I}$.

You saw yesterday with Lucas that setting ξ equal to a white noise on $\mathbb{R}^d$ and $\xi_\varepsilon := \rho_\varepsilon * \xi$ where $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers, the definition

$$\Pi_x^\varepsilon \mathbf{I} := (G * \xi_\varepsilon - G * \xi_\varepsilon(x))\, \xi_\varepsilon$$

does not give a convergent family as $\varepsilon \to 0$.

On the other hand, if we define $\hat\Pi_x^\varepsilon \tau = \Pi_x^\varepsilon \tau$ for $\tau \neq \mathbf{I}$ and

$$\hat\Pi_x^\varepsilon \mathbf{I} := (G * \xi_\varepsilon - G * \xi_\varepsilon(x))\, \xi_\varepsilon - C_\varepsilon$$

with $C_\epsilon = -\frac{1}{2\pi} \log |\epsilon| + O(1)$, then $\hat\Pi_x^\varepsilon \mathbf{I}$ converges in probability to a distribution $\hat\Pi_x \mathbf{I}$ as $\varepsilon \to 0$ and this allows to define a model $(\hat\Pi, \hat\Gamma)$.

Let me recall you this statement.

Theorem

Let (Π, Γ) be a model as in the example, and $U \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$ be defined by

$$U = \overline{\mathcal{K}}\boldsymbol{\cdot}(\alpha + \beta U).$$

Then the map $(\Pi, \Gamma) \mapsto u$ is continuous.

The topology of (Π, Γ) has to be made clear, but can be understood as the topology of each component Π^i_x in $\mathcal{C}^{\alpha_i}$.

In fact when we let a model vary, we have to specify in what class it is allowed to vary.

This is where regularity structures play a role.

According to what we saw above, a (somewhat provocative) definition in a simplified setting could be

Definition

A regularity structure is a finite set $(\alpha_i)_{i\in I}$ of real numbers.

But then one has to impose some conditions on the model.

Regularity structures

A more accurate definition (in a reasonably simplified setting) would be

Definition

A regularity structure is a pair (A, G) where

- $A = (\alpha_i)_{i \in I}$ is a finite set of real numbers
- G is a subgroup of the linear automorphisms of $\mathbb{R}^I$ s.t. for all $\Gamma \in G$

$$\Gamma^{ii} = 1, \qquad \Gamma^{ij} = 0 \quad \text{if} \quad \alpha_i \geq \alpha_j \quad \text{and} \quad i \neq j.$$

Models

Then the following definition is in order

Definition

A model of a given regularity structure (A, G) is a pair (Π, Γ) s.t.

1. $\Pi = (\Pi^i)_{i \in I}$ is a family of germs s.t. for any integer $r > -\min(\alpha_i, i \in I)$

$$|\Pi^i_x(\psi^{\varepsilon_n}_x)| \lesssim \varepsilon_n^{\alpha_i},$$

uniformly over x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(B(0,1))$ with $\|\psi\|_{C^r} \leq 1$.

2. $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \Gamma \in G$ is such that

$$\Gamma_{xy} \circ \Gamma_{yz} = \Gamma_{xz}, \qquad \Pi^j_y = \sum_{i \in I} \Pi^i_x \, \Gamma^{ij}_{xy}, \qquad j \in I, \; x, y, z \in \mathbb{R}^d,$$

and $|\Gamma^{ij}_{xy}| \lesssim |x - y|^{\alpha_j - \alpha_i}$ if $\alpha_i < \alpha_j$.

Products

The product we wrote

$$\bullet\left(v(x)\mathbf{1}+V^{\stackrel{\bullet}{|}}(x)\,\stackrel{\bullet}{|}+\sum_{i=1}^{2}V^{X_i}(x)\,X_i\right)=v(x)\bullet+V^{\stackrel{\bullet}{|}}(x)\,\stackrel{\bullet}{\underset{\bullet}{|}}+\sum_{i=1}^{2}V^{X_i}(x)\bullet X_i$$

is only an example of a more general and beautiful result on products of modelled distributions.

Note first that we should not look for a generic product of modelled distributions: for example we do not expect to define $\bullet^2=\bullet\bullet$

Let us say this informally: suppose that for $i=1,2$, $U^i\in\mathcal{D}^{\gamma_i}_{(\Pi,\Gamma)}$ with

$$U^i(x)=\sum_{j\in A_i}a^i_j(x)\,j,\qquad A_i\subset A,$$

and we suppose that there is a product rule

$$A_1\times A_2\ni(j_1,j_2)\to j_1j_2\in A$$

Products

If Γ has proper compatibility properties with A_1, A_2 and the product rule, then setting $\overline{\alpha}_i := \min(\alpha_j, \in A_i)$ we have

Theorem

There is a unique bilinear extension of the product rule to a map

$$(U^1, U^2) \mapsto U^1 U^2 \in \mathcal{D}^{(\gamma_1+\overline{\alpha}_2)\wedge(\gamma_2+\overline{\alpha}_1)}_{(\Pi,\Gamma)}$$

where $U^i \in \mathcal{D}^{\gamma_i}_{(\Pi,\Gamma)}$ *with*

$$U^i(x) = \sum_{j\in A_i} a^i_j(x)\, j, \qquad A_i \subset A,$$

$$U^1 U^2(x) = \sum_{j_1\in A_1, j_2\in A_2} \mathbb{1}_{(\alpha_{j_1}+\alpha_{j_2}<(\gamma_1+\overline{\alpha}_2)\wedge(\gamma_2+\overline{\alpha}_1))}\, a^1_{j_1}(x)\, a^2_{j_2}(x)\, j_1 j_2.$$

In the example $\gamma_1 = 1 + 2\kappa, \alpha_1 = 0, \gamma_2 = +\infty, \alpha_2 = -1 - \kappa$, which gives $\gamma = \kappa$.

Since $U^1, U^2, U^1 U^2$ are all modelled distributions, we can reconstruct them

$$u^1 := \mathcal{R}U^1, \qquad u^2 := \mathcal{R}U^2, \qquad u^1 \star u^2 := \mathcal{R}(U^1 U^2).$$

This gives a notion of product of (certain) distributions associated with a model.

Again, one has to insist on the fact that $U^1, U^2, U^1 U^2$ are mere collections of coefficients, and that it is the reconstruction theorem which produces concrete distributions from them.

The definition $u^1 \star u^2 := \mathcal{R}(U^1 U^2)$ is surprisingly easy, based on the result that $U^1 U^2$ is a modelled distribution. The hard work is in fact in the construction of

$$\Pi_x(j_1 j_2), \qquad j_1 \in A_1, \; j_2 \in A_2,$$

as we mentioned in the case $\Pi_x \text{⁝} = \Pi_x(\bullet\,\text{ǃ})$.

In general $\Pi_x(j_1 j_2) \neq \Pi_x(j_1)\Pi_x(j_2)$ and the latter expression may not even make sense.

Renormalisation

So let us go back to the definition $\hat{\Pi}^{\varepsilon}_x \tau = \Pi^{\varepsilon}_x \tau$ for $\tau \neq \mathbf{!}$ and

$$\hat{\Pi}^{\varepsilon}_x \mathbf{!} := \left(G * \xi_\varepsilon - G * \xi_\varepsilon(x)\right) \xi_\varepsilon - C_\varepsilon$$

with $C_\epsilon = -\frac{1}{2\pi} \log|\epsilon| + O(1)$.

Note that indeed $\hat{\Pi}^{\varepsilon}_x(j_1 j_2) \neq \hat{\Pi}^{\varepsilon}_x(j_1)\hat{\Pi}^{\varepsilon}_x(j_2)$, while $\Pi^{\varepsilon}_x(j_1 j_2) = \Pi^{\varepsilon}_x(j_1)\Pi^{\varepsilon}_x(j_2)$.

This is probably the most fascinating and difficult part of this theory: the need of a renormalisation procedure.

Here we have a constant $C \in \mathbb{R}$ acting on the model additively.

In more complicated situations, one tries to construct a group $\mathcal{G}^-$, called the renormalisation group, with an action on the space of models over a fixed regularity structure.

This group should possibly be finite-dimensional.

Renormalised equation

We would like to find u as the fixed point of a map

$$v \mapsto G * ((\alpha + \beta v)\xi)\,.$$

We lift the equation to a space of modelled distributions. We look for a $V \in \mathcal{D}^{1+2\kappa}_{(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)}$ of the form

$$V(x) = v(x)\mathbf{1} + V^{\dot{\vert}}(x)\,\dot{\vert} + \sum_{i=1}^{2} V^{X_i}(x)\,X_i, \qquad v = \mathcal{R}\langle \hat{\Pi}^\varepsilon, V, \rangle,$$

we write

$$\bullet V(x) = v(x)\,\bullet + V^{\dot{\vert}}(x)\,\underset{\bullet}{\dot{\vert}} + \sum_{i=1}^{2} V^{X_i}(x)\,\bullet X_i.$$

We know that $\bullet V \in \mathcal{D}^{\kappa}_{(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)}$ and also $W := \bullet(\alpha + \beta V) \in \mathcal{D}^{\kappa}_{(\hat{\Pi}^\varepsilon, \hat{\Gamma}^\varepsilon)}$.

Renormalised equation

Now we have

$$
\begin{aligned}
\mathcal{R}W &= \langle \hat{\Pi}^{\varepsilon}, W \rangle_x(x) \\
&= (\alpha + \beta v(x))\, \hat{\Pi}^{\varepsilon}_x \bullet (x) + V^{\overset{\bullet}{|}}(x)\, \hat{\Pi}^{\varepsilon}_x \overset{\bullet}{\underset{\bullet}{|}}(x) + \sum_{i=1}^{2} V^{X_i}(x) \hat{\Pi}^{\varepsilon}_x \bullet X_i(x) \\
&= (\alpha + \beta v(x))\, \xi_\varepsilon(x) + V^{\overset{\bullet}{|}}(x)\, [(G * \xi_\varepsilon(x) - G * \xi_\varepsilon(x))\, \xi_\varepsilon(x) - C_\varepsilon] \\
&= (\alpha + \beta v(x))\, \xi_\varepsilon(x) - C_\varepsilon V^{\overset{\bullet}{|}}(x).
\end{aligned}
$$

We compute $\overline{\mathcal{K}}\bullet(\alpha + \beta V) \in \mathcal{D}^{1+2\kappa}_{(\hat{\Pi}^{\varepsilon}, \hat{\Gamma}^{\varepsilon})}$, which has the form

$$
\overline{\mathcal{K}}W(x) = G * ((\alpha + \beta v)\, \xi_\varepsilon - C_\varepsilon V^{\overset{\bullet}{|}})(x)\, \mathbf{1} + (\alpha + \beta v(x))\, \overset{\bullet}{|} + \sum_{i=1}^{2} f_i(x)\, X_i
$$

Renormalised equation

A fixed point $U^\varepsilon = \overline{\mathcal{K}}_{\bullet}(\alpha + \beta U)$ with

$$\overline{\mathcal{K}}_{\bullet}(\alpha + \beta V)(x) = G * ((\alpha + \beta v)\,\xi_\varepsilon - C_\varepsilon V^{\dagger})(x)\,\mathbf{1} + (\alpha + \beta v(x))\,\dagger + \sum_{i=1}^{2} f_i(x)\,X_i$$

must then satisfy

$$U^{\dagger}(x) = (\alpha + \beta \hat{u}^\varepsilon(x)), \qquad \hat{u}^\varepsilon(x) = \mathcal{R}\langle \hat{\Pi}^\varepsilon, U\rangle_x(x),$$

and therefore

$$\begin{aligned}\hat{u}^\varepsilon(x) &= G * ((\alpha + \beta \hat{u}^\varepsilon)\,\xi_\varepsilon - C_\varepsilon(\alpha + \beta \hat{u}^\varepsilon))(x) \\ &= G * ((\alpha + \beta \hat{u}^\varepsilon)\,(\xi_\varepsilon - C_\varepsilon))(x).\end{aligned}$$

Further results

Convergence of the renormalised model...