Invariant Theory of Ore Extensions

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Introduction

This is a course on rings that are similar to the polynomial rings: Ore extensions.

We begin by remembering results of classical invariant theory.

An interesting direction in noncommutative invariant theory, generalizing the case of the polynomial algebra, is the invariant theory for free and relatively free algebras. For these:

$\diamond$: E. Formanek. Noncommutative invariant theory. Contemp. Math., 43:87-119, 1985.

Introduction

Pre-requisites:

- Some commutative algebra, such as in Introduction to Commutative Algebra by Atiyah, Macdonald.
- Some algebraic geometry - such as Chapter I from The Red Book of Varieties and Schemes by Mumford, Chapter I from Algebraic Geometry by Hartshorne; and elementary aspects of quotients of varieties, such as in Lectures in Invariant Theory by Dolgachev.
- Theory of Ore Extensions, Ore localizations and Noetherian Rings, such as in Introduction to Noncommutative Noetherian Rings by Goodearl and Warfield.

- However, most of this course can be understood with a very modest previous knowledge.
- All theorems, even those proved during the course, will have indicated the place where the proof was borrowed from.
- The choice of topics followed three rules: importance in general invariant theory; beauty; and personal interest of the lecturer.
- A detailed bibliography can be found in the end of these slides.

We begin with a classical result, which contaitns the seed of great generality.

Theorem

Let R be any unital ring, and call $P_n(R) = R[x_1, \ldots, x_n]$ the polynomial algebra in n indeterminates over R. Consider the action of the symmetric group S_n on $P_n(R)$: $\sigma \in S_n$ fix R and send x_i to $x_{\sigma(i)}$. Then $P_n(R)^{S_n} = R[p_1, \ldots, p_n]$, where the p_i are the elementary symmetric polynomials; moreover, they are algebraically independent.

Proof.

Jacobson, Basic Algebra I. □

The base ring from now on is an infinite field k.

Definition

Let G be a finite group of automorphisms of $k[x_1, \dots, x_n]$ that is contained in $GL_n(k)$; that is, every x_i is sent to a linear combination of the indeterminates. Then G is called a finite group of linear automorphisms. In an equivalent way: we have a representation of G in a finite dimensional vector space V and make G act as algebra automorphisms of $S(V^*)$.

Classical Invariant Theory: Finiteness of Generators

$\diamond$: is $P_n(k)^G$ a finitely generated algebra?

- Hilbert considered the problem of finite generation of the algebra of invariants for the action of $SL_n(\mathbb{C})$ in symmetric powers of the natural representation. His proof implies the finite generation of invariants for finite groups of linear automorphisms acting on the polynomial algebra.
- Noether later showed ($char\ k = 0$) that the degree of the generators is at most $|G|$. So the number of generators can be choosen to be at most $\binom{n+|G|}{n}$

These proofs can be found in Benson, Polynomial Invariants of Finite Groups.

Classical Invariant Theory: Finiteness of Generators

Theorem

(Noether) Let R be a commutative finitely generated k-algebra and G a finite group of algebra automorphisms of R. Then R^G is also a finitely generated k-algebra, and moreover R is a finitely generated R^G module.

Proof.

Dolgachev, Lectures on Invariant Theory. □

Classical Invariant Theory: Chevalley-Shephard-Todd Theorem

Definition

A linear automorphism g of a finite dimensional vector space V which fixes pointwise a hyperplane and has finite order is called a pseudo-reflection (if the order is 2, it is called a reflection). A finite group of automorphisms of V is called a pseudo-reflection group (reflection group) if it is generated by the pseudo-reflections (reflections) it contains.

When $k = \mathbb{C}$ we have the unitary reflection groups of Shephard and Todd. When k is $\mathbb{R}$ we can only have reflections, and the class coincide with the finite Coxeter groups. When k is $\mathbb{Q}$, we have the Weyl groups.

Classical Invariant Theory: Chevalley-Shephard-Todd Theorem

Theorem

(Chevalley-Shephard-Todd) Let $P_n(k)$ be the polynomial algebra and $G \subset GL_n(k)$ a finite group o linear automorphisms of it, with $|G|$ coprime to char k. Then the following are equivalent:

- $P_n(k)^G \cong P_n(k)$ *(geometrically: the varieties $\mathbb{A}^n/G$ and $\mathbb{A}^n$ are isomorphic);*
- G *is a pseudo-reflection group in its natural representation;*
- $P_n(k)^G$ *is a regular ring (geometrically: the quotient variety $\mathbb{A}^n/G$ is smooth);*
- $P_n(k)$ *is a free/projective/flat $P_n(k)^G$ module (geometrically: the quotient map $\mathbb{A}^n \to \mathbb{A}^n/G$ is flat).*

Proof.

Benson, Polynomial Invariants of Finite Groups. □

And how about the question of when $\mathbb{A}^n/G$ is a rational variety?

Problem

(Noether's Problem, 1913) Given a finite group G acting linearly on the rational function field $k(x_1, \ldots, x_n)$, when $k(x_1, \ldots, x_n)^G \cong k(x_1, \ldots, x_n)$?

Noether's Problem

- Related to rationality questions of many moduli spaces.
- Related to the question of rationality of the center of the division ring of generic matrices (Procesi).
- Related to the Inverse Problem of Galois Theory — one of Noether original motivations.

Here are some important cases of positive solution. In this slide, *char* $k = 0$.

- When $n = 1$ and 2; and when k is algebraically closed, for $n = 3$ (Burnside).
- When the natural representation of G decomposes as a direct sum of one dimensional representations (Fischer).
- For finite abelian groups acting by transitive permutations of the variables $x_1, \ldots, x_n$, the problem is settled by the work of Lenstra.
- For $k(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the symmetric group S_n permutes the variables y_i, x_i simultaneously (Mattuck).
- For the alternating groups $\mathcal{A}_3, \mathcal{A}_4$, and $\mathcal{A}_5$ (Maeda). Open question for $n > 5$.

- When G is a psudo-reflections group (by Chevalley-Shephard-Todd Theorem and †).

Counter-examples are also known.

†: Let D be a commutative domain with field of fractions F, and let G be a finite group of automorphisms of D. Then $\mathit{Frac}\, D^G = F^G$.

This remains true in the noncommutative world but it is much more difficult to prove as we shall see.

Miyata's Theorem

A key tool in the modern treatment of these questions is:

Theorem

(Miyata)
Let K be a field, $S = K[x]$ with F as field of fractions. Let G be a subgroup of ring automorphisms of S, not necessarely finite, such that $G(K) \subseteq K$. Then two cases are possible:

- *$S^G \subseteq K$, and then $S^G = F^G = K^G$.*
- *S^G is not contained in K. In this case, let u be any element of S^G not belonging to K of minimal degree in x, among all the invariants in S^G not in K. Then $S^G = K^G[u]$ and $F^G = K^G(u)$.*

Proof.

T. Miyata, Invariants of certain groups I, Nagoya Math. J., 42: 68-73, 1971. □

We shall now discuss the structure of $Aut_k\, P_n(k)$.

Definition

The set $Aff_n(k)$ of affine automorphisms of $P_n(k) = k[x_1, \ldots, x_n]$ consists of automorphisms g which acts in the following form:

$$g(x_i) = \sum_{j=1}^{n} a_{ij} x_j + b_i,\ i = 1, \ldots, n,$$

$$(a_{ij})_{i,j=1,\ldots,n} \in GL_n(k) \in, b_i \in k, i = 1, \ldots, n.$$

Definition

The set $J_n(k)$ (from Jonqueries) consists of the automorphisms of $k[x_1, \ldots, x_n]$ of the following form, called triangular:

$$g(x_i) = \lambda_i x_i + f(x_{i+1}, \ldots, x_n), \lambda_i \neq 0 \in k,$$

$$f \in k[x_{i+1}, \ldots, x_n], i = 1, \ldots, n-1;$$

$$g(x_n) = \lambda_n x_n + f_n, \lambda_n \neq 0 \in k, f_n \in k.$$

- For $n = 2$ we have the celebrated Jung-van der Kulk Theorem, that says that $Aut_k\, P_2(k)$ is the amalgamated free product of $Aff_2(k)$ and $J_2(k)$ over their intersection. For a proof: A. van der Essen, Polynomial automorphisms and the Jacobian Conjecture.
- In the general, the subgroup of $Aut_k\, P_n(k)$ generated by $Aff_n(k)$ and $J_n(k)$ is called the group of tame automorphisms. That not all automorphisms are tame was conjectured by Nagata, who suggested the following counterexample (for $n = 3$), later confirmed by Urmibaev and Shestakov:

$$x_1 \mapsto x_1 - 2x_2(x_3x_1 - x_2^2) - x_3(x_3x_1 - x_2^2)^2,$$

$$x_2 \mapsto x_2 + (x_3x_1 + x_2)^2, x_3 \mapsto x_3.$$

Let *char* $k = 0$.

Theorem

Every finite group of automorphisms G of $P_2(k)$ is conjugated inside $Aut_k\, P_2(k)$ to a group of linear automorphisms.

Proof.

F. Dumas, An introduction to noncommutative polynomial invariants. □

Heuristics: a good "noncommutative polynomial extension" of a ring R, let's say S, would be freely generated on the left as a R-module by powers of a certain element $x \in S$: $1, x, \ldots, x^n, \ldots$. We should have a well behaved notion of degree:

- $deg(R) = 0$;
- $deg(x^n) = n$;
- $deg(ss') = deg(s) + deg(s'), s, s' \in S$;
- $deg(s + s') \leq max\{deg(s), deg(s')\}, s, s' \in S$.

Definition

Let R be any ring, α a ring automorphism of R, and δ an α-derivation: $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$. A ring $S = R[x, \alpha, \delta]$, containing R, being a free left R-module with basis $\{1, x, \ldots, x^n, \ldots\}$ and such that $xr = \alpha(r)x + \delta(r)$ always exists and it is unique modulo isomorphism. Such a ring is called an Ore extension of R.

If $\alpha = id$ we write $R[x, \delta]$; if $\delta = 0$ we write $R[x, \alpha]$.

Let's present our main examples; the base field allways having zero characteristics.

Definition

A mutiplicatively antisymmetric matix over a field k is a $n \times n$ matrix $Q = (q_{ij})$ with entries in k^* such that $q_{ii} = 1, q_{ij} = q_{ji}^{-1}$ for all i, j. Given such a matrix, the quantum affine space of dimension n is the algebra $O_Q(k^n)$ given by generators $x_1, \ldots, x_n$ and relations $x_i x_j = q_{ij} x_j x_i$, for all i, j.

Since the Gelfand-Kirillov dimension — or the global dimension — of such algebras is n, the size of the matrix can be recovered.

Their realization as Ore extensions is a follows:

$$k[x_1][x_2,\alpha_2]\dots[x_n,\alpha_n];\ \alpha_i(x_j)=q_{ij}x_j, j<i.$$

We find useful to introduce a special notation for matrices of size 2×2, parametrized by a scalar $q\neq 1\in k^*$: $k_q[x,y]$ have the generators x,y and relation $yx=qxy$. This ring is called the quantum plane.

char $k = 0$, allways, when discussing:

Definition

(Weyl Algebra) Call $A_n(k)$ *the iterated Ore extension* $k[x_1, \ldots, x_n][y_1, \partial_1] \ldots [y_n, \partial_n]$, *where* ∂_i *is a derivation with relation to* x_i *which sends every* y_j *with* $j < i$ *to* 0.

It is usefull to give two other characterizations of the Weyl Algebra.

- It is the quotient of the free associative algebra in variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ by the relations $[x_i, x_j] = [y_i, y_j] = 0$, $i, j = 1, \ldots, n$ and $[y_i, x_j] = \delta_{ij}$, $i, j = 1, \ldots, n$.

Definition

(Differential operator rings) Let A be a finitely generated commutative k-algebra. Define, inductively,
$D(A)_0 = \{d \in End_k(A) | [d, a] = 0, \forall a \in A\}$, *and for all* $i > 0$,
$D(A)_i = \{d \in End_k(A) | [d, a] \in D(A)_{i-1}, \forall a \in A\}$. *Clearly we have* $D(A)_i \subseteq D(A)_j$ *if* $i < j$ *and* $D(A)_i D(A)_j \subseteq D(A)_{i+j}$. *We call* $D(A) = \bigcup_{i=0}^{\infty} D(A)_i$, *and the associative algebra thus obtained is called the ring of differential operators on A.*

- $A_n(k) = D(k[x_1, \ldots, x_n])$.

$\diamond$: The Weyl Algebras $A_n(k), A_m(k)$, for $m \neq n$ are pairwise non-isomorphic: take any dimension you like.

Our last main example is the first quantum Weyl Algebra $A_1^q(k)$, given by generators x, y and relation $yx - qxy = 1$, $q \neq 1 \in k^*$. As an Ore extension, it is $k[x][y, \alpha, \delta]$, where $\alpha(x) = qx$, and δ a α-derivation, the Jackson derivative:

$$\delta(f(x)) = \frac{f(qx) - f(x)}{qx - x},\ f \in k[x].$$

The n-th quantum Weyl Algebra is $A_n^q(k) = A_1^q(k) \otimes \ldots \otimes A_1^q(k)$, n times.

Lema

$x^n r = \alpha^n(r)x^n + \ldots + \delta^n(r), r \in R.$

Proposition

If R is a domain and S is an Ore extension of it, then it is also a domain; it have the same units as R.

Proposition

If R is a division ring and S is an Ore extension of it, then given $a, b \in S$ there exists unique $q, r \in S$ with $a = qb + r$ with $\deg r < \deg b$; a right-hand version of the Euclidian algorithm also holds.

Corollary

The Weyl Algebra $A_n(k)$ has only k as units. In particular, it has no inner automorphisms besides the identity. Recall also that the Weyl Algebras are simple rings.

Proof.

Björk, Rings of Differential Operators. □

One last important result to recall:

Theorem

(Extended Hilbert Basis Theorem) Let R be a ring, and $S = R[x, \alpha, \delta]$ be an Ore extension. If R is left (right) Noetherian, then so is S.

We are now going to see a result of Montgomery and Small which generalizes the Hilbert-Noether theorem in the noncommutative setting — and in particular for Ore extensions. To do so, and for many subsequent work, we need the notion of a skew group ring.

Definition

Let R be a ring and G be a finite group that acts on R by automorphisms. The skew group ring, $R * G$, is freely generated as left R-module by the elements of G, with multiplication of basis elements defined as follows: $rgsh = rg^{-1}(s)gh$, $r, s \in R, g, h \in G$. For convenience, we write $rg = gr^g$.

Skew Group Rings

Lema

Consider a skew group ring $S = R * G$. Suppose $|G|$ is invertible in R, and let $e = 1/|G| \sum_{g \in G} g$. Then:

1. $e^2 = e$
2. $eg = e$, $g \in G$.
3. $eS = eR$
4. $eSe = eR^G \cong R^G$

Proof.

Montgomery, Fixed Rings of Finite Groups of Automorphisms of Associative Rings. □

Montgomery-Small Theorem

Theorem

(Montgomery-Small) Let R be a finitely generated k-algebra, non necessarely commutative, and let G be a finite group of algebra automorphisms of R, with $|G|^{-1} \in R$. If R is left or right Noetherian, R^G is a finitely generated k-algebra.

Proof.

F. Dumas, An introduction to noncommutative polynomial invariants. □

We now specialize our discussion to invariants of the Weyl Algebra. We begin by recalling some facts from Morita Theory.

Proposition

Two rings T and S are Morita equivalent if, and only if, there exists $n \geq 1$ and an indempotent $e \in M_n(S)$, such that $T \cong eM_n(S)e$ and $M_n(S)eM_n(S) = M_n(S)$.

Proof.

Jacobson, Basic Algebra II. □

Proposition

Let V be a right A-module, let $B = End_A(V)$, and consider V as a left B-module in the natural way. Then the following are equivalent:

- *V is a generator for Mod_A;*
- *V is a finitely generated projective B-module and $A = End_B(V)$.*

Proof.

Montgomery, Fixed Rings of Finite Groups of Automorphisms of Associative Rings. □

More on Skew Group Rings

Let $S = R * G$ be a skew group ring. Then R has a natural structure of right S-module.

Proposition

1. $End_S(R) \cong R^G$;
2. *If R is simple and G consists of outer automorphisms, then S is simple;*
3. *Under the same conditios as above, R is an S generator.*
4. *Under the conditions as above, and if $|G|^{-1} \in R$, then R^G and S are Morita Equivalent;*

Proof.

Montgomery, Fixed Rings of Finite Groups of Automorphisms of Associative Rings. □

Corollary

Let G be any finite group of automorphisms of the Weyl Algebra $A_n(k)$. Then:

1. $A_n(k)^G$ *is finitely generated, simple, Noetherian algebra.*
2. $A_n(k)$ *is a finitely generated projective* $A_n(k)^G$*-module.*

The second item of last theorem resembles one of the equivalent conditions of Chevalley-Shephard-Todd theorem for the polynomial algebra. However, if we ask for groups G such that $A_n(k)^G \cong A_n(k)$, we find:

Theorem

(Alev-Polo) Suppose k algebraically closed. Let G be a finite group of $\mathrm{Aut}_k A_n(k)$. If $A_n(k)^G \cong A_n(k)$, then G is trivial.

Proof.

J. Alev, P. Polo, A Rigidity Theorem for Finite Group of Automorphisms on Enveloping Algebras of Semissimple Lie Algebras, Advances in Mathematics, 111:208-226, 1995. □

In fact even more is true:

Theorem

(Tikaradze) If Γ is a noncommutative $\mathbb{C}$-domain and G a finite subgroup of algebra automorphisms of it, then $\Gamma^G \cong A_n(\mathbb{C})$ for some Weyl Algebra implies $G = id$ and $\Gamma \cong A_n(\mathbb{C})$.

Proof.

arXiv:1708.07923. □

We are now going to see a geometric fact that gives us some heuristics on why the Weyl Algebra is rigid.

Definition

Let G be a finite group of automorphisms of the polynomial algebra $P_n(k)$. It can be extended to a group of algebra automorphisms of the Weyl Algebra $A_n(k)$, seem as a ring of differential operators on $P_n(k)$; $gD(f) = g(D(g^{-1}f))$, $f \in P_n(k), g \in G, D \in A_n(k)$. Groups of automorphisms of the Weyl Algebra that arise in such a way from groups of linear automorphisms of the polynomial algebra are again called linear.

Theorem

Suppose k algebraically closed, and let G be a finite group of linear automorphisms of the Weyl Algebra $A_n(k)$. The inclusion $P_n(k)^G \to P_n(k)$ induces, by restriction of domain, a map $A_n(k)^G \to D(P_n(k)^G)$ that is allways injective, and is an isomorphism if, and only if, G contains no pseudo-reflections appart from id.

Proof.

Levasseur, Anneaus d'operateurs differentiels, LNM 867. □

Remark: $g(D(f)) = gDg^{-1}(gf) = D(f)$.

Invariant Theory of The Weyl Algebra: Finding Generators and Relations

We now move to the task of finding generators and relations for the invariants of the Weyl Algebra. In case we are interested in a minimal generating set, we have the following impressive result.

Theorem

(Levasseur, Stafford) $A_n(k)^G$, when G is a finite group of linear automorphisms, is generated by $k[x_1, \ldots, x_n]^G$ and $k[y_1, \ldots, y_n]^G$, which are invariants of two polynomial algebras sitting inside $A_n(k)$: the multiplication operators and constant coefficients differential operators.

Proof.

Levasseur and Stafford, Invariant Differential Operators and an homomorphism of Harish-Chandra, J. Am. Math. Soc. 8:365-372,1995. □

Invariant Theory of The Weyl Algebra: Finding Generators and Relations

The Weyl Algebra $A_n(k)$ have two natural associated filtrations: $\mathcal{F}$, the filtration by order of differential operators, and $\mathcal{B}$, the Bernstein filtration.

- The graded associated algebra, in both cases, is the polynomial algebra in $2n$ indeterminates;
- Let G be a finite group of linear automorphisms acting on $A_n(k)$. Then $A_n(k)^G$ inherits two natural filtrations from $\mathcal{F}$ and $\mathcal{B}$, and in both cases we have that with this induced filtration, $gr\, A_n(k)^G \cong (gr\, A_n(k))^G$.

By passing to graded associated algebra, generators and relations can be found, and these can lifted to $A_n(k)^G$ to obtain generators and relations for it. A detailed discussion of this can be found in:

$\diamond$: W. N. Traves, Invariant Theory and Differential Operators, Radon Series Comp. Appl. Math., 1:1-24, 2007.

A last proposition showing differences between invariants of the polynomial algebra and the Weyl Algebra.

Proposition

Let G be any — non necessarely finite — group of linear automorphisms of $A_n(k)$. Then the element $x = \sum_{i=1}^{n} x_i y_i$ is always invariant.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

We are now going to consider the group $Aut_{\mathbb{C}}\, A_1(\mathbb{C})$.

The Automorphism Group of the Weyl Algebra

Every element $g \in SL_2(\mathbb{C})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ induces an $\mathbb{C}$-algebra automorphism of $A_1(\mathbb{C}) = \mathbb{C}[x][y, \partial]$ by sending $x \to ax + by$, $y \to cx + dy$. Call $S(\mathbb{C})$ the subgroup of $\mathbb{C}$-automorphisms obtained in this manner (called linear admissible), and call $J(\mathbb{C})$ the subgroup of triangular automorphisms of $A_1(\mathbb{C})$, which are of the form: $x \to \lambda^{-1}x + p(y)$, $y \to \lambda y + \mu$, $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, $p \in \mathbb{C}[y]$.

Theorem

Aut $A_1(\mathbb{C})$ is the amalgamated free product of $J(\mathbb{C})$ and $S(\mathbb{C})$ over their intersection.

Proof.

Alev, Un Automorfisme non modéré de U(g3), Commun. Algebra, 14:1365-1378, 1985. □

- In a similar fashion to the polynomial algebra in two variables, it can be shown that all finite subgroups of automorphisms of $A_1(\mathbb{C})$ are conjugated to a finite subgroup of $S(\mathbb{C})$.
- The finite subgroups of $SL_2(\mathbb{C})$ are classified: they are the binary dihedral groups. For a proof of the classification, T. A. Springer, Invariant Theory, LNM 585.

We now proceed to the birational study of invariants of the Weyl Algebra. As we shall see, despite the rigidity result, a plethora of interesting phenomena happens.

Ore Localization: Basic Facts

Definition

Let A be a noncommutative domain, X a multiplicatively closed subset of A containing 1. A_X is a right localization for A with respect to X if there is an injection ι of A into A_X (allowing the identification of A with a subring of F), such that every element of X is invertible, and every element of A_X is of the form rx^{-1}, for $r \in A, x \in X$. Symmetrically, one can define left localizations.

Definition

Let A be a domain, X a multiplicatively closed set ($x, y, \in X$ implies $xy \in X$) containing 1. Then X satisfies the left (right) Ore condition if for every $r \in A, x \in X$, $Rx \cap Xr \neq \varnothing$ ($xR \cap Xr \neq \varnothing$).

Ore Localization: Basic Facts

Theorem

- *Let A be a domain, X a multiplicatively closed subset of A. Then the left (right) localization A_X exists if and only if X satisfies the left (right) Ore condition. In case $X = A - \{0\}$, we call the localizations left (right) quotient ring, and we call A a left (right) Ore domain. We omit the adjectives left in right in case both apply.*
- *A left (right) Noetherian A domain satisfies the left (right) Ore condition for the set $X = A - \{0\}$.*
- *If the left and right quotient rings exist, then they coincide.*

$\diamond$: The quotient ring of an Ore domain A will be denoted *Frac* A.

Hence the Weyl Algebra $A_n(k)$ has a quotient ring, called the Weyl Field, and denoted by $\mathcal{F}_n(k)$.

Let's focus on the case of Ore extensions in general. Let R be an left and right Noetherian domain; let $A = R[x, \alpha, \delta]$ be an Ore extension. As we saw *Frac* A exists. Let $K =$ *Frac* R: α extends to an automorphism of K: $\alpha(xy^{-1}) = \alpha(x)(\alpha(y))^{-1}$, $x, y \in R$, and δ to an α-derivation of the same ring, using $\delta(s^{-1}) = -\alpha(s)^{-1}\delta(s)s^{-1}$, for $s \neq 0 \in R$.

Hence we can form the Ore extension $K[x, \alpha, \delta]$ and we have:

Proposition

Frac A $=$ *Frac* $K[x, \alpha, \delta]$. *The common value of these skew-fields will be denoted by* $K(x, \alpha, \delta)$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

Call $w = yx \in A_1(k)$ and consider the subalgebra generated by it and y. It is clear it has the same quotient ring as $A_1(k)$. Since $yw = wy + y$, this subalgebra is the Ore extension $k[w][y, \theta]$, where θ is the automorphism $w \mapsto w + 1$.

In general we have, calling $w_i = y_i x_i, i = 1, \dots, n$.

Proposition

$\mathcal{F}_n(k) \cong k(w_1, \dots, w_n)(y_1; \theta_1) \dots (y_n; \theta_n)$, where $\theta_i(w_j) = w_j + \delta_{ij}$ and it fix y_j when $j < i$.

The Weyl Fields

$\diamond$: A canonical idea: we can embedd a the ring $k(x, \alpha)$ in a skew Laurent series ring: $k((x, \alpha))$.

Proposition

Let K be a division ring with center $Z(K)$. Let α be an automorphism of K such that α^k is not inner, $k \in \mathbb{Z} - \{0\}$; $K^\alpha = \{k \in K | \alpha(k) = k\}$. Then the center of $K(x, \alpha)$ is $Z(K) \cap K^\alpha$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

Corollary

The center of $\mathcal{F}_n(k)$ is k.

$\diamond$: The skew-fields $\mathcal{F}_n(k(x_1, \ldots, x_l))$ and $\mathcal{F}_m(k(x_1, \ldots, x_t))$ are isomorphic if, and only if, $m = n, l = t$. Take the transcendence degree of the center; then take the Gelfand-Kirillov transcendence degree over the center.

Noncommutative Birational equivalence; Gelfand-Kirillov Conjecture

Definition

We say two Ore domains are birationally equivalent if the have isomorphic quotient rings.

The research on noncommutative birationality began in the work of Gelfand and Kirillov, who made the spetacular Conjecture that the birational class of envelopping algebras of Lie algebras should be parametrized by just two integers:

Conjecture

(Gelfand-Kirillov) Let k be an algebraically closed field of zero characteristics, and let L be a finite dimensional algebraic Lie algebra. Then the quotient ring of $U(L)$ is isomorphic to a Weyl Field $\mathcal{F}_n(k(x_1, \ldots, x_l))$.

Noncommutative Birational equivalence; Gelfand-Kirillov Conjecture

- True for gl_n, sl_n: Gelfand-Kirillov, 1966;
- Nilpotent Lie algebras: Gelfand-Kirillov, 1966;
- Solvable Lie algebras: Joseph, Borho-Gabriel-Rentschler, McConnell-Robson, 1973;
- false for mixed types: Alev-Ooms-Van den Berg, 1996;
- True for any Lie algebra of dimension at most 8: Alev-Ooms-Van den Berg, 2000;
- False for types B, D, F, E: Premet, 2010;
- Open for type C, G;

Noncommutative Birational equivalence; Gelfand-Kirillov Conjecture

It is true after some modifications:

- All simple L after tensoring over the center with a suitable finite field extension of its field of fractions - Gelfand, Kirillov, 1968; conjectured to be true for all algebraic Lie algebras (Alev, Ooms, van den Berg, 2000);
- For simple L and maximal primitive quotients (Conze, 1974).

$\diamond$: Gelfand-Kirillov Philosophy says that the Weyl fields are important non-commutative analogues of the field of rational functions.

A Result by Faith

Theorem

Let R be a left and right Ore domain with quotient ring F. Let G be a finite group of automorphisms of R such that $|G|^{-1} \in R$. Then R^G is a left and right Ore domain and $\operatorname{Frac} R^G = F^G$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

Given this, Jacques Alev and François Dumas introduced the following:

Problem (J. Alev, F. Dumas, 2006)

(Noncommutative Noether's Problem) Let G be a finite group of linear automorphisms of the Weyl Algebra $A_n(k)$, with action extended to $\mathcal{F}_n(k)$. When we have $\mathcal{F}_n(k)^G \cong \mathcal{F}_n(k)$?

$\diamond$: Important for the study of the quotient ring of many algebras: it reproves the Gelfand-Kirillov Conjecture for gl_n in a novel way; it implies the Gelfand-Kirillov Conjectue for spherical subalgebras of rational Cherednik Algebras; it has many applications to the important class of Galois algebras.

See Noncommutative Noether's Problem for Complex Reflection Groups, F. Eshmatov, V. Futorny, S. Ovsienko, J. Schwarz, Proc. Amer. Math. Soc. 145 (2017), 5043-5052.

Original cases of positive solution:

- $n = 1, 2$ (Alev, Dumas, 2006);
- The natural representation of G decomposes as a direct sum of one dimensional representations (Alev, Dumas, 2006);
- The permutation action of S_n in $A_n(k)$ over an algebraically closed field (Futorny, Molev, Ovsienko, 2010);
- All unitary reflection groups actions over the complex numbers (Eshmatov, Futorny, Ovsienko, Schwarz, 2015).

Miyata's Theorem for Ore Extensions

Theorem

(Alev, Dumas) Let K be a field (commutative or not), α an automorphism and δ a α-derivation of K. Let $S = K[x, \alpha, \delta]$ be an Ore extension, with quotient ring D. Let G be a subgroup of ring automorphisms of S, not necesserely finite, such that $G(K) \subseteq K$. Then two cases are possible:

- $S^G \subseteq K$, *and then* $D^G = S^G = K^G$.
- S^G *is not contained in* K. *In this case, let* u *be any element of* S^G *not belonging to* K *of minimal degree in* x, *among all the invariants in* S^G *not in* K. *Then there is an automorphism* α' *and an* α'*-derivation* δ' *of* K^G *such that* $S^G = K^G[u, \alpha', \delta']$ *and* $D^G = \mathit{Frac} S^G$. *If* K *is commutative, then* $\alpha' = \alpha^{\deg_x u}$.

Lema

Let K be a division ring and $S = K[x, \alpha, \delta]$ an Ore extension of it. Let $u \in S$ with degree in x bigger than 0.

- *Let L be a sub division ring of K, and let $\mathcal{U} = \{u^i | i \in \mathbb{N}\}$. Then $\mathcal{U}$ is linearly independent on the left and on the right over L.*
- *Let T be the free left L-module generated by $\mathcal{U}$; and similarly T' on the right. Suppose $T = T'$ and both equals a ring S'. Then there exists an α' automorphism of L and an α'-derivation δ' of L such that $S' = L[u, \alpha', \delta']$.*
- *If K is commutative, then $\alpha' = \alpha^{\deg u}$.*

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

We discovered a strong connection between Noether's Problem and its Noncommutative Version.

Theorem (V. Futorny, J. Schwarz, 2018)

Let G be a finite group that acts linearly on $P_n(k)$ and hence on the Weyl Algebra $A_n(k)$. If Noether's Problem has a positive solution, then $\mathcal{F}_n(k)^G \cong \mathcal{F}_n(k)$. In particular, all previous cases of positive solution to Noncommutative Noether's Problem are reproved, and we obtain a positive solution for $n = 3$ $(k = \overline{k})$ and the alternating groups $\mathcal{A}_i$, $i = 3, 4, 5$.

Lema

- *If A is an Ore domain and X a multiplicatively closed subset of it such that A_X exists, then they have the same quotient ring;*
- *If A is a commutative domain finitely generated over k, and S a multiplicatively closed subset of A, then $D(A)_S$ exists and is isomorphic to $D(A_S)$.*

Proof.

A. J. Muhasky. The differential operator ring of an affine curve. Trans. of the American Math. Society, 307:705723, 1988. □

Lema

Let $\mathcal{B}$ be the subalgebra of $A_n(k)^G$ generated by $k[x_1, \dots, x_n]^G$ and $k[y_1, \dots, y_n]^G$; let S be the set of regular elements in $k[x_1, \dots, x_n]^G$, and F its field of fractions. Then $\mathcal{B}_S = A_n(k)^G_S = D(F)$.

Proof.

Levasseur and Stafford, Invariant Differential Operators and an homomorphism of Harish-Chandra, J. Am. Math. Soc. 8:365-372,1995. □

Lema

Let $R \subset S$ be two Noetherian domains, with the same quotient ring. Assume S is simple and that it is finitely generated as left and right R-module. Then $R = S$.

Proof.

Levasseur, Stafford, Invariant Differential Operators and an Homomorphism of Harish-Chandra, Journal of the American Mathematical Society, 8:365-372. □

Theorem

Let G be any finite group of automorphisms of $A_1(\mathbb{C})$. Then $\mathcal{F}_1(\mathbb{C})^G \cong \mathcal{F}_1(\mathbb{C})$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

Invariant Theory of The Quantum affine Space: Chevalley-Shephard-Todd Theorem

Assume *char* $k = 0$ and k algebraically closed. Consider a quantum affine space $O_Q(k^n)$ with its canonical graduation: every x_i has degree 1.

Definition

Let g be a graded automorphism of $O_Q(k^n)$. Then g is a quasi-reflection if it has finite order and one of the following holds:

- *Let O_1 the component of degree 1 of the quantum affine space — it is an n-dimensional vector space. Then, g, in its restriction to O_1, is a pseudo-reflection.*
- *g has order 4 and there is a basis $\{y_1, \ldots, y_n\}$ of O_1 such that g fix $y_j, j < n-1$ and $g(y_{n-1}) = iy_{n-1}$ e $g(y_n) = -iy_n$, where i is such that $i^2 = -1$. In this case g is called a mystic relfection.*

Invariant Theory of The Quantum affine Space: Chevalley-Shephard-Todd Theorem

Theorem

Let G be a finite group of graded automorphisms of $O_Q(k^n)$. Then the following are equivalent:

- $O_Q(k^n)^G \cong O_{Q'}(k^n)$ *for a possibly distinct $n \times n$ multiplicatively antisymmetric matrix Q'.*
- *G is generated by the quasi-reflections it contains.*
- $O_Q(k^n)^G$ *has finite global dimension.*

Proof.

E. Kirkman, J. Kuzmanovich , J. Zhang. Shephard-Todd-Chevalley theorem for skew polynomial rings. Algebras and Representation Theory, 13:127-158, 2010. □

Mystic Reflections?

Let's see what more can be said of the new guys here: the mystic reflections.

Proposition

- *If a finite group G is generated by mystic reflections, then it does not contain any element which acts as a pseudo-reflection on O_1.*
- *There exists finite groups G generated by mystic reflections which cannot be realized as a classical pseudo-reflection group.*

Proof.

E. Kirkman, J. Kuzmanovich , J. Zhang. Shephard-Todd-Chevalley theorem for skew polynomial rings. Algebras and Representation Theory, 13:127-158, 2010. □

$\diamond$: This topic is related on the study of yet another class of noncommutative rings which resemble the polynomial algebra (mainly it's homological aspects); the Artin-Schelter regular algebras. For more: E. E. Kirkman, Invariant Theory of Artin-Schelter Regular Algebras: A survey, arXiv:1506.06121.

Theorem

- *The full automorphism group of* $\mathbb{C}_q[x, y]$ *is* $(\mathbb{C}^*)^2$ *acting by* (a, b) *sending* x *to* ax *and* y *to* by*.*
- *The full automorphism group of* $A_1^q(\mathbb{C})$ *is* $\mathbb{C}^*$*, acting by* a *sending* x *to* ax*,* y *to* $a^{-1}y$*.*

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

Proposition

Frac $k_q[x, y] \cong$ *Frac* $A_1^q(k)$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants; Brown, Goodearl, Lectures on Algebraic Quantum Groups. □

$\diamond$: Let $\overline{q} = (q_1, \ldots, q_n) \in k^n$ be a n-uple which each entry is non-null and a non-root of unity. We call $O_{\overline{q}}(k^{2n})$ the tensor product $k_{q_1}[x_1, y_1] \otimes \ldots \otimes k_{q_n}[x_n, y_n]$; if every entry in $\overline{q}$ is the same parameter q, we use the notation $O_q(k^{2n})$.

q-Difference Noether's Problem

Problem (V. Futorny, J. Hartwig, 2014)

(q-Difference Noether's Problem) Let G be a finite group of automorphisms of $O_q(k^{2n})$. When there exists $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, with $\overline{q} = (q^{k_1}, \ldots, q^{k_n})$ and $Frac\ O_q(k^{2n})^G \cong Frac\ O_{\overline{q}}(k^{2n})$?

$\diamond$: Provides a novel proof, with aditional information, for the Quantum Gelfand-Kirillov Conjecture for $U_q(gl_n)$ and $U_q(sl_n)$; also proves a Quantum Gelfand-Kirillov Conjecture for many classes of Galois Algebras.

$\diamond$: The groups $A(m, p, n)$, $m > 1, n \geq 1, p|m$, are a subset of $G_m^{\otimes n}$, $G_m \subset k$ the cyclic group in m elements. It consist of elements $(g_1, \ldots, g_m)$ such that $(\prod_{i=1}^n g_i)^{m/p} = 1$. $G(m, p, n)$ is the semidirect product of $A(m, p, n)$ and S_n.

$\diamond$: They are all the irreducible unitary reflection groups (except $G(1, 1, n) \cong S_n$ in its natural representation; and $G(2, 2, 2)$, the Klein group), plus 34 exceptional groups.

q-Difference Noether's Problem

$\diamond$: The groups $G = G(m, p, n)$ act as follows on $O_q(k^{2n})$:
$h = (g, \pi) \in G$, $g = (g_1, \ldots, g_n) \in G_m^{\otimes n}$, $\pi \in S_n$: $h(x_i) = g_i x_{\pi(i)}$, $h(y_i) = y_{\pi(i)}$, $i = 1, \ldots, n$.

Theorem

$$\mathit{Frac}\, O_q(k^{2n})^{G(m,p,n)} \simeq \mathit{Frac}\left(k_{q^{m/p}}[x, y] \otimes k_{q^m}[x, y]^{\otimes n-1}\right).$$

Proof.

J. Hartwig, The q-difference Noether problem for complex reflection groups and quantum OGZ algebras, Comm. Alg. 45, 1166-1176, 2017. □

Theorem

For every finite group G of automorphisms of $\mathbb{C}_q[x, y]$, we have $\mathrm{Frac}\,\mathbb{C}_q[x, y]^G \cong \mathrm{Frac}\,\mathbb{C}_{q^{|G|}}[x, y]$.

Proof.

F. Dumas, An Introduction to Noncommutative Polynomial Invariants. □

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