Regularity structures – Exercise sessions

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Goal of those sessions

Consider gaussian white-noise ξ, parameters $\alpha, \beta \in \mathbb{R}$.
Throughout these sessions we will study

$$(-\Delta)\phi = (\alpha + \beta\phi)\xi, \quad x \in \mathbb{R}^d. \tag{E}$$

We will be interested in $d = 1$, $d = 2$.
Goal: Show some useful tools and explain why

1. if $d = 1$: (E) admits (local) solutions;
2. if $d = 2$: for natural smooth approximations ξ_ϵ of ξ, there exist $C_\epsilon \in \mathbb{R}$ s.t. the renormalised equations

$$(-\Delta)\phi_\epsilon = (\alpha + \beta\phi_\epsilon)(\xi_\epsilon - \beta C_\epsilon),$$

admit solutions converging to some (non-trivial) ϕ.

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Equation (E) is a “pretext” to illustrate on a simple case some of the

- analytic aspects,
- probabilistic aspects,

of regularity structures.

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Session 1

First remarks on (E)

A few remarks

$$(-\Delta)\phi = (\alpha + \beta\phi)\xi, \quad x \in \mathbb{R}^d. \tag{E}$$

Some (heuristic) remarks:

- this is an elliptic equation;
- we might have to play with α, β to obtain contractivity for Picard fixed-point;
- we might want to work on torus $\mathbb{T}^d$ rather than $\mathbb{R}^d$ to benefit from boundedness;
- the regularity of ξ decreases when d increases so we expect the equation to be more difficult to discuss;
- for simplicity we will not consider questions of uniqueness/boundary conditions.

Mild formulation

We will replace (E) by its mild formulation

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi), \tag{M}$$

where $\mathsf{K} =$ "$(-\Delta)^{-1}$" is the fundamental solution of the Laplacian,

$$\mathsf{K}(x) = -\frac{1}{2}|x| \quad (\mathrm{d} = 1), \qquad \mathsf{K}(x) = -\frac{1}{2\pi}\log|x| \quad (\mathrm{d} = 2).$$

Question: in which space could we set up a fixed-point for (M)?

Classical Hölder functions

Definition

Let $\alpha \in (0,1)$, and $\phi\colon \mathbb{R}^d \to \mathbb{R}$. We say that ϕ is α-Hölder if

$$|\phi(y) - \phi(x)| \lesssim |y-x|^\alpha.$$

More generally.

Definition

Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, and $\phi\colon \mathbb{R}^d \to \mathbb{R}$. We say that ϕ is α-Hölder if

$$\left|\phi(y) - \sum_{|k|\le\alpha} \frac{\phi^{(k)}(x)}{k!}(y-x)^k\right| \lesssim |y-x|^\alpha.$$

What about distributions?

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What about distributions?

Quick calculation: if $|\phi(x) - \phi(y)| \lesssim |y - x|^\alpha$, then

$$\phi(\eta_x^\lambda) = \int \phi(y)\eta_x^\lambda(y)dy = \int \phi(x + \lambda y)\eta(y)dy = O(1).$$

If $\int \eta = 0$, then

$$\phi(\eta_x^\lambda) = \int (\phi(x + \lambda y) - \phi(x))\eta(y)dy = O(\lambda^\alpha).$$

Measuring regularity: the Hölder spaces

Generally: define

$$\mathscr{B}_\alpha^r := \left\{ \eta \in \mathcal{D}(B(0,1)), \|\eta\|_{C^r} \leq 1, \int \eta(x) x^k dx = 0 \text{ for } 0 \leq |k| \leq \alpha \right\}.$$

And for $\alpha \in \mathbb{R}$, $r > -\alpha$, $K \subset \mathbb{R}^d$

$$\|\phi\|_{\mathcal{C}_K^\alpha} = \sup_{x \in K, \eta \in \mathscr{B}^r} |\phi(\eta_x)| + \sup_{x \in K, \lambda \in (0,1], \eta \in \mathscr{B}_\alpha^r} \frac{|\phi(\eta_x^\lambda)|}{\lambda^\alpha}, \qquad (1)$$

Definition (Local Hölder spaces)

For $\alpha \in \mathbb{R}$, $\mathcal{C}_{\text{loc}}^\alpha$ is the complete metrizable space corresponding to the family of seminorms (1).

When $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, $\mathcal{C}_{\text{loc}}^\alpha$ coincides with the classical Hölder functions.

Some natural questions about Hölder spaces:

1. what is gaussian white-noise ξ and what is its Hölder regularity?
2. when is pointwise multiplication $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\gamma}_{\mathrm{loc}}$ well-defined and continuous?
3. when is convolution $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ well-defined and continuous?

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Gaussian white noise

Definition

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Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. A *gaussian white noise on H* is a linear isometry

$$\begin{aligned} \xi \colon H &\to L^2(\Omega, \mathcal{F}, \mathbb{P}) \\ h &\mapsto \xi(h), \end{aligned}$$

on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $h \in H$, $\xi(h)$ is a real-valued centered gaussian variable.

NB: the isometry property means that for $h_1, h_2 \in H$, $\mathbb{E}[\xi(h_1)\xi(h_2)] = \langle h_1, h_2\rangle$. In particular, $\xi(h) \sim \mathcal{N}(0, \|h\|^2)$.

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Kolmogorov's continuity theorem for distributions

ξ eats test-functions. Is it a Hölder distribution?

Theorem (Kolmogorov's continuity for distributions)

Let $X : L^2(\mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a continuous map. Assume there exist $\alpha \in \mathbb{R}$, $p \geq 1$ such that for all $\eta \in L^2(\mathbb{R}^d)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\big|X(\eta_x^{2^{-k}})\big|^p\Big] \leq C_{p,\eta} 2^{-k\alpha p}.$$

Then there exists $\tilde{X} : L^2(\mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. *for all $\eta \in L^2(\mathbb{R}^d)$, $\tilde{X}(\eta) = X(\eta)$ almost surely,*
2. *for all $\omega \in \Omega$, and $\alpha' < \alpha - d/p$, $\tilde{X}(\omega) \in \mathcal{C}^{\alpha'}_{\mathrm{loc}}(\mathbb{R}^d)$.*

Technique of proof: use dyadic decompositions, such as wavelets or Littlewood-Paley decompositions.

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Let us check that ξ satisfies the condition for $\alpha = -d/2$.

$$\begin{aligned}\mathbb{E}\left[\left|\xi(\eta_x^{2^{-k}})\right|^p\right] &\leq C_p \mathbb{E}\left[\left|\xi(\eta_x^{2^{-k}})\right|^2\right]^{p/2} && \text{(gaussian moments)}\\ &= C_p \|\eta_x^{2^{-k}}\|_{L^2}^p && \text{(isometry)}\end{aligned}$$

But:

$$\begin{aligned}\|\eta_x^{2^{-k}}\|_{L^2}^2 &= \int \eta_x^{2^{-k}}(y)\eta_x^{2^{-k}}(y)dy\\ &= \int 2^{kd}\eta(2^k(y-x))2^{kd}\eta(2^k(y-x))dy\\ &= \int 2^{kd}\eta(u)^2 du = 2^{kd}\|\eta\|_{L^2}^2.\end{aligned}$$

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Consequence:

Proposition

Any gaussian white-noise ξ on $\mathbb{R}^d$ has a modification which belongs to $\mathcal{C}_{\mathrm{loc}}^{-d/2-\kappa}(\mathbb{R}^d)$ for all $\kappa > 0$.

In fact, exploiting gaussianity in a smarter way one can prove

Proposition

Any gaussian white-noise ξ on $\mathbb{R}^d$ has a modification which belongs to the Besov space $\mathcal{B}_{p,\infty;\mathrm{loc}}^{-d/2}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

This is slightly stronger because $\mathcal{B}_{p,\infty;\mathrm{loc}}^{-d/2}(\mathbb{R}^d) \subset \mathcal{C}_{\mathrm{loc}}^{-d/2-d/p}(\mathbb{R}^d)$.

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Direct Picard iteration works in $d = 1$

Situation

Recall the situation:

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi). \tag{M}$$

We want to solve (M) as a Picard fixed-point in some $\mathcal{C}^{\alpha}_{\text{loc}}(\mathbb{R}^d)$. In this section we start with the case $d = 1$.

We asked three questions about Hölder spaces:

1. what is gaussian white-noise ξ and what is its Hölder regularity?
2. when is pointwise multiplication $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\gamma}_{\mathrm{loc}}$ well-defined and continuous?
3. when is convolution $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ well-defined and continuous?

Hölder spaces question 1: white noise

Question

What is gaussian white-noise ξ and what is its Hölder regularity?

Answer

$\xi \in \mathcal{C}_{\mathrm{loc}}^{-d/2-\kappa}$ *for any* $\kappa > 0$. *For* $d = 1$*:* $\xi \in \mathcal{C}_{\mathrm{loc}}^{-1/2-\kappa}$.

Hölder spaces question 2: Young multiplication

Question

When is pointwise multiplication $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\gamma}_{\mathrm{loc}}$ *well-defined and continuous?*

Answer (Young multiplication)

The product $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\min(\alpha,\beta)}_{\mathrm{loc}}$ *is canonically well-defined iff* $\alpha + \beta > 0$*, with continuity bounds*

$$\|fg\|_{\mathcal{C}^{\min(\alpha,\beta)}_{K}} \leq C_{\alpha,\beta} \|f\|_{\mathcal{C}^{\alpha}_{K \oplus B(0,A)}} \|g\|_{\mathcal{C}^{\beta}_{K \oplus B(0,A)}},$$

for some $A \geq 0$.

Hölder spaces question 3: classical Schauder estimates

Question

When is convolution $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ *well-defined and continuous?*

Answer (Schauder estimates)

For any $\alpha \in \mathbb{R}$, $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\alpha+2}_{\mathrm{loc}}$ *with continuity bounds*

$$\|\mathsf{K} * f\|_{\mathcal{C}^{\alpha+2}_{K}} \leq C_\alpha \|f\|_{\mathcal{C}^{\alpha}_{K \oplus B(0,A)}},$$

for some $A \geq 0$.

First terms of Picard iteration

Let us perform Picard iteration (recall $\xi \in \mathcal{C}_{\mathrm{loc}}^{-1/2-\kappa}$):

$$\begin{aligned}
\phi_{(0)} &:= 0 && \in C^\infty, \\
\phi_{(1)} &:= \mathsf{K} * ((\alpha + \beta\phi_{(0)})\xi) \\
&= \alpha \mathsf{K} * \xi && \in \mathcal{C}_{\mathrm{loc}}^{3/2-\kappa} \text{ (Schauder)}, \\
\phi_{(2)} &:= \mathsf{K} * ((\alpha + \beta\phi_{(1)})\xi) && \in \mathcal{C}_{\mathrm{loc}}^{3/2-\kappa} \text{ (Schauder + Young)}.
\end{aligned}$$

Young multiplication is justified because

$$\phi_{(1)} \in \mathcal{C}_{\mathrm{loc}}^{3/2-\kappa}, \quad \xi \in \mathcal{C}_{\mathrm{loc}}^{-1/2-\kappa}, \quad \frac{3}{2} - \kappa + \frac{-1}{2} - \kappa > 0.$$

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Conclusion: in $d = 1$, the iteration map

$$\begin{aligned}\mathcal{P} \colon C^{3/2-\kappa}_{\mathrm{loc}} &\to C^{3/2-\kappa}_{\mathrm{loc}} \\ \phi &\mapsto \mathsf{K} * ((\alpha + \beta\phi)\xi),\end{aligned}$$

is well-defined with continuity estimates

$$\left\|\mathcal{P}(\phi_1) - \mathcal{P}(\phi_2)\right\|_{\mathcal{C}^{3/2-\kappa}_K} \leq C\beta\|\xi\|_{\mathcal{C}^{-1/2-\kappa}_{K\oplus B(0,A)}}\|\phi_1 - \phi_2\|_{\mathcal{C}^{3/2-\kappa}_{K\oplus B(0,A)}}.$$

Subtlety: the Lipschitz constant depends on the compact K via the norm of the noise. Possible workarounds:

- construct solutions on bounded space (e.g. on the torus);
- work with weighted Hölder spaces.

In any case, $\mathcal{P}$ becomes a contraction when setting β small enough.
NB: β might be random to construct ϕ a.s.; or if β is deterministic ϕ exists only with positive probability.

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Direct Picard iteration does not work in $d = 2$

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Problem: $\phi_{(2)}$ has no canonical meaning because

$$\phi_{(1)} \in \mathcal{C}^{1-\kappa}_{\mathrm{loc}}, \quad \xi \in \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}, \quad (1-\kappa) + (-1-\kappa) \leq 0.$$

There is no canonical product $\mathcal{C}^{1-\kappa}_{\mathrm{loc}} \times \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}$.

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Problem: $\phi_{(2)}$ has no canonical meaning because

$$\phi_{(1)} \in \mathcal{C}^{1-\kappa}_{\text{loc}}, \quad \xi \in \mathcal{C}^{-1-\kappa}_{\text{loc}}, \quad (1-\kappa) + (-1-\kappa) \leq 0.$$

There is no canonical product $\mathcal{C}^{1-\kappa}_{\text{loc}} \times \mathcal{C}^{-1-\kappa}_{\text{loc}}$.

Da Prato–Debussche trick?

In this kind of situation, one could try:

Trick (Da Prato–Debussche, 2003)

Consider $\psi := \phi - \phi_{(1)}$ and try Picard iteration on ψ.

Interpretation: remove the term of "worst regularity".

The fixed-point equation on ψ:

$$\psi = \phi - \phi_{(1)} = \mathsf{K} * ((\alpha + \beta\phi)\xi) - \alpha\mathsf{K} * \xi = \beta\mathsf{K} * (\phi\xi).$$

i.e.

$$\psi = \beta\mathsf{K} * (\phi_{(1)}\xi) + \beta\mathsf{K} * (\psi\xi).$$

NB: Recall $\phi_{(1)} = \alpha\mathsf{K} * \xi$. We assume we can give a meaning to $\phi_{(1)}\xi \in \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}$ by probabilistic techniques.

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Regularity vs homogeneity

Observation: if f, g are α resp. β-Hölder for $\alpha, \beta \in (0,1)$, then

$$|f(y)g(y) - f(x)g(x)| \lesssim |y-x|^{\min(\alpha,\beta)},$$
$$|f(y) - f(x)||g(y) - g(x)| \lesssim |y-x|^{\alpha+\beta},$$

where $\alpha + \beta > \min(\alpha, \beta)$.

Interpretation: the *germ* $\left(f(\cdot) - f(x)\right)_{x\in\mathbb{R}^d}$ admits "nice" multiplicativity properties with respect to its *homogeneity*.

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Some reminders on germs

Germs: definitions

Definition (Germ)

A *germ* is a family $F = (F_x)_{x\in\mathbb{R}^d}$ of distributions $F_x \in \mathcal{D}'(\mathbb{R}^d)$.

Interpretation: a germ is a family of local approximations.

Basic example: if f is a $0 < \gamma$-Hölder function, its Taylor germ:

$$T(f)_x(\cdot) = \sum_{|k|\leq\gamma} \frac{f^k(x)}{k!}(\cdot - x)^k.$$

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Germs: definitions

Definition (Coherence)

A germ F is (α, γ)-coherent if
$|(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\alpha(|y-x|+\lambda)^{\gamma-\alpha}$

Definition (Homogeneity)

A germ F is $\bar{\alpha}$-homogeneous if $|F_x(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}}$.

Example: if f is γ-Hölder, $T(f)$ is $(0, \gamma)$-coherent and 0-homogeneous.

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Germs: reconstruction

A bridge between germs and distributions is given by:

Theorem (Reconstruction, Hairer 2014, Caravenna-Zambotti 2020)

If F is (α, γ)-coherent for some $\gamma > 0$, then there exists a unique distribution $\mathcal{R}(F)$ such that $(\mathcal{R}(F) - F_x)_x$ is γ-homogeneous.

Example: if f is γ-Hölder, $\mathcal{R}(T(f)) = f$.

Useful particular case: if $x, y \mapsto F_x(y)$ is continuous, then

$$\mathcal{R}(F)(x) = F_x(x).$$

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Operations on germs

We want to perform a fixed-point argument at the level of germs.
Question: can one perform the operations of convolution and multiplication at the level of germs?

Germs: Schauder estimates

Let K be the fundamental solution of the Laplacian.

Theorem (Schauder estimates for germs, L.B, F. Caravenna, L. Zambotti 2022)

Let F be an (α, γ)-coherent germ. Then the germ

$$\mathcal{K}(F)_x := \mathsf{K} * F_x - \sum_{|k| < \gamma + 2} \frac{(\mathsf{K} * \{F_x - \mathcal{R}(F)\})^{(k)}(x)}{k!} (\cdot - x)^k,$$

is well-defined, $((\alpha + 2) \wedge 0, \gamma + 2)$-coherent, and $\mathcal{R}(\mathcal{K}(F)) = \mathsf{K} * \mathcal{R}(F)$.

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This can be generalised to a class of β-regularising kernels.

Germs: products?

In general there is no canonical way of multiplying germs.

However, for continuous germs $x, y \mapsto F_x(y)$, $x, y \mapsto G_x(y)$, the germ $(FG)_x(y) \coloneqq F_x(y)G_x(y)$ satisfies (if reconstruction is applicable)

$$\mathcal{R}(FG)(x) = F_x(x)G_x(x) = \mathcal{R}(F)(x)\mathcal{R}(G)(x).$$

$d = 2$: the model

Basis germs II

Setting

Recall the situation:

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi), \quad x \in \mathbb{R}^2.$$

Let ξ be gaussian white noise, ρ a test-function with $\int \rho = 1$, and $\xi_\epsilon = \xi * \rho^\epsilon$: this is a smooth approximation of ξ.

We consider the fixed-point equation on germs

$$F_\epsilon = \mathcal{K}((\alpha + \beta F_\epsilon)\xi_\epsilon).$$

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Pursuing the iteration: the germs of interest are of the form

$$\sum_i f_i(x)\Pi_x^{i,\epsilon},$$

for explicit homogeneous basis germ $\Pi^{i,\epsilon}$.

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Germ $\Pi_x^{i,\epsilon}(\cdot) =$	Hom. α	Symbol
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Symbolic notation

We will use Hairer's symbolic notations:

- blue dot $\bullet$: represents the noise;
- bar: represents convolution (with K);
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The notations simplify the expressions. Example:

$$\xi_\epsilon(y) + 3x(\mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(y) = \Pi_x^\epsilon(\bullet + 3x\underset{\bullet}{\overset{\bullet}{|}})(y).$$

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Thank you for your attention!

Session 2

Recap

We consider gaussian white-noise ξ, coefficients $\alpha, \beta \in \mathbb{R}$, and:

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi), \quad x \in \mathbb{R}^d. \tag{M}$$

We have seen that:

1. in $d = 1$: Picard fixed-point can be performed in "classical" spaces of Hölder distributions;
2. in $d = 2$: the same approach fails. However, we heuristically motivated that (M) should be lifted at the level of *germs* i.e. families of local approximations $(F_x)_{x \in \mathbb{R}^d} \subset \mathcal{D}'$.

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$\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x)$	$1-\kappa$	
$(\cdot_1 - x_1)$	1	X_1
$(\cdot_2 - x_2)$	1	X_2

Symbolic notation

We will use Hairer's symbolic notations:

- blue dot $\bullet$: represents the noise;
- bar: represents convolution (with K);
- symbolic multiplication (e.g. $\overset{\bullet}{\underset{\bullet}{|}} = \bullet\,\overset{\bullet}{|}$, $\mathbb{1}\overset{\bullet}{|} = \overset{\bullet}{|}$) represents multiplication.

The notations simplify the expressions. Example:

$$\xi_\epsilon(y) + 3x(\mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(y) = \Pi_x^\epsilon(\bullet + 3x\overset{\bullet}{\underset{\bullet}{|}})(y).$$

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Multiplicativity of Π

Remark: for all (applicable) symbols τ, τ',

$$\Pi_x^\epsilon(\tau\tau') = \Pi_x^\epsilon(\tau)\Pi_x^\epsilon(\tau'),$$

(as a product of smooth functions). Example:

$$\Pi_x^\epsilon(\overset{\bullet}{\underset{\bullet}{|}})(\cdot) = (\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(\cdot) = \Pi_x^\epsilon(\overset{\bullet}{|})(\cdot)\Pi_x^\epsilon(\bullet)(\cdot).$$

Note: we do not expect this to remain true at the limit $\epsilon \to 0$. In fact, we expect $\Pi_x^\epsilon(\overset{\bullet}{\underset{\bullet}{|}})(\cdot)$ to diverge:

$$\begin{aligned}\mathbb{E}[\Pi_x^\epsilon(\overset{\bullet}{\underset{\bullet}{|}})(y)] &= \mathbb{E}[(\mathsf{K} * \xi_\epsilon)(y)\xi_\epsilon(y)] \\ &= \mathbb{E}[\xi((\mathsf{K}_\epsilon)_y)\,\xi(\rho_y^\epsilon)] \\ &= \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \to \infty.\end{aligned}$$

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&= \mathbb{E}[\xi((\mathsf{K}_\epsilon)_y)\,\xi(\rho_y^\epsilon)] \\
&= \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \to \infty.
\end{aligned}$$

Reexpansion operator Γ

Reexpansion

Fact

There exists a reexpansion operator Γ^ϵ defined by the relations

$$\Pi_y^{i,\epsilon} = \sum_j \Pi_x^{j,\epsilon} \Gamma_{x,y}^{j,i,\epsilon},$$

or in symbolic notation

$$\Pi_y^\epsilon(\tau) = \Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\tau)).$$

Let us calculate Γ^ϵ on some examples.

Example 0 (Polynomials)

Consider $\tau = X^k$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(X^k)) = \Pi_y^\epsilon(X^k)$, but

$$\begin{aligned}\Pi_y^\epsilon(X^k) &= (\cdot - y)^k \\ &= \sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} (\cdot - x)^k \\ &= \Pi_x^\epsilon\Big(\sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} X^l \Big).\end{aligned}$$

whence we set $\Gamma_{x,y}^\epsilon(X^k) = \sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} X^l$.

Example 1 (Noise)

Consider $\tau = \bullet$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\bullet)) = \Pi_y^\epsilon(\bullet)$, but

$$\Pi_y^\epsilon(\bullet) = \xi_\epsilon = \Pi_x^\epsilon(\bullet)$$

whence we set $\Gamma_{x,y}^\epsilon(\bullet) = \bullet$.

Example 2 (Convolution)

Consider $\tau = \rotatebox{180}{!}$.
We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\rotatebox{180}{!})) = \Pi^\epsilon_y(\rotatebox{180}{!})$ but

$$\begin{aligned}\Pi^\epsilon_y(\rotatebox{180}{!}) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(y) \\ &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\ &= \Pi^\epsilon_x\big(\rotatebox{180}{!} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}\big),\end{aligned}$$

whence we set $\Gamma^\epsilon_{x,y}(\rotatebox{180}{!}) = \rotatebox{180}{!} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}$.

Example 2 (Convolution)

Consider $\tau = \mathbf{\dagger}$.
We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathbf{\dagger})) = \Pi^\epsilon_y(\mathbf{\dagger})$ but

$$\begin{aligned}\Pi^\epsilon_y(\mathbf{\dagger}) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(y) \\ &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\ &= \Pi^\epsilon_x\big(\mathbf{\dagger} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}\big),\end{aligned}$$

whence we set $\Gamma^\epsilon_{x,y}(\mathbf{\dagger}) = \mathbf{\dagger} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}$.

Example 2 (Convolution)

Consider $\tau = \mathbf{I}$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\mathbf{I})) = \Pi_y^\epsilon(\mathbf{I})$ but

$$\begin{aligned}
\Pi_y^\epsilon(\mathbf{I}) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(y) \\
&= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\
&= \Pi_x^\epsilon\big(\mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}\big),
\end{aligned}$$

whence we set $\Gamma_{x,y}^\epsilon(\mathbf{I}) = \mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}$.

Example 3 (Product)

Consider $\tau = \mathbf{I}$.
We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathbf{I})) = \Pi^\epsilon_y(\mathbf{I})$ but

$$
\begin{aligned}
\Pi^\epsilon_y(\mathbf{I}) &= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= \Pi^\epsilon_x(\mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\,\bullet),
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\end{aligned}$$

whence we set $\Gamma^\epsilon_{x,y}(\mathbf{I}) = \mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\,\bullet$.

All calculations done, we can write explicitly:

$$
\Gamma^{\epsilon}_{x,y} = \begin{array}{c} j\backslash i \\ \bullet \\ \overset{\bullet}{\underset{\bullet}{|}} \\ \bullet X_1 \\ \bullet X_2 \\ \mathbb{1} \\ \overset{\bullet}{|} \\ X_1 \\ X_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} \bullet & \overset{\bullet}{\underset{\bullet}{|}} & \bullet X_1 & \bullet X_2 & \mathbb{1} & \overset{\bullet}{|} & X_1 & X_2 \end{array} \\
\begin{pmatrix}
1 & \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) & x_1 - y_1 & x_2 - y_2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array},
$$

Some first properties

The reexpansion operator Γ^ϵ

- is triangular with 1 on the diagonal;
- is Hölder continuous: $|\Gamma^{\sigma,\tau}_{x,y}| \lesssim |y-x|^{\alpha_\tau - \alpha_\sigma}$;
- satisfies the group property: $\Gamma^\epsilon_{x,y}\Gamma^\epsilon_{y,z} = \Gamma^\epsilon_{x,z}$.

Consequence: the pair $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ is a *model*.

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Consequence: the pair $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ is a *model*.

Multiplicativity of Γ

Note that for all (applicable) symbols τ, τ',

$$\Gamma^{\epsilon}_{x,y}(\tau\tau') = \Gamma^{\epsilon}_{x,y}(\tau)\Gamma^{\epsilon}_{x,y}(\tau').$$

Example:

$$\begin{aligned}\Gamma^{\epsilon}_{x,y}\big(\overset{\bullet}{\underset{\bullet}{|}}\big) &= \overset{\bullet}{\underset{\bullet}{|}} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\,\bullet \\ &= \big(\overset{\bullet}{|} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}\big)\,\bullet \\ &= \Gamma^{\epsilon}_{x,y}\big(\overset{\bullet}{|}\big)\Gamma^{\epsilon}_{x,y}(\bullet).\end{aligned}$$

NB: Multiplicativity of Γ is guaranteed in general by the algebraic construction.

Admissibility of (Π, Γ)

The model M^ϵ satisfies:

$$\begin{aligned}
\Pi^\epsilon_x(\mathcal{I}) &= \mathsf{K} * \Pi^\epsilon_x(\bullet)(\cdot) - \mathsf{K} * \Pi^\epsilon_x(\bullet)(x),\\
\Gamma^\epsilon_{x,y}(\mathcal{I}) &= \mathcal{I} - \Pi^\epsilon_x(\mathcal{I})(y)\mathbb{1},\\
\Gamma^\epsilon_{x,y}(\tau\tau') &= \Gamma^\epsilon_{x,y}(\tau)\Gamma^\epsilon_{x,y}(\tau') \quad \tau, \tau' \in T.
\end{aligned}$$

We say that M^ϵ is *admissible*, and note $\mathscr{M}_{\mathrm{adm}}$ the corresponding set.
(NB: In this condition, Γ is multiplicative but not necessarily Π.)

Consequence: for $M = (\Pi, \Gamma) \in \mathscr{M}_{\mathrm{adm}}$, Γ is determined by Π i.e. we may identify M and Π.

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The model M^ϵ satisfies:

$$\begin{aligned} \Pi^\epsilon_x(\uparrow) &= \mathsf{K} * \Pi^\epsilon_x(\bullet)(\cdot) - \mathsf{K} * \Pi^\epsilon_x(\bullet)(x), \\ \Gamma^\epsilon_{x,y}(\uparrow) &= \uparrow - \Pi^\epsilon_x(\uparrow)(y)\mathbb{1}, \\ \Gamma^\epsilon_{x,y}(\tau\tau') &= \Gamma^\epsilon_{x,y}(\tau)\Gamma^\epsilon_{x,y}(\tau') \quad \tau, \tau' \in T. \end{aligned}$$

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Stationarity of (Π, Γ)

For φ, h, τ, the random processes

$$x \mapsto \Pi^{\epsilon}_x(\tau)(\varphi_x), \qquad x \mapsto \Gamma^{\epsilon}_{x,x+h}(\tau),$$

are stationary i.e. their distribution do not depend on x.

Example:

$$\begin{aligned}\Pi_x(\dot{|})(\varphi_x) &= \mathsf{K} * \xi_\epsilon(\varphi_x) - \mathsf{K} * \xi_\epsilon(x) \\ &= \xi\big((\mathsf{K} * \rho^\epsilon * \varphi - \mathsf{K} * \rho^\epsilon)_x\big).\end{aligned}$$

Useful consequence: for all $x \in \mathbb{R}^d$,

$$\mathbb{E}[|\Pi^{\epsilon}_x(\tau)(\varphi^{\lambda}_x)|^2] = \mathbb{E}[|\Pi^{\epsilon}_0(\tau)(\varphi^{\lambda})|^2].$$

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$d = 2$: the modelled distributions

Modelled distributions

We are interested in germs of the form $F_x = \sum_\tau f_\tau(x)\Pi_x^\epsilon(\tau)$.

Definition (Modelled distributions)

Let $M \in \mathscr{M}_{\mathrm{adm}}$ be an admissible model. The space $\mathcal{D}^\gamma$ of modelled distributions for M is the space of functions $f(x) = \sum_\tau f_\tau(x)\tau$ with

$$|f_\tau(x)| \lesssim 1, \quad \Big| f_\tau(x) - \sum_\sigma \Gamma_{x,y}^{\tau,\sigma} f_\sigma(y) \Big| \lesssim |y-x|^{\gamma-\alpha_\tau}.$$

We note $\|\cdot\|_{\mathcal{D}_K^\gamma}$ the corresponding seminorm on the compact K.

Then: the germ $F_x := \sum_\tau f_\tau(x)\Pi_x(\tau)$ is (α,γ)-coherent for $\alpha = \min_\tau(\alpha_\tau) = -1-\kappa$.

Modelled distributions

Iterating the equation: we want to set up the fixed-point in the subspace $\mathcal{D}^\gamma_{\mathrm{fp}}$ of $\mathcal{D}^\gamma$ of modelled distributions of the form

$$f(x) = a(x)\mathbb{1} + b(x)\,\mathbf{I} + c_1(x)X_1 + c_2(x)X_2.$$

For the model M^ϵ, this corresponds to germs

$$\begin{aligned} F_x(\cdot) &= \Pi^\epsilon_x\big(f(x)\big)(\cdot) \\ &= a(x) + b(x)\big(\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x)\big) \\ &\quad + c_1(x)(\cdot_1 - x_1) + c_2(x)(\cdot_2 - x_2). \end{aligned}$$

Question: are the spaces $\mathcal{D}^\gamma$ stable by multiplication? convolution?

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Multiplication operator

Theorem (Multiplication)

Let $f(x) = a(x)\mathbf{1} + b(x)\mathcal{I}(\bullet) + c_1(x)X_1 + c_2(x)X_2$ *be in* $\mathcal{D}^{\gamma}_{\mathrm{fp}}$ *for some* $\gamma > 1$. *Set*

$$f\bullet(x) := a(x)\bullet + b(x)\bullet\mathcal{I}(\bullet) + c_1(x)\bullet X_1 + c_2(x)\bullet X_2.$$

Then $f\bullet \in \mathcal{D}^{\gamma-1-\kappa}$ *with continuity bounds*

$$\|f\bullet\|_{\mathcal{D}_K^{\gamma-1-\kappa}} = \|f\|_{\mathcal{D}_K^{\gamma}}.$$

NB: this is a particular case of a general multiplication result for modelled distributions, see [Hai14, Theorem 4.7].

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Convolution operator

Let K be the fundamental solution of the Laplacian.

Theorem (Multi-level Schauder estimates)

There is an operator $\mathcal{K}\colon \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+2}$ such that $\mathcal{R} \circ \mathcal{K} = \mathsf{K} * \mathcal{R}$. Furthermore, $\mathcal{K}$ is explicit in function of M and $\mathcal{R}$, with continuity bounds

$$\|\mathcal{K}(f)\|_{\mathcal{D}_K^{\gamma+2}} \leq C \|\Pi\|_{\mathcal{M}_{K \oplus B(0,A)}} \|f\|_{\mathcal{D}^\gamma_{K \oplus B(0,A)}},$$

Truncation

Theorem (Truncation of modelled distributions)

Let $f \in \mathcal{D}^\gamma$ and $\gamma' < \gamma$, then truncating f at level γ' gives a modelled distribution $f^{\leq \gamma'} \in \mathcal{D}^{\gamma'}$, with continuity bounds

$$\|f^{\leq \gamma'}\|_{\mathcal{D}_K^{\gamma'}} \leq C\big(1 + \|\Gamma\|_K\big)\|f\|_{\mathcal{D}_K^{\gamma}},$$

Fixed-point

Consequence: the operator

$$\begin{aligned}\mathcal{P}\colon \mathcal{D}^\gamma_{\mathrm{fp}} &\to \mathcal{D}^\gamma_{\mathrm{fp}} \\ f &\mapsto \mathcal{K}^{\gamma-1-\kappa}(\alpha\bullet + \beta f\bullet),\end{aligned}$$

is well defined, with continuity bounds

$$\|\mathcal{P}(f)-\mathcal{P}(g)\|_{\mathcal{D}^\gamma_K} \leq \beta C\Big(1+\|M\|_{\mathcal{M}_{K\oplus B(0,A)}}\Big)^{k_0}\|f-g\|_{\mathcal{D}^\gamma_{K\oplus B(0,A)}},$$

Slight problem as in $d=1$: the Lipschitz constant depends on the compact K.

Forgetting this technical difficulty: $\mathcal{P}$ becomes a contraction for β small enough.

For the reconstruction $\phi_\epsilon := \mathcal{R}^\epsilon(f^\epsilon)$:

$$\phi_\epsilon = \mathsf{K} * ((\alpha + \beta\phi_\epsilon)\xi_\epsilon).$$

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Enhanced continuity

For the operations described above, there are "enhanced" continuity estimates: comparing modelled distributions over different models.
Consequence: there exists a continous solution map

$$S\colon U \to \bigsqcup_{M \in \mathscr{M}_{\mathrm{adm}}} \mathcal{D}^{\gamma}_{M},$$

for some open set $U = U_\beta \subset \mathscr{M}_{\mathrm{adm}}$, corresponding to the fixed-point above.

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$d = 2$: convergence of models

Kolmogorov's criterion for models

Theorem ("Kolmogorov's criterion for stationary models")

Let $(M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon))_{\epsilon > 0}$ *be a sequence of stationary admissible models in* $\mathscr{M}_{\mathrm{adm}}$*, defined on* $(\Omega, \mathcal{F}, \mathbb{P})$*.*
Assume that there exist $\kappa, \theta > 0$ *such that for all symbols* τ *of negative homogeneity, all test-functions* φ*, and all* $p \geq 1$*,* $\epsilon, \epsilon_1, \epsilon_2 \in (0,1)$*,*

$$\mathbb{E}\big[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p} \lambda^{p(\alpha_\tau + \kappa)},$$

$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p} (\epsilon_1 + \epsilon_2)^{p\theta} \lambda^{p(\alpha_\tau + \kappa)}.$$

Then for all $p \geq 1$*, the sequence* $(M^\epsilon)_{\epsilon > 0}$ *converges in* $L^p\big((\Omega, \mathcal{F}, \mathbb{P}); \mathscr{M}_{\mathrm{adm}}\big)$*.*

Proof: uses wavelets. See [Hai14, Theorem 10.7]

Kolmogorov's criterion for models

Theorem ("Kolmogorov's criterion for stationary models")

Let $(M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon))_{\epsilon>0}$ be a sequence of stationary admissible models in $\mathscr{M}_{\mathrm{adm}}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
Assume that there exist $\kappa, \theta > 0$ such that for all symbols τ of negative homogeneity, all test-functions φ, and all $p \geq 1$, $\epsilon, \epsilon_1, \epsilon_2 \in (0,1)$,

$$\mathbb{E}\big[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p} \lambda^{p(\alpha_\tau+\kappa)},$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p}(\epsilon_1+\epsilon_2)^{p\theta} \lambda^{p(\alpha_\tau+\kappa)}.$$

Then for all $p \geq 1$, the sequence $(M^\epsilon)_{\epsilon>0}$ converges in $L^p\big((\Omega, \mathcal{F}, \mathbb{P}); \mathscr{M}_{\mathrm{adm}}\big)$.

Proof: uses wavelets. See [Hai14, Theorem 10.7]

Remark: gaussianity assumptions usually give equivalence of moments. In our case it will suffice to obtain the bounds only for $p = 2$:

$$\mathbb{E}\big[|\Pi_0^{\epsilon}(\tau)(\varphi^{\lambda})|^2\big] \leq C_{\tau,\varphi}\lambda^{2\alpha_\tau+\kappa},$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^{\lambda})|^2\big] \leq C_{\tau,\varphi}(\epsilon_1+\epsilon_2)^{\theta}\lambda^{2\alpha_\tau+\kappa}.$$

In our case, the concerned symbols are

$$\tau \in \{\bullet, \bullet X_1, \bullet X_2, \vdots\}.$$

$\bullet, \bullet X_1, \bullet X_2$ are simpler; $\vdots$ is a bit more tedious.

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In our case, the concerned symbols are

$$\tau \in \{\bullet, \bullet X_1, \bullet X_2, \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\}.$$

$\bullet, \bullet X_1, \bullet X_2$ are simpler; $\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}$ is a bit more tedious.

Example 1

Let us consider the symbol $\tau = \bullet$. By definition,

$$\Pi_0^\epsilon(\bullet)(\varphi^\lambda) = \xi_\epsilon(\varphi^\lambda) = \xi * \rho^\epsilon(\varphi^\lambda) = \xi(\rho^\epsilon * \varphi^\lambda).$$

So: $\mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] = \|\rho^\epsilon * \varphi^\lambda\|_{L^2}^2$ (isometry of ξ).

Recall Young's convolution inequality in $\mathbb{R}^d$:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{if} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p, q, r \leq +\infty.$$

Consequence:

$$\begin{aligned} \mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] &\leq \|\rho^\epsilon\|_{L^1}^2 \|\varphi^\lambda\|_{L^2}^2 \quad \text{(Young)} \\ &= \|\rho\|_{L^1}^2 \|\varphi\|_{L^2}^2 \lambda^{-2}, \end{aligned}$$

This gives the first bound.

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Example 1

The second bound concerns

$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\cdot) - \Pi_0^{\epsilon_2}(\cdot))(\varphi^\lambda)|^2\big] = \|(\rho^{\epsilon_1} - \rho^{\epsilon_2}) * \varphi^\lambda\|_{L^2}^2.$$

The function $\rho^{\epsilon_1,\epsilon_2} := \rho^{\epsilon_1} - \rho^{\epsilon_2}$ has vanishing integral:

$$\begin{aligned}\rho^{\epsilon_1,\epsilon_2} * \varphi^\lambda(x) &= \int \rho^{\epsilon_1,\epsilon_2}(z)\varphi^\lambda(x-z)dz \\ &= \int \rho^{\epsilon_1,\epsilon_2}(z)(\varphi^\lambda(x-z) - \varphi^\lambda(x))dz.\end{aligned}$$

Consequence:

$$\|\rho^{\epsilon_1,\epsilon_2} * \varphi^\lambda\|_{L^2}^2 = \int\int \rho^{\epsilon_1,\epsilon_2}(z_1)\rho^{\epsilon_1,\epsilon_2}(z_2)F_\lambda(z_1,z_2)dz_1dz_2,$$

for $F_\lambda(z_1,z_2) := \int(\varphi^\lambda(x-z_1) - \varphi^\lambda(x))(\varphi^\lambda(x-z_2) - \varphi^\lambda(x))dx$.

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$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\cdot) - \Pi_0^{\epsilon_2}(\cdot))(\varphi^\lambda)|^2\big] = \|(\rho^{\epsilon_1} - \rho^{\epsilon_2}) * \varphi^\lambda\|_{L^2}^2.$$

Change of variable:

$$F_\lambda(z_1, z_2) = \lambda^{-2} \int (\varphi(u - \frac{z_1}{\lambda}) - \varphi(u))(\varphi(u - \frac{z_2}{\lambda}) - \varphi(u)) du.$$

Since φ is θ-Hölder for any $\theta \in (0, 1)$:

$$|F_\lambda(z_1, z_2)| \leq C_{\theta,\varphi} \lambda^{-d} \Big|\frac{z_1}{\lambda}\Big|^\theta \Big|\frac{z_2}{\lambda}\Big|^\theta = C_{\theta,\varphi} \lambda^{-d-2\theta} |z_1|^\theta |z_2|^\theta.$$

This implies as wanted

$$\|(\rho^{\epsilon_1} - \rho^{\epsilon_2}) * \varphi^\lambda\|_{L^2}^2 \leq C_{\theta,\varphi,\rho} \lambda^{-2-2\theta} (\epsilon_1 + \epsilon_2)^{2\theta}.$$

Example 2

Let us consider $\tau = \bullet X_1$. By definition,

$$\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda) = \xi_\epsilon(\cdot_1 \varphi^\lambda(\cdot)) = \xi(\rho^\epsilon * (\cdot_1 \varphi^\lambda(\cdot))).$$

Define $\eta(\cdot) \coloneqq \cdot_1 \varphi(\cdot)$. Then $\cdot_1 \varphi^\lambda(\cdot) = \lambda \eta^\lambda(\cdot)$, thus

$$\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda) = \lambda \Pi_0^\epsilon(\bullet)(\eta^\lambda).$$

Consequence: the estimates for $\bullet X_1$, $\bullet X_2$ follow from those for $\bullet$.

Example 3

It remains to consider $\tau = \mathbf{I}$. By definition,

$$\begin{aligned}\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda) &= \int \big(\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(0)\big)\xi_\epsilon(x)\varphi^\lambda(x)dx \\ &= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx\end{aligned}$$

This is less immediate to estimate. Useful tool: chaos decomposition.

Example 3

It remains to consider $\tau = \dot{\mathsf{I}}$. By definition,

$$\begin{aligned}
\Pi_0^\epsilon(\dot{\mathsf{I}})(\varphi^\lambda) &= \int \big(\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(0)\big)\xi_\epsilon(x)\varphi^\lambda(x)dx \\
&= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx
\end{aligned}$$

This is less immediate to estimate. Useful tool: chaos decomposition.

Chaos decomposition

Because ξ is gaussian white noise:

Fact (Wiener's isometry)

There exists an (explicit) linear map

$$I\colon \bigoplus_{n\geq 0} \left(L^2(\mathbb{R}^d)\right)^{\otimes n} \to L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

such that

- *for* $f \in L^2(\mathbb{R}^d)$, $I_1(f) = \xi(f)$;
- *for* $f, g \in L^2(\mathbb{R}^d)$, $I_2(f \otimes g) = I_1(f) I_1(g) - \langle f, g\rangle$ *(NB: there is a general product formula);*
- *for* $f \in \left(L^2(\mathbb{R}^d)\right)^{\otimes n}$, $\mathbb{E}[|I_n(f)|^2] \leq n! \|f\|_{L^2}^2$.

Now we can decompose:

$$
\begin{aligned}
\Pi^{\epsilon}_0(\;)(\varphi^\lambda) &= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx \\
&= \int I_1\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)I_1\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx \\
&= \int \Big(I_2\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\big) \\
&\qquad + \langle \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot), \rho^\epsilon(\cdot - x)\rangle\Big)\varphi^\lambda(x)dx \\
&= I_2\Big(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx\Big) \\
&\qquad + \int \langle \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot), \rho^\epsilon(\cdot - x)\rangle\varphi^\lambda(x)dx
\end{aligned}
$$

Back to example 3

We get the chaos decomposition:

$$\Pi_0^{\epsilon}(\mathbf{I})(\varphi^{\lambda}) = \langle \mathsf{K}_{\epsilon}, \rho^{\epsilon} \rangle \int \varphi^{\lambda}(x)dx - \int \mathsf{K}_{\epsilon} * \rho^{\epsilon}(x)\varphi^{\lambda}(x)dx$$
$$+ I_2\Big(\int \mathsf{K}_{\epsilon}(\cdot - x) - \mathsf{K}_{\epsilon}(\cdot) \otimes \rho^{\epsilon}(\cdot - x)\varphi^{\lambda}(x)dx \Big).$$

We estimate the terms separately: recall $\mathsf{K}(x) = -\frac{1}{2\pi}\log|x|$:

$$\begin{aligned} C_{\epsilon} &:= \langle \mathsf{K}_{\epsilon}, \rho^{\epsilon} \rangle = \int\int K(\epsilon z)\rho(x-z)\rho(x)dxdz \\ &= -\int\int \frac{1}{2\pi}\log|\epsilon z|\rho(x-z)\rho(x)dxdz \\ &= -\frac{1}{2\pi}\log|\epsilon| + \mathrm{cst}_{\rho}, \end{aligned}$$

Back to example 3

We get the chaos decomposition:

$$\Pi_0^\epsilon(\overset{\bullet}{\underset{\bullet}{|}})(\varphi^\lambda) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\left(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \right).$$

For the second term, we exploit the singularity at origin of K:

$$|\mathsf{K}(x)| \lesssim |x|^{-\kappa}.$$

This is preserved by mollification: $|\mathsf{K}_\epsilon * \rho^\epsilon(x)| \lesssim |x|^{-\kappa}$. Thus:

$$\left| \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx \right|^2 \lesssim \lambda^{-2\kappa}.$$

Back to example 3

We get the chaos decomposition:

$$\Pi_0^\epsilon({\color{blue}\mathbf{I}})(\varphi^\lambda) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\Big(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \Big).$$

For the third term, we use the isometry property of I:

$$\mathbb{E}\big[\big|I_2(\cdot)\big|^2\big] \lesssim \Big\| \int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \Big\|_{L^2}^2.$$

This is "explicit" in function of K, ρ. Using again the singularity at origin this is $\lesssim \lambda^{-2\kappa}$.

We get the chaos decomposition:

$$\Pi_0^{\epsilon}(\overset{\bullet}{\underset{\bullet}{|}})(\varphi^{\lambda}) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\left(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \right).$$

Conclusion:

$$\Pi_0^{\epsilon}(\overset{\bullet}{\underset{\bullet}{|}})(\varphi^{\lambda}) = \Pi_x^\epsilon(C_\epsilon \mathbb{1})(\varphi^\lambda) + \mathrm{OK},$$

with $C_\epsilon = -\frac{1}{2\pi}\log|\epsilon| + O(1) \xrightarrow[\epsilon \to 0]{} \infty.$

$d = 2$: renormalisation

The conclusion of the previous section: $(M^\epsilon)_{\epsilon>0}$ diverges, but if we consider the *renormalised model* $\hat{M}^\epsilon$ defined by

$$\hat{\Pi}^\epsilon_x(\tau) := \begin{cases} \Pi^\epsilon_x(\overset{\bullet}{\underset{\bullet}{|}} - C_\epsilon \mathbb{1}) & \text{if } \tau = \overset{\bullet}{\underset{\bullet}{|}}, \\ \Pi^\epsilon_x(\tau) & \text{else,} \end{cases} \qquad \hat{\Gamma}^\epsilon = \Gamma^\epsilon,$$

with $C_\epsilon = -\frac{1}{2\pi}\log|\epsilon| + O(1)$ defined above, then $(\hat{M}^\epsilon)_{\epsilon>0}$ converges.

NB: By continuity of the solution map $\mathscr{M}_{\mathrm{adm}} \to \mathcal{D}^\gamma$ and of the reconstruction operator $\mathcal{D}^\gamma \to \mathcal{D}'$, the sequence $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) \in \mathcal{D}'$ with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha \bullet + \hat{f}_\epsilon \bullet),$$

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$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet),$$

converges.

The renormalised model

Explicitly:

$$
\begin{aligned}
\hat{\Pi}_x^\epsilon(\bullet)(\cdot) &= \xi_\epsilon(\cdot),\\
\hat{\Pi}_x^\epsilon(\mathcal{I}(\bullet)\bullet)(\cdot) &= (\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi(x))\xi_\epsilon(\cdot) - C_\epsilon,\\
\hat{\Pi}_x^\epsilon(\bullet X_1)(\cdot) &= \xi_\epsilon(\cdot)(\cdot_1 - x_1),\\
\hat{\Pi}_x^\epsilon(\bullet X_2)(\cdot) &= \xi_\epsilon(\cdot)(\cdot_2 - x_2),\\
\hat{\Pi}_x^\epsilon(\mathbb{1})(\cdot) &= 1,\\
\hat{\Pi}_x^\epsilon(\mathcal{I}(\bullet))(\cdot) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x),\\
\hat{\Pi}_x^\epsilon(X_1)(\cdot) &= \cdot_1 - x_1,\\
\hat{\Pi}_x^\epsilon(X_2)(\cdot) &= \cdot_2 - x_2.
\end{aligned}
$$

- $\hat{\Gamma}^\epsilon = \Gamma^\epsilon$: Π is not uniquely determined by Γ for admissible models;
- $\hat{\Gamma}^\epsilon = \Gamma^\epsilon$ is true in this case but not in general;
- $\hat{\Pi}^\epsilon$ is not multiplicative anymore: $\hat{\Pi}^\epsilon_x(\bullet)\,\hat{\Pi}^\epsilon_x(\uparrow) \neq \hat{\Pi}^\epsilon(\updownarrow)$.

Question: to what equation does the model $\hat{\Pi}^\epsilon$ correspond?
More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha \color{blue}{\bullet} \color{black} + \hat{f}_\epsilon \color{blue}{\bullet}\color{black}). \tag{13}$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

First remark: by iterating (13):

$$\hat{f}_\epsilon(x) = a_\epsilon(x)\color{blue}{\mathbb{1}}\color{black} + (\alpha + \beta a_\epsilon(x))\color{blue}{\overset{\bullet}{|}}\color{black} + c_{1,\epsilon}(x)\color{blue}{X_1}\color{black} + c_{2,\epsilon}(x)\color{blue}{X_2}\color{black},$$

So:

$$\hat{\phi}_\epsilon(x) = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)(x) = \hat{\Pi}^\epsilon_x(\hat{f}_\epsilon(x))(x) = a_\epsilon(x).$$

The renormalised equations

Question: to what equation does the model $\hat{\Pi}^\epsilon$ correspond?
More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha \bullet + \hat{f}_\epsilon \bullet).$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

Second remark:

$$\hat{\phi}_\epsilon = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) = \hat{\mathcal{R}}^\epsilon(\hat{\mathcal{K}}^\epsilon(\alpha \bullet + \hat{f}_\epsilon \bullet)) = \mathsf{K} * \big(\hat{\mathcal{R}}^\epsilon(\alpha \bullet + \beta \hat{f}_\epsilon \bullet)\big).$$

But $\hat{\Pi}^\epsilon$ is explicit:

$$\hat{\mathcal{R}}^\epsilon(\alpha \bullet + \beta \hat{f}_\epsilon \bullet)(x) = \hat{\Pi}^\epsilon_x(\alpha \bullet + \beta \hat{f}_\epsilon(x) \bullet)(x) = (\alpha + \beta \hat{\phi}_\epsilon(x))(\xi_\epsilon(x) - \beta C_\epsilon).$$

The renormalised equations

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More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet).$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

Conclusion: $\hat{\phi}_\epsilon$ solves

$$\hat{\phi}_\epsilon = \mathsf{K} * \big((\alpha + \beta\hat{\phi}_\epsilon(x))(\xi_\epsilon(x) - \beta C_\epsilon)\big).$$

Family of solutions

NB: we can replace C_ϵ by $C_\epsilon + c$ for any constant $c \in \mathbb{R}$ and still obtain convergence.
This gives a whole family of solutions indexed by $\mathbb{R}$.

Thank you for your attention!