Regularity structures – Exercise sessions

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Goal of those sessions

Consider gaussian white-noise ξ, parameters $\alpha, \beta \in \mathbb{R}$.
Throughout these sessions we will study

$$(-\Delta)\phi = (\alpha + \beta\phi)\xi, \quad x \in \mathbb{R}^d. \tag{E}$$

We will be interested in $d = 1$, $d = 2$.
Goal: Show some useful tools and explain why

1. if $d = 1$: (E) admits (local) solutions;
2. if $d = 2$: for natural smooth approximations ξ_ϵ of ξ, there exist $C_\epsilon \in \mathbb{R}$ s.t. the renormalised equations

$$(-\Delta)\phi_\epsilon = (\alpha + \beta\phi_\epsilon)(\xi_\epsilon - \beta C_\epsilon),$$

admit solutions converging to some (non-trivial) ϕ.

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Equation (E) is a "pretext" to illustrate on a simple case some of the

- analytic aspects,
- probabilistic aspects,

of regularity structures.

However: we will not discuss the algebraic aspects.

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Session 1

First remarks on (E)

A few remarks

$$(-\Delta)\phi = (\alpha + \beta\phi)\xi, \quad x \in \mathbb{R}^d. \tag{E}$$

Some (heuristic) remarks:

- this is an elliptic equation;
- we might have to play with α, β to obtain contractivity for Picard fixed-point;
- we might want to work on torus $\mathbb{T}^d$ rather than $\mathbb{R}^d$ to benefit from boundedness;
- the regularity of ξ decreases when d increases so we expect the equation to be more difficult to discuss;
- for simplicity we will not consider questions of uniqueness/boundary conditions.

Mild formulation

We will replace (E) by its mild formulation

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi), \tag{M}$$

where $\mathsf{K} =$ "$(-\Delta)^{-1}$" is the fundamental solution of the Laplacian,

$$\mathsf{K}(x) = -\frac{1}{2}|x| \quad (\mathrm{d} = 1), \qquad \mathsf{K}(x) = -\frac{1}{2\pi}\log|x| \quad (\mathrm{d} = 2).$$

Question: in which space could we set up a fixed-point for (M)?

Classical Hölder functions

Definition

Let $\alpha \in (0,1)$, and $\phi\colon \mathbb{R}^d \to \mathbb{R}$. We say that ϕ is α-Hölder if

$$|\phi(y) - \phi(x)| \lesssim |y - x|^\alpha.$$

More generally.

Definition

Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, and $\phi\colon \mathbb{R}^d \to \mathbb{R}$. We say that ϕ is α-Hölder if

$$\left|\phi(y) - \sum_{|k|\le\alpha} \frac{\phi^{(k)}(x)}{k!}(y-x)^k\right| \lesssim |y - x|^\alpha.$$

What about distributions?

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What about distributions?

Measuring regularity: the Hölder spaces

Quick calculation: if $|\phi(x) - \phi(y)| \lesssim |y - x|^\alpha$, then

$$\phi(\eta_x^\lambda) = \int \phi(y)\eta_x^\lambda(y)dy = \int \phi(x + \lambda y)\eta(y)dy = O(1).$$

If $\int \eta = 0$, then

$$\phi(\eta_x^\lambda) = \int (\phi(x + \lambda y) - \phi(x))\eta(y)dy = O(\lambda^\alpha).$$

Measuring regularity: the Hölder spaces

Generally: define

$$\mathscr{B}^r_\alpha := \left\{\eta \in \mathcal{D}(B(0,1)), \|\eta\|_{C^r} \leq 1, \int \eta(x) x^k dx = 0 \text{ for } 0 \leq |k| \leq \alpha \right\}.$$

And for $\alpha \in \mathbb{R}$, $r > -\alpha$, $K \subset \mathbb{R}^d$

$$\|\phi\|_{\mathcal{C}^\alpha_K} = \sup_{x\in K, \eta \in \mathscr{B}^r} |\phi(\eta_x)| + \sup_{x \in K, \lambda \in (0,1], \eta \in \mathscr{B}^r_\alpha} \frac{|\phi(\eta^\lambda_x)|}{\lambda^\alpha}, \qquad (1)$$

Definition (Local Hölder spaces)

For $\alpha \in \mathbb{R}$, $\mathcal{C}^\alpha_{\text{loc}}$ is the complete metrizable space corresponding to the family of seminorms (1).

When $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, $\mathcal{C}^\alpha_{\text{loc}}$ coincides with the classical Hölder functions.

Some natural questions about Hölder spaces:

1. what is gaussian white-noise ξ and what is its Hölder regularity?
2. when is pointwise multiplication $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\gamma}_{\mathrm{loc}}$ well-defined and continuous?
3. when is convolution $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ well-defined and continuous?

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Gaussian white noise

Definition

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Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. A *gaussian white noise on H* is a linear isometry

$$\begin{aligned} \xi \colon H &\to L^2(\Omega, \mathcal{F}, \mathbb{P}) \\ h &\mapsto \xi(h), \end{aligned}$$

on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $h \in H$, $\xi(h)$ is a real-valued centered gaussian variable.

NB: the isometry property means that for $h_1, h_2 \in H$, $\mathbb{E}[\xi(h_1)\xi(h_2)] = \langle h_1, h_2 \rangle$. In particular, $\xi(h) \sim \mathcal{N}(0, \|h\|^2)$.

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Kolmogorov's continuity theorem for distributions

ξ eats test-functions. Is it a Hölder distribution?

Theorem (Kolmogorov's continuity for distributions)

Let $X : L^2(\mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a continuous map. Assume there exist $\alpha \in \mathbb{R}$, $p \geq 1$ such that for all $\eta \in L^2(\mathbb{R}^d)$, $k \in \mathbb{N}$, $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[\left|X(\eta_x^{2^{-k}})\right|^p\right] \leq C_{p,\eta} 2^{-k\alpha p}.$$

Then there exists $\tilde{X} : L^2(\mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. *for all $\eta \in L^2(\mathbb{R}^d)$, $\tilde{X}(\eta) = X(\eta)$ almost surely,*
2. *for all $\omega \in \Omega$, and $\alpha' < \alpha - d/p$, $\tilde{X}(\omega) \in \mathcal{C}^{\alpha'}_{\mathrm{loc}}(\mathbb{R}^d)$.*

Technique of proof: use dyadic decompositions, such as wavelets or Littlewood-Paley decompositions.

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Let us check that ξ satisfies the condition for $\alpha = -d/2$.

$$\begin{aligned} \mathbb{E}\left[\left|\xi(\eta_x^{2^{-k}})\right|^p\right] &\leq C_p \mathbb{E}\left[\left|\xi(\eta_x^{2^{-k}})\right|^2\right]^{p/2} && \text{(gaussian moments)} \\ &= C_p \|\eta_x^{2^{-k}}\|_{L^2}^p && \text{(isometry)} \end{aligned}$$

But:

$$\begin{aligned} \|\eta_x^{2^{-k}}\|_{L^2}^2 &= \int \eta_x^{2^{-k}}(y)\eta_x^{2^{-k}}(y)dy \\ &= \int 2^{kd}\eta(2^k(y-x))2^{kd}\eta(2^k(y-x))dy \\ &= \int 2^{kd}\eta(u)^2 du = 2^{kd}\|\eta\|_{L^2}^2. \end{aligned}$$

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Consequence:

Proposition

Any gaussian white-noise ξ on $\mathbb{R}^d$ has a modification which belongs to $\mathcal{C}^{-d/2-\kappa}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $\kappa > 0$.

In fact, exploiting gaussianity in a smarter way one can prove

Proposition

Any gaussian white-noise ξ on $\mathbb{R}^d$ has a modification which belongs to the Besov space $\mathcal{B}^{-d/2}_{p,\infty;\mathrm{loc}}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

This is slightly stronger because $\mathcal{B}^{-d/2}_{p,\infty;\mathrm{loc}}(\mathbb{R}^d) \subset \mathcal{C}^{-d/2-d/p}_{\mathrm{loc}}(\mathbb{R}^d)$.

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Any gaussian white-noise ξ on $\mathbb{R}^d$ has a modification which belongs to $\mathcal{C}_{\mathrm{loc}}^{-d/2-\kappa}(\mathbb{R}^d)$ for all $\kappa > 0$.

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Direct Picard iteration works in $d = 1$

Situation

Recall the situation:

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi). \tag{M}$$

We want to solve (M) as a Picard fixed-point in some $\mathcal{C}^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$. In this section we start with the case $d = 1$.

We asked three questions about Hölder spaces:

1. what is gaussian white-noise ξ and what is its Hölder regularity?
2. when is pointwise multiplication $\mathcal{C}^{\alpha}_{\mathrm{loc}} \times \mathcal{C}^{\beta}_{\mathrm{loc}} \to \mathcal{C}^{\gamma}_{\mathrm{loc}}$ well-defined and continuous?
3. when is convolution $\mathsf{K} \colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ well-defined and continuous?

Hölder spaces question 1: white noise

Question

What is gaussian white-noise ξ and what is its Hölder regularity?

Answer

$\xi \in \mathcal{C}_{\mathrm{loc}}^{-d/2-\kappa}$ *for any* $\kappa > 0$. *For* $d = 1$*:* $\xi \in \mathcal{C}_{\mathrm{loc}}^{-1/2-\kappa}$.

Hölder spaces question 2: Young multiplication

Question

When is pointwise multiplication $\mathcal{C}^\alpha_{\mathrm{loc}} \times \mathcal{C}^\beta_{\mathrm{loc}} \to \mathcal{C}^\gamma_{\mathrm{loc}}$ well-defined and continuous?

Answer (Young multiplication)

The product $\mathcal{C}^\alpha_{\mathrm{loc}} \times \mathcal{C}^\beta_{\mathrm{loc}} \to \mathcal{C}^{\min(\alpha,\beta)}_{\mathrm{loc}}$ is canonically well-defined iff $\alpha + \beta > 0$, with continuity bounds

$$\|fg\|_{\mathcal{C}_K^{\min(\alpha,\beta)}} \leq C_{\alpha,\beta} \|f\|_{\mathcal{C}^\alpha_{K \oplus B(0,A)}} \|g\|_{\mathcal{C}^\beta_{K \oplus B(0,A)}},$$

for some $A \geq 0$.

Hölder spaces question 3: classical Schauder estimates

Question

When is convolution $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\beta}_{\mathrm{loc}}$ *well-defined and continuous?*

Answer (Schauder estimates)

For any $\alpha \in \mathbb{R}$, $\mathsf{K}\colon \mathcal{C}^{\alpha}_{\mathrm{loc}} \to \mathcal{C}^{\alpha+2}_{\mathrm{loc}}$ *with continuity bounds*

$$\|\mathsf{K} * f\|_{\mathcal{C}^{\alpha+2}_{K}} \leq C_{\alpha} \|f\|_{\mathcal{C}^{\alpha}_{K \oplus B(0,A)}},$$

for some $A \geq 0$.

First terms of Picard iteration

Let us perform Picard iteration (recall $\xi \in \mathcal{C}^{-1/2-\kappa}_{\mathrm{loc}}$):

$$\begin{aligned}
\phi_{(0)} &:= 0 && \in C^\infty, \\
\phi_{(1)} &:= \mathsf{K} * ((\alpha + \beta\phi_{(0)})\xi) && \\
&= \alpha \mathsf{K} * \xi && \in \mathcal{C}^{3/2-\kappa}_{\mathrm{loc}} \text{ (Schauder)}, \\
\phi_{(2)} &:= \mathsf{K} * ((\alpha + \beta\phi_{(1)})\xi) && \in \mathcal{C}^{3/2-\kappa}_{\mathrm{loc}} \text{ (Schauder + Young)}.
\end{aligned}$$

Young multiplication is justified because

$$\phi_{(1)} \in \mathcal{C}^{3/2-\kappa}_{\mathrm{loc}}, \quad \xi \in \mathcal{C}^{-1/2-\kappa}_{\mathrm{loc}}, \quad \frac{3}{2} - \kappa + \frac{-1}{2} - \kappa > 0.$$

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Conclusion: in $d = 1$, the iteration map

$$\mathcal{P}\colon C^{3/2-\kappa}_{\mathrm{loc}} \to C^{3/2-\kappa}_{\mathrm{loc}}$$
$$\phi \mapsto \mathsf{K} * ((\alpha + \beta\phi)\xi),$$

is well-defined with continuity estimates

$$\|\mathcal{P}(\phi_1) - \mathcal{P}(\phi_2)\|_{\mathcal{C}^{3/2-\kappa}_K} \leq C\beta\|\xi\|_{\mathcal{C}^{-1/2-\kappa}_{K\oplus B(0,A)}}\|\phi_1 - \phi_2\|_{\mathcal{C}^{3/2-\kappa}_{K\oplus B(0,A)}}.$$

Subtlety: the Lipschitz constant depends on the compact K via the norm of the noise. Possible workarounds:

- construct solutions on bounded space (e.g. on the torus);
- work with weighted Hölder spaces.

In any case, $\mathcal{P}$ becomes a contraction when setting β small enough.
NB: β might be random to construct ϕ a.s.; or if β is deterministic ϕ exists only with positive probability.

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Direct Picard iteration does not work in $d = 2$

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\phi_{(2)} &\overset{?}{:=} \mathsf{K} * ((\alpha + \beta\phi_{(1)})\xi) && \in ?
\end{aligned}$$

Problem: $\phi_{(2)}$ has no canonical meaning because

$$\phi_{(1)} \in \mathcal{C}^{1-\kappa}_{\mathrm{loc}}, \quad \xi \in \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}, \quad (1-\kappa) + (-1-\kappa) \leq 0.$$

There is no canonical product $\mathcal{C}^{1-\kappa}_{\mathrm{loc}} \times \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}$.

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Problem: $\phi_{(2)}$ has no canonical meaning because

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\phi_{(1)} \in \mathcal{C}^{1-\kappa}_{\mathrm{loc}}, \quad \xi \in \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}, \quad (1-\kappa) + (-1-\kappa) \leq 0.
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There is no canonical product $\mathcal{C}^{1-\kappa}_{\mathrm{loc}} \times \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}$.

Da Prato–Debussche trick?

In this kind of situation, one could try:

Trick (Da Prato–Debussche, 2003)

Consider $\psi := \phi - \phi_{(1)}$ and try Picard iteration on ψ.

Interpretation: remove the term of "worst regularity".

The fixed-point equation on ψ:

$$\psi = \phi - \phi_{(1)} = \mathsf{K} * ((\alpha + \beta\phi)\xi) - \alpha \mathsf{K} * \xi = \beta \mathsf{K} * (\phi\xi).$$

i.e.

$$\psi = \beta \mathsf{K} * (\phi_{(1)}\xi) + \beta \mathsf{K} * (\psi\xi).$$

NB: Recall $\phi_{(1)} = \alpha \mathsf{K} * \xi$. We assume we can give a meaning to $\phi_{(1)}\xi \in \mathcal{C}^{-1-\kappa}_{\mathrm{loc}}$ by probabilistic techniques.

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Regularity vs homogeneity

Observation: if f, g are α resp. β-Hölder for $\alpha, \beta \in (0,1)$, then

$$|f(y)g(y) - f(x)g(x)| \lesssim |y-x|^{\min(\alpha,\beta)},$$
$$|f(y) - f(x)||g(y) - g(x)| \lesssim |y-x|^{\alpha+\beta},$$

where $\alpha + \beta > \min(\alpha, \beta)$.

Interpretation: the *germ* $(f(\cdot) - f(x))_{x \in \mathbb{R}^d}$ admits "nice" multiplicativity properties with respect to its *homogeneity*.

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Some reminders on germs

Germs: definitions

Definition (Germ)

A *germ* is a family $F = (F_x)_{x \in \mathbb{R}^d}$ of distributions $F_x \in \mathcal{D}'(\mathbb{R}^d)$.

Interpretation: a germ is a family of local approximations.

Basic example: if f is a $0 < \gamma$-Hölder function, its Taylor germ:

$$T(f)_x(\cdot) = \sum_{|k| \leq \gamma} \frac{f^k(x)}{k!}(\cdot - x)^k.$$

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Germs: definitions

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A germ F is (α, γ)-coherent if
$|(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\alpha(|y-x| + \lambda)^{\gamma-\alpha}$

Definition (Homogeneity)

A germ F is $\bar{\alpha}$-homogeneous if $|F_x(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}}$.

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Germs: reconstruction

A bridge between germs and distributions is given by:

Theorem (Reconstruction, Hairer 2014, Caravenna-Zambotti 2020)

If F is (α, γ)-coherent for some $\gamma > 0$, then there exists a unique distribution $\mathcal{R}(F)$ such that $(\mathcal{R}(F) - F_x)_x$ is γ-homogeneous.

Example: if f is γ-Hölder, $\mathcal{R}(T(f)) = f$.

Useful particular case: if $x, y \mapsto F_x(y)$ is continuous, then

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Operations on germs

We want to perform a fixed-point argument at the level of germs.
Question: can one perform the operations of convolution and multiplication at the level of germs?

Germs: Schauder estimates

Let K be the fundamental solution of the Laplacian.

Theorem (Schauder estimates for germs, L.B, F. Caravenna, L. Zambotti 2022)

Let F be an (α, γ)-coherent germ. Then the germ

$$\mathcal{K}(F)_x := \mathsf{K} * F_x - \sum_{|k|<\gamma+2} \frac{(\mathsf{K} * \{F_x - \mathcal{R}(F)\})^{(k)}(x)}{k!} (\cdot - x)^k,$$

is well-defined, $((\alpha+2) \wedge 0, \gamma+2)$-coherent, and $\mathcal{R}(\mathcal{K}(F)) = \mathsf{K} * \mathcal{R}(F)$.

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Germs: products?

In general there is no canonical way of multiplying germs.

However, for continuous germs $x, y \mapsto F_x(y)$, $x, y \mapsto G_x(y)$, the germ $(FG)_x(y) \coloneqq F_x(y)G_x(y)$ satisfies (if reconstruction is applicable)

$$\mathcal{R}(FG)(x) = F_x(x)G_x(x) = \mathcal{R}(F)(x)\mathcal{R}(G)(x).$$

Thank you for your attention!

Session 2

$d = 2$: the model

Basis germs II

Setting

Recall the situation:

$$\phi = \mathsf{K} * ((\alpha + \beta\phi)\xi), \quad x \in \mathbb{R}^2.$$

Let ξ be gaussian white noise, ρ a test-function with $\int \rho = 1$, and $\xi_\epsilon = \xi * \rho^\epsilon$: this is a smooth approximation of ξ.

We consider the fixed-point equation on germs

$$F_\epsilon = \mathcal{K}((\alpha + \beta F_\epsilon)\xi_\epsilon).$$

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Form of F_ϵ

$$F_\epsilon = \mathcal{K}((\alpha + \beta F_\epsilon)\xi_\epsilon).$$

We need a nice complete metric space of germs in which to set up this fixed-point.

Let us try the Picard iteration:

$$\begin{aligned}
(F_{\epsilon,(0)})_x &:= 0, \\
(F_{\epsilon,(1)})_x &:= \mathcal{K}((\alpha + \beta F_{\epsilon,(0)})\xi_\epsilon)_x \\
&= \alpha K * \xi_\epsilon \\
&= \alpha(\underbrace{K * \xi_\epsilon - K * \xi_\epsilon(x)}_{1-\kappa \text{ homogeneous germ}}) + \alpha K * \xi_\epsilon(x) \underbrace{1}_{0-hom.}.
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Pursuing the iteration: the germs of interest are of the form

$$\sum_i f_i(x)\Pi_x^{i,\epsilon},$$

for explicit homogeneous basis germ $\Pi^{i,\epsilon}$.

The first germs by order of homogeneity:

Germ $\Pi_x^{i,\epsilon}(\cdot) =$	Hom. α	Symbol
$\xi_\epsilon(\cdot)$	$-1-\kappa$	$\bullet$
$(\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(\cdot)$	-2κ	
$\xi_\epsilon(\cdot)(\cdot_1 - x_1)$	$-\kappa$	$\bullet X_1$
$\xi_\epsilon(\cdot)(\cdot_2 - x_2)$	$-\kappa$	$\bullet X_2$
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Symbolic notation

We will use Hairer's symbolic notations:

- blue dot $\bullet$: represents the noise;
- bar: represents convolution (with K);
- symbolic multiplication (e.g. $⁝ = \bullet ⫯$, $\mathbb{1} ⫯ = ⫯$) represents multiplication.

The notations simplify the expressions. Example:

$$\xi_\epsilon(y) + 3x(\mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(y) = \Pi_x^\epsilon(\bullet + 3x ⁝)(y).$$

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Multiplicativity of Π

Remark: for all (applicable) symbols τ, τ',

$$\Pi_x^\epsilon(\tau\tau') = \Pi_x^\epsilon(\tau)\Pi_x^\epsilon(\tau'),$$

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Reexpansion operator Γ

Reexpansion

Fact

There exists a reexpansion operator Γ^ϵ defined by the relations

$$\Pi_y^{i,\epsilon} = \sum_j \Pi_x^{j,\epsilon} \Gamma_{x,y}^{j,i,\epsilon},$$

or in symbolic notation

$$\Pi_y^\epsilon(\tau) = \Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\tau)).$$

Let us calculate Γ^ϵ on some examples.

Example 0 (Polynomials)

Consider $\tau = X^k$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(X^k)) = \Pi_y^\epsilon(X^k)$, but

$$\begin{aligned}
\Pi_y^\epsilon(X^k) &= (\cdot - y)^k \\
&= \sum_{0 \le l \le k} \binom{k}{l} (x-y)^{l-k} (\cdot - x)^k \\
&= \Pi_x^\epsilon\Big(\sum_{0 \le l \le k} \binom{k}{l} (x-y)^{l-k} X^l \Big).
\end{aligned}$$

whence we set $\Gamma_{x,y}^\epsilon(X^k) = \sum_{0 \le l \le k} \binom{k}{l} (x-y)^{l-k} X^l$.

Example 1 (Noise)

Consider $\tau = \bullet$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\bullet)) = \Pi_y^\epsilon(\bullet)$, but

$$\Pi_y^\epsilon(\bullet) = \xi_\epsilon = \Pi_x^\epsilon(\bullet)$$

whence we set $\Gamma_{x,y}^\epsilon(\bullet) = \bullet$.

Example 2 (Convolution)

Consider $\tau = \dot{\mid}$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\dot{\mid})) = \Pi_y^\epsilon(\dot{\mid})$ but

$$\begin{aligned}\Pi_y^\epsilon(\dot{\mid}) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(y) \\ &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\ &= \Pi_x^\epsilon(\dot{\mid} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}),\end{aligned}$$

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Consider $\tau = \mathbf{I}$.
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&= \Pi_x^\epsilon(\mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}),
\end{aligned}$$

whence we set $\Gamma_{x,y}^\epsilon(\mathbf{I}) = \mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}$.

Example 2 (Convolution)

Consider $\tau = \uparrow$.
We want $\Pi_x^\epsilon(\Gamma_{x,y}^\epsilon(\uparrow)) = \Pi_y^\epsilon(\uparrow)$ but

$$\begin{aligned}
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&= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\
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whence we set $\Gamma_{x,y}^\epsilon(\uparrow) = \uparrow + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}$.

Example 3 (Product)

Consider $\tau = \mathsf{I}$.
We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathsf{I})) = \Pi^\epsilon_y(\mathsf{I})$ but

$$
\begin{aligned}
\Pi^\epsilon_y(\mathsf{I}) &= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= \Pi^\epsilon_x(\mathsf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet),
\end{aligned}
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whence we set $\Gamma^\epsilon_{x,y}(\mathsf{I}) = \mathsf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet$.

Example 3 (Product)

Consider $\tau = \dot{\underset{\bullet}{|}}$.
We want $\Pi^{\epsilon}_{x}(\Gamma^{\epsilon}_{x,y}(\dot{\underset{\bullet}{|}})) = \Pi^{\epsilon}_{y}(\dot{\underset{\bullet}{|}})$ but

$$
\begin{aligned}
\Pi^{\epsilon}_{y}(\dot{\underset{\bullet}{|}}) &= (\mathsf{K} * \xi_{\epsilon} - \mathsf{K} * \xi_{\epsilon}(y))\xi_{\epsilon} \\
&= \big(\mathsf{K} * \xi_{\epsilon} - \mathsf{K} * \xi_{\epsilon}(x) + \mathsf{K} * \xi_{\epsilon}(x) - \mathsf{K} * \xi_{\epsilon}(y)\big)\xi_{\epsilon} \\
&= (\mathsf{K} * \xi_{\epsilon} - \mathsf{K} * \xi_{\epsilon}(x))\xi_{\epsilon} + (\mathsf{K} * \xi_{\epsilon}(x) - \mathsf{K} * \xi_{\epsilon}(y))\xi_{\epsilon} \\
&= \Pi^{\epsilon}_{x}\big(\dot{\underset{\bullet}{|}} + (\mathsf{K} * \xi_{\epsilon}(x) - \mathsf{K} * \xi_{\epsilon}(y))\,\bullet\big),
\end{aligned}
$$

whence we set $\Gamma^{\epsilon}_{x,y}(\dot{\underset{\bullet}{|}}) = \dot{\underset{\bullet}{|}} + (\mathsf{K} * \xi_{\epsilon}(x) - \mathsf{K} * \xi_{\epsilon}(y))\,\bullet$.

Example 3 (Product)

Consider $\tau = \mathbf{I}$.
We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathbf{I})) = \Pi^\epsilon_y(\mathbf{I})$ but

$$\begin{aligned}
\Pi^\epsilon_y(\mathbf{I}) &= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= \big(\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y)\big)\xi_\epsilon \\
&= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\
&= \Pi^\epsilon_x\big(\mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet\big),
\end{aligned}$$

whence we set $\Gamma^\epsilon_{x,y}(\mathbf{I}) = \mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet$.

All calculations done, we can write explicitly:

$$\Gamma^{\epsilon}_{x,y} = \begin{array}{c} j\backslash i \\ \bullet \\ \bullet\mathcal{I}(\bullet) \\ \bullet X_1 \\ \bullet X_2 \\ \mathbb{1} \\ \mathcal{I}(\bullet) \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cccccccc} \bullet & \bullet\mathcal{I}(\bullet) & \bullet X_1 & \bullet X_2 & \mathbb{1} & \mathcal{I}(\bullet) & X_1 & X_2 \end{array} \\ \begin{pmatrix} 1 & \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) & x_1 - y_1 & x_2 - y_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) & x_1 - y_1 & x_2 - y_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

Some first properties

The reexpansion operator Γ^ϵ

- is triangular with 1 on the diagonal;
- is Hölder continuous: $|\Gamma^{\sigma,\tau}_{x,y}| \lesssim |y-x|^{\alpha_\tau - \alpha_\sigma}$;
- satisfies the group property: $\Gamma^\epsilon_{x,y}\Gamma^\epsilon_{y,z} = \Gamma^\epsilon_{x,z}$.

Consequence: the pair $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ is a *model*.

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Multiplicativity of Γ

Note that for all (applicable) symbols τ, τ',

$$\Gamma^{\epsilon}_{x,y}(\tau\tau') = \Gamma^{\epsilon}_{x,y}(\tau)\Gamma^{\epsilon}_{x,y}(\tau').$$

Example:

$$\begin{aligned}
\Gamma^{\epsilon}_{x,y}(\mathcal{I}(\Xi)\Xi) &= \mathcal{I}(\Xi)\Xi + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\,\Xi \\
&= \left(\mathcal{I}(\Xi) + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbb{1}\right)\Xi \\
&= \Gamma^{\epsilon}_{x,y}(\mathcal{I}(\Xi))\Gamma^{\epsilon}_{x,y}(\Xi).
\end{aligned}$$

NB: Multiplicativity of Γ is guaranteed in general by the algebraic construction.

Admissibility of (Π, Γ)

The model M^ϵ satisfies:

$$\begin{aligned}
\Pi_x^\epsilon(\upharpoonleft) &= \mathsf{K} * \Pi_x^\epsilon(\bullet)(\cdot) - K * \Pi_x^\epsilon(\bullet)(x),\\
\Gamma_{x,y}^\epsilon(\upharpoonleft) &= \upharpoonleft - \Pi_x^\epsilon(\upharpoonleft)(y)\mathbb{1},\\
\Gamma_{x,y}^\epsilon(\tau\tau') &= \Gamma_{x,y}^\epsilon(\tau)\Gamma_{x,y}^\epsilon(\tau') \quad \tau, \tau' \in T.
\end{aligned}$$

We say that M^ϵ is *admissible*, and note $\mathscr{M}_{\mathrm{adm}}$ the corresponding set.
(NB: In this condition, Γ is multiplicative but not necessarily Π.)

Consequence: for $M = (\Pi, \Gamma) \in \mathscr{M}_{\mathrm{adm}}$, Γ is determined by Π i.e. we may identify M and Π.

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Stationarity of (Π, Γ)

For φ, h, τ, the random processes

$$x \mapsto \Pi_x^\epsilon(\tau)(\varphi_x), \qquad x \mapsto \Gamma^\epsilon_{x,x+h}(\tau),$$

are stationary i.e. their distribution do not depend on x.

Example:

$$\begin{aligned}\Pi_x(\dot{\mid})(\varphi_x) &= \mathsf{K} * \xi_\epsilon(\varphi_x) - \mathsf{K} * \xi_\epsilon(x) \\ &= \xi\big((\mathsf{K} * \rho^\epsilon * \varphi - \mathsf{K} * \rho^\epsilon)_x\big).\end{aligned}$$

Useful consequence: for all $x \in \mathbb{R}^d$,

$$\mathbb{E}[|\Pi_x^\epsilon(\tau)(\varphi_x^\lambda)|^2] = \mathbb{E}[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^2].$$

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$$\begin{aligned}\Pi_x(\mathfrak{I})(\varphi_x) &= \mathsf{K} * \xi_\epsilon(\varphi_x) - \mathsf{K} * \xi_\epsilon(x) \\ &= \xi\big((\mathsf{K} * \rho^\epsilon * \varphi - \mathsf{K} * \rho^\epsilon)_x\big).\end{aligned}$$

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$d = 2$: the modelled distributions

Modelled distributions

We fix $M \in \mathscr{M}_{\mathrm{adm}}$ an admissible model.

Definition (Modelled distributions)

The space $\mathcal{D}^\gamma$ of modelled distributions for M is the space of functions $f(x) = \sum_\tau f_\tau(x)\tau$ with

$$|f_\tau(x)| \lesssim 1, \quad \left|f_\tau(x) - \sum_\sigma \Gamma^{\tau,\sigma}_{x,y} f_\sigma(y)\right| \lesssim |y-x|^{\gamma-\alpha_\tau}.$$

We will note $\|\cdot\|_{\mathcal{D}^\gamma_K}$ the corresponding seminorm on the compact K.

Then: the germ $F_x \coloneqq \sum_\tau f_\tau(x)\Pi_x(\tau) =: \langle \Pi, f\rangle_x$, is (α,γ)-coherent for $\alpha = \min_\tau(\alpha_\tau) = -1-\kappa$.

Modelled distributions

Iterating the equation: we want to set up the fixed-point in the subspace $\mathcal{D}^\gamma_{\mathrm{fp}}$ of $\mathcal{D}^\gamma$ of modelled distributions of the form

$$f(x) = a(x)\mathbb{1} + b(x)\stackrel{\bullet}{|} + c_1(x)X_1 + c_2(x)X_2.$$

For the model M^ϵ, this corresponds to germs

$$\begin{aligned} F_x(\cdot) &= \Pi^\epsilon_x\big(f(x)\big)(\cdot) \\ &= a(x) + b(x)\big(\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x)\big) \\ &\quad + c_1(x)(\cdot_1 - x_1) + c_2(x)(\cdot_2 - x_2). \end{aligned}$$

Multiplication operator

Theorem (Multiplication)

Let $f(x) = a(x)\mathbb{1} + b(x)\,\text{ꜝ} + c_1(x)X_1 + c_2(x)X_2$ *be in* $\mathcal{D}^{\gamma}_{\mathrm{fp}}$ *for some* $\gamma > 1$. *Set*

$$f\bullet(x) := a(x)\bullet + b(x)\,\text{ꜝ}\bullet + c_1(x)\bullet X_1 + c_2(x)\bullet X_2.$$

Then $f\bullet \in \mathcal{D}^{\gamma-1-\kappa}$ *with continuity bounds*

$$\|f\bullet\|_{\mathcal{D}_K^{\gamma-1-\kappa}} = \|f\|_{\mathcal{D}_K^{\gamma}}.$$

NB: this is a particular case of a general multiplication result for modelled distributions, see [Hai14, Theorem 4.7].

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Convolution operator

Let K be the fundamental solution of the Laplacian.

Theorem (Multi-level Schauder estimates)

There is an operator $\mathcal{K}\colon \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+2}$ such that $\mathcal{R} \circ \mathcal{K} = \mathsf{K} * \mathcal{R}$. Furthermore, $\mathcal{K}$ is explicit in function of M and $\mathcal{R}$, with continuity bounds

$$\|\mathcal{K}(f)\|_{\mathcal{D}_K^{\gamma+2}} \leq C \|\Pi\|_{\mathcal{M}_{K \oplus B(0,A)}} \|f\|_{\mathcal{D}^\gamma_{K \oplus B(0,A)}},$$

Truncation

Theorem (Truncation of modelled distributions)

Let $f \in \mathcal{D}^\gamma$ and $\gamma' < \gamma$, then truncating f at level γ' gives a modelled distribution $f^{\leq \gamma'} \in \mathcal{D}^{\gamma'}$, with continuity bounds

$$\|f^{\leq \gamma'}\|_{\mathcal{D}_K^{\gamma'}} \leq C\big(1 + \|\Gamma\|_K\big)\|f\|_{\mathcal{D}_K^\gamma},$$

Fixed-point

Consequence: the operator

$$\begin{aligned}\mathcal{P} \colon \mathcal{D}^{\gamma}_{\mathrm{fp}} &\to \mathcal{D}^{\gamma}_{\mathrm{fp}} \\ f &\mapsto \mathcal{K}^{\gamma-1-\kappa}(\alpha\bullet + \beta f\bullet),\end{aligned}$$

is well defined, with continuity bounds

$$\|\mathcal{P}(f) - \mathcal{P}(g)\|_{\mathcal{D}^{\gamma}_{K}} \leq \beta C\left(1 + \|M\|_{\mathcal{M}_{K\oplus B(0,A)}}\right)^{k_0} \|f - g\|_{\mathcal{D}^{\gamma}_{K\oplus B(0,A)}},$$

Slight problem as in $d = 1$: the Lipschitz constant depends on the compact K.

Forgetting this technical difficulty: $\mathcal{P}$ becomes a contraction for β small enough.

For the reconstruction $\phi_\epsilon \coloneqq \mathcal{R}^{\epsilon}(f^{\epsilon})$:

$$\phi_\epsilon = \mathsf{K} * ((\alpha + \beta\phi_\epsilon)\xi_\epsilon).$$

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Enhanced continuity

For the operations described above, there are "enhanced" continuity estimates: comparing modelled distributions over different models.

Consequence: there exists a continous solution map

$$S\colon U \to \bigsqcup_{M \in \mathscr{M}_{\mathrm{adm}}} \mathcal{D}^{\gamma}_{M},$$

for some open set $U = U_\beta \subset \mathscr{M}_{\mathrm{adm}}$, corresponding to the fixed-point above.

Question: does the (random) sequence $(M^\epsilon)_{\epsilon>0}$ converge in $\mathscr{M}_{\mathrm{adm}}$?

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$d = 2$: convergence of models

Kolmogorov's criterion for models

Theorem ("Kolmogorov's criterion for stationary models")

Let $(M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon))_{\epsilon>0}$ be a sequence of stationary admissible models in $\mathscr{M}_{\mathrm{adm}}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
Assume that there exist $\kappa, \theta > 0$ such that for all symbols τ of negative homogeneity, all test-functions φ, and all $p \geq 1$, $\epsilon, \epsilon_1, \epsilon_2 \in (0,1)$,

$$\mathbb{E}\big[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p} \lambda^{p(\alpha_\tau + \kappa)},$$

$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p} (\epsilon_1 + \epsilon_2)^{p\theta} \lambda^{p(\alpha_\tau + \kappa)}.$$

Then for all $p \geq 1$, the sequence $(M^\epsilon)_{\epsilon>0}$ converges in $L^p\big((\Omega, \mathcal{F}, \mathbb{P}); \mathscr{M}_{\mathrm{adm}}\big)$.

Proof: uses wavelets. See [Hai14, Theorem 10.7]

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Remark: gaussianity assumptions usually give equivalence of moments. In our case it will suffice to obtain the bounds only for $p = 2$:

$$\mathbb{E}\big[|\Pi_0^{\epsilon}(\tau)(\varphi^{\lambda})|^2\big] \le C_{\tau,\varphi}\lambda^{2\alpha_\tau+\kappa},$$
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In our case, the concerned symbols are

$$\tau \in \{\bullet, \bullet X_1, \bullet X_2, \overset{\bullet}{\underset{\bullet}{\vdots}}\}.$$

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Example 1

Let us consider the symbol $\tau = \bullet$. By definition,

$$\Pi_0^\epsilon(\bullet)(\varphi^\lambda) = \xi_\epsilon(\varphi^\lambda) = \xi * \rho^\epsilon(\varphi^\lambda) = \xi(\rho^\epsilon * \varphi^\lambda).$$

So: $\mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] = \|\rho^\epsilon * \varphi^\lambda\|_{L^2}^2$ (isometry of ξ).

Recall Young's convolution inequality in $\mathbb{R}^d$:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{if} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p, q, r \leq +\infty.$$

Consequence:

$$\begin{aligned}\mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] &\leq \|\rho^\epsilon\|_{L^1}^2 \|\varphi^\lambda\|_{L^2}^2 \quad \text{(Young)} \\ &= \|\rho\|_{L^1}^2 \|\varphi\|_{L^2}^2 \lambda^{-2},\end{aligned}$$

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Example 1

The second bound concerns

$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\cdot) - \Pi_0^{\epsilon_2}(\cdot))(\varphi^\lambda)|^2\big] = \|(\rho^{\epsilon_1} - \rho^{\epsilon_2}) * \varphi^\lambda\|_{L^2}^2.$$

The function $\rho^{\epsilon_1,\epsilon_2} := \rho^{\epsilon_1} - \rho^{\epsilon_2}$ has vanishing integral:

$$\begin{aligned}\rho^{\epsilon_1,\epsilon_2} * \varphi^\lambda(x) &= \int \rho^{\epsilon_1,\epsilon_2}(z)\varphi^\lambda(x-z)dz \\ &= \int \rho^{\epsilon_1,\epsilon_2}(z)(\varphi^\lambda(x-z) - \varphi^\lambda(x))dz.\end{aligned}$$

Consequence:

$$\|\rho^{\epsilon_1,\epsilon_2} * \varphi^\lambda\|_{L^2}^2 = \int\int \rho^{\epsilon_1,\epsilon_2}(z_1)\rho^{\epsilon_1,\epsilon_2}(z_2)F_\lambda(z_1,z_2)dz_1dz_2,$$

for $F_\lambda(z_1,z_2) := \int(\varphi^\lambda(x-z_1) - \varphi^\lambda(x))(\varphi^\lambda(x-z_2) - \varphi^\lambda(x))dx$.

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Change of variable:

$$F_\lambda(z_1, z_2) = \lambda^{-2} \int (\varphi(u - \frac{z_1}{\lambda}) - \varphi(u))(\varphi(u - \frac{z_2}{\lambda}) - \varphi(u))du.$$

Since φ is θ-Hölder for any $\theta \in (0,1)$:

$$|F_\lambda(z_1, z_2)| \leq C_{\theta,\varphi}\lambda^{-d}\Big|\frac{z_1}{\lambda}\Big|^\theta \Big|\frac{z_2}{\lambda}\Big|^\theta = C_{\theta,\varphi}\lambda^{-d-2\theta}|z_1|^\theta|z_2|^\theta.$$

This implies as wanted

$$\|(\rho^{\epsilon_1} - \rho^{\epsilon_2}) * \varphi^\lambda\|_{L^2}^2 \leq C_{\theta,\varphi,\rho}\lambda^{-2-2\theta}(\epsilon_1 + \epsilon_2)^{2\theta}.$$

Example 2

Let us consider $\tau = \bullet X_1$. By definition,

$$\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda) = \xi_\epsilon(\cdot_1\varphi^\lambda(\cdot)) = \xi(\rho^\epsilon * (\cdot_1\varphi^\lambda(\cdot))).$$

Define $\eta(\cdot) \coloneqq \cdot_1\varphi(\cdot)$. Then $\cdot_1\varphi^\lambda(\cdot) = \lambda\eta^\lambda(\cdot)$, thus

$$\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda) = \lambda\Pi_0^\epsilon(\bullet)(\eta^\lambda).$$

Consequence: the estimates for $\bullet X_1$, $\bullet X_2$ follow from those for $\bullet$.

Example 3

It remains to consider $\tau = \mathbf{I}$. By definition,

$$\begin{aligned}\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda) &= \int \big(\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(0)\big)\xi_\epsilon(x)\varphi^\lambda(x)dx \\ &= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx\end{aligned}$$

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Chaos decomposition

Because ξ is gaussian white noise:

Fact (Wiener's isometry)

There exists an (explicit) linear map

$$I\colon \bigoplus_{n\geq 0} \left(L^2(\mathbb{R}^d)\right)^{\otimes n} \to L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

such that

- *for* $f \in L^2(\mathbb{R}^d)$, $I_1(f) = \xi(f)$;
- *for* $f, g \in L^2(\mathbb{R}^d)$, $I_2(f \otimes g) = I_1(f) I_1(g) - \langle f, g\rangle$ *(NB: there is a general product formula);*
- *for* $f \in \left(L^2(\mathbb{R}^d)\right)^{\otimes n}$, $\mathbb{E}[|I_n(f)|^2] \leq n! \|f\|_{L^2}^2$.

Back to example 3

Now we can decompose:

$$
\begin{aligned}
\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda) &= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx \\
&= \int I_1\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)I_1\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx \\
&= \int \Big(I_2\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\big) \\
&\qquad + \langle \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot), \rho^\epsilon(\cdot - x)\rangle\Big)\varphi^\lambda(x)dx \\
&= I_2\Big(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx\Big) \\
&\qquad + \int \langle \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot), \rho^\epsilon(\cdot - x)\rangle\varphi^\lambda(x)dx
\end{aligned}
$$

We get the chaos decomposition:

$$\Pi_0^{\epsilon}(\,{\vdots}\,)(\varphi^{\lambda}) = \langle \mathsf{K}_{\epsilon}, \rho^{\epsilon} \rangle \int \varphi^{\lambda}(x)dx - \int \mathsf{K}_{\epsilon} * \rho^{\epsilon}(x)\varphi^{\lambda}(x)dx$$
$$+ I_2\bigg(\int \mathsf{K}_{\epsilon}(\cdot - x) - \mathsf{K}_{\epsilon}(\cdot) \otimes \rho^{\epsilon}(\cdot - x)\varphi^{\lambda}(x)dx \bigg).$$

We estimate the terms separately: recall $\mathsf{K}(x) = -\frac{1}{2\pi}\log|x|$:

$$\begin{aligned} C_{\epsilon} := \langle \mathsf{K}_{\epsilon}, \rho^{\epsilon} \rangle &= \int\int K(\epsilon z)\rho(x-z)\rho(x)dxdz \\ &= -\int\int \frac{1}{2\pi}\log|\epsilon z|\rho(x-z)\rho(x)dxdz \\ &= -\frac{1}{2\pi}\log|\epsilon| + \mathrm{cst}_{\rho}, \end{aligned}$$

We get the chaos decomposition:

$$\Pi_0^\epsilon(\;)(\varphi^\lambda) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\Big(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \Big).$$

For the second term, we exploit the singularity at origin of K:

$$|\mathsf{K}(x)| \lesssim |x|^{-\kappa}.$$

This is preserved by mollification: $|\mathsf{K}_\epsilon * \rho^\epsilon(x)| \lesssim |x|^{-\kappa}$. Thus:

$$\Big| \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx \Big|^2 \lesssim \lambda^{-2\kappa}.$$

Back to example 3

We get the chaos decomposition:

$$\Pi_0^\epsilon(\mathsf{I})(\varphi^\lambda) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\Big(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \Big).$$

For the third term, we use the isometry property of I:

$$\mathbb{E}\big[\big|I_2(\cdot)\big|^2\big] \lesssim \Big\| \int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \Big\|_{L^2}^2 .$$

This is “explicit” in function of K, ρ. Using again the singularity at origin this is $\lesssim \lambda^{-2\kappa}$.

Back to example 3

We get the chaos decomposition:

$$\Pi_0^\epsilon({\color{blue}\mathrm{I}})(\varphi^\lambda) = \langle \mathsf{K}_\epsilon, \rho^\epsilon \rangle \int \varphi^\lambda(x)dx - \int \mathsf{K}_\epsilon * \rho^\epsilon(x)\varphi^\lambda(x)dx$$
$$+ I_2\left(\int \mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot) \otimes \rho^\epsilon(\cdot - x)\varphi^\lambda(x)dx \right).$$

Conclusion:

$$\Pi_0^\epsilon({\color{blue}\mathrm{I}})(\varphi^\lambda) = \Pi_x^\epsilon(C_\epsilon {\color{blue}\mathbb{1}})(\varphi^\lambda) + \mathrm{OK},$$

with $C_\epsilon = -\frac{1}{2\pi}\log|\epsilon| + O(1) \xrightarrow[\epsilon \to 0]{} \infty$.

$d = 2$: renormalisation

The conclusion of the previous section: if we consider the *renormalised model* $\hat{M}^\epsilon$ defined by

$$\hat{\Pi}^\epsilon_x(\tau) := \begin{cases} \Pi^\epsilon_x(\mathbf{I} - C_\epsilon \mathbb{1}) & \text{if } \tau = \mathbf{I}, \\ \Pi^\epsilon_x(\tau) & \text{else,} \end{cases} \qquad \hat{\Gamma}^\epsilon = \Gamma^\epsilon,$$

with $C_\epsilon = -\frac{1}{2\pi}\log|\epsilon| + O(1)$ defined above, then $(\hat{M}^\epsilon)_{\epsilon>0}$ converges.

NB: By continuity of the solution map $\mathscr{M}_{\mathrm{adm}} \to \mathcal{D}^\gamma$ and of the reconstruction operator $\mathcal{D}^\gamma \to \mathcal{D}'$, the sequence $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) \in \mathcal{D}'$ with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha \bullet + \hat{f}_\epsilon \bullet),$$

converges.

The conclusion of the previous section: if we consider the *renormalised model* $\hat{M}^\epsilon$ defined by

$$\hat{\Pi}^\epsilon_x(\tau) := \begin{cases} \Pi^\epsilon_x(\Psi - C_\epsilon \mathbb{1}) & \text{if } \tau = \Psi, \\ \Pi^\epsilon_x(\tau) & \text{else}, \end{cases} \qquad \hat{\Gamma}^\epsilon = \Gamma^\epsilon,$$

with $C_\epsilon = -\frac{1}{2\pi} \log|\epsilon| + O(1)$ defined above, then $(\hat{M}^\epsilon)_{\epsilon>0}$ converges.

NB: By continuity of the solution map $\mathscr{M}_{\text{adm}} \to \mathcal{D}^\gamma$ and of the reconstruction operator $\mathcal{D}^\gamma \to \mathcal{D}'$, the sequence $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) \in \mathcal{D}'$ with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet),$$

converges.

The renormalised model

Explicitly:

$$
\begin{aligned}
\hat{\Pi}^{\epsilon}_x(\bullet)(\cdot) &= \xi_\epsilon(\cdot),\\
\hat{\Pi}^{\epsilon}_x(\overset{\bullet}{\underset{\bullet}{|}})(\cdot) &= (\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi(x))\xi_\epsilon(\cdot) - C_\epsilon,\\
\hat{\Pi}^{\epsilon}_x(\bullet X_1)(\cdot) &= \xi_\epsilon(\cdot)(\cdot_1 - x_1),\\
\hat{\Pi}^{\epsilon}_x(\bullet X_2)(\cdot) &= \xi_\epsilon(\cdot)(\cdot_2 - x_2),\\
\hat{\Pi}^{\epsilon}_x(\mathbb{1})(\cdot) &= 1,\\
\hat{\Pi}^{\epsilon}_x(\overset{\bullet}{|})(\cdot) &= \mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x),\\
\hat{\Pi}^{\epsilon}_x(X_1)(\cdot) &= \cdot_1 - x_1,\\
\hat{\Pi}^{\epsilon}_x(X_2)(\cdot) &= \cdot_2 - x_2.
\end{aligned}
$$

- $\hat{\Gamma}^{\epsilon} = \Gamma^{\epsilon}$: Π is not uniquely determined by Γ for admissible models;
- $\hat{\Gamma}^{\epsilon} = \Gamma^{\epsilon}$ is true in this case but not in general;
- $\hat{\Pi}^{\epsilon}$ is not multiplicative anymore: $\hat{\Pi}^{\epsilon}(\bullet\,\uparrow) \neq \hat{\Pi}^{\epsilon}(\updownarrow)$.

Question: to what equation does the model $\hat{\Pi}^\epsilon$ correspond?
More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha \bullet + \hat{f}_\epsilon \bullet). \tag{50}$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

First remark: by iterating (50):

$$\hat{f}_\epsilon(x) = a_\epsilon(x)\mathbf{1} + (\alpha + \beta a_\epsilon(x))\overset{\bullet}{|} + c_{1,\epsilon}(x)X_1 + c_{2,\epsilon}(x)X_2,$$

So:

$$\hat{\phi}_\epsilon(x) = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)(x) = \hat{\Pi}^\epsilon_x(\hat{f}_\epsilon(x))(x) = a_\epsilon(x).$$

The renormalised equations

Question: to what equation does the model $\hat{\Pi}^\epsilon$ correspond?
More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet).$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

Second remark:

$$\hat{\phi}_\epsilon = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) = \hat{\mathcal{R}}^\epsilon(\hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet)) = \mathsf{K} * \left(\hat{\mathcal{R}}^\epsilon(\alpha\bullet + \beta\hat{f}_\epsilon\bullet)\right).$$

But $\hat{\Pi}^\epsilon$ is explicit:

$$\hat{\mathcal{R}}^\epsilon(\alpha\bullet+\beta\hat{f}_\epsilon\bullet)(x) = \hat{\Pi}^\epsilon_x(\alpha\bullet+\beta\hat{f}_\epsilon(x)\bullet)(x) = (\alpha+\beta\hat{\phi}_\epsilon(x))(\xi_\epsilon(x)-\beta C_\epsilon).$$

The renormalised equations

Question: to what equation does the model $\hat{\Pi}^\epsilon$ correspond?
More precisely: let $\hat{f}_\epsilon$ be a modelled distribution with

$$\hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(\alpha\bullet + \hat{f}_\epsilon\bullet).$$

What equation does $\hat{\phi}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)$ satisfy?

Conclusion: $\hat{\phi}_\epsilon$ solves

$$\hat{\phi}_\epsilon = \mathsf{K} * \big((\alpha + \beta\hat{\phi}_\epsilon(x))(\xi_\epsilon(x) - \beta C_\epsilon)\big).$$

Family of solutions

NB: we can replace C_ϵ by $C_\epsilon + c$ for any constant $c \in \mathbb{R}$ and still obtain convergence.
This gives a whole family of solutions indexed by $\mathbb{R}$.

Thank you for your attention!