Kronecker Product and the vec Operator

Definition 1. *Let $\boldsymbol{A}$ be an $n \times p$ matrix and $\boldsymbol{B}$ an $m \times q$ matrix. The $mn \times pq$ matrix*

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{1,1}\boldsymbol{B} & a_{1,2}\boldsymbol{B} & \cdots & a_{1,p}\boldsymbol{B} \\ a_{2,1}\boldsymbol{B} & a_{2,2}\boldsymbol{B} & \cdots & a_{2,p}\boldsymbol{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}\boldsymbol{B} & a_{n,2}\boldsymbol{B} & \cdots & a_{n,p}\boldsymbol{B} \end{bmatrix}$$

is called the **Kronecker product** *of $\boldsymbol{A}$ and $\boldsymbol{B}$. It is also called the direct product or the tensor product.*

Some properties of the Kronecker product:

1. $\boldsymbol{A} \otimes (\boldsymbol{B} \otimes \boldsymbol{C}) = (\boldsymbol{A} \otimes \boldsymbol{B}) \otimes \boldsymbol{C}$ associativity,
2. $\boldsymbol{A} \otimes (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} \otimes \boldsymbol{B}) + (\boldsymbol{A} \otimes \boldsymbol{C})$ distributivity
   $(\boldsymbol{A} + \boldsymbol{B}) \otimes \boldsymbol{C} = (\boldsymbol{A} \otimes \boldsymbol{C}) + (\boldsymbol{B} \otimes \boldsymbol{C})$,
3. For scalar a, $a \otimes \boldsymbol{A} = \boldsymbol{A} \otimes a = a\boldsymbol{A}$,
4. For scalars a and b, $a\boldsymbol{A} \otimes b\boldsymbol{B} = ab\ \boldsymbol{A} \otimes \boldsymbol{B}$,
5. For conforming matrices, $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{A}\boldsymbol{C} \otimes \boldsymbol{B}\boldsymbol{D}$,
6. $(\boldsymbol{A} \otimes \boldsymbol{B})^T = \boldsymbol{A}^T \otimes \boldsymbol{B}^T$, $\quad (\boldsymbol{A} \otimes \boldsymbol{B})^H = \boldsymbol{A}^H \otimes \boldsymbol{B}^H$,
7. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}^T \otimes \boldsymbol{b} = \boldsymbol{b}\boldsymbol{a}^T = \boldsymbol{b} \otimes \boldsymbol{a}^T$
   (note $\boldsymbol{a}\boldsymbol{a}^T = \boldsymbol{a} \otimes \boldsymbol{a}^T$),

8. For partitioned matrices, $[\boldsymbol{A}_1, \boldsymbol{A}_2] \otimes \boldsymbol{B} = [\boldsymbol{A}_1 \otimes \boldsymbol{B}, \boldsymbol{A}_2 \otimes \boldsymbol{B}]$, but $\boldsymbol{A} \otimes [\boldsymbol{B}_1, \boldsymbol{B}_2] \neq [\boldsymbol{A} \otimes \boldsymbol{B}_1, \boldsymbol{A} \otimes \boldsymbol{B}_2]$,

9. For square nonsingular matrices $\boldsymbol{A}$ and $\boldsymbol{B}$: $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$,

10. For $m \times m$ matrix $\boldsymbol{A}$ and $n \times n$ matrix $\boldsymbol{B}$: $|\boldsymbol{A} \otimes \boldsymbol{B}| = |\boldsymbol{A}|^n |\boldsymbol{B}|^m$,

11. $\operatorname{tr}(\boldsymbol{A} \otimes \boldsymbol{B}) = \operatorname{tr}(\boldsymbol{A})\operatorname{tr}(\boldsymbol{B})$,

12. $\operatorname{rank}(\boldsymbol{A} \otimes \boldsymbol{B}) = \operatorname{rank}(\boldsymbol{A}) \operatorname{rank}(\boldsymbol{B})$.

Definition 2. *The* vec *operator creates a column vector from a matrix* $\boldsymbol{A}$ *by stacking the column vectors of* $\boldsymbol{A} = [\boldsymbol{a}_1 \boldsymbol{a}_2 \cdots \boldsymbol{a}_n]$ *below one another:*

$$\operatorname{vec}(\boldsymbol{A}) = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}.$$

Theorem 1.

$$\operatorname{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{B}^T \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{X}).$$

Proof. Let $\boldsymbol{B} = [\boldsymbol{b}_1 \ \boldsymbol{b}_2 \cdots \boldsymbol{b}_n]$ (of size $m \times n$) and $\boldsymbol{X} =$

$[\boldsymbol{x}_1\ \boldsymbol{x}_2 \cdots \boldsymbol{x}_m]$. Then, the kth column of $\boldsymbol{AXB}$ is

$$
\begin{aligned}
(\boldsymbol{AXB})_{:,k} &= \boldsymbol{AXb}_k = \boldsymbol{A}\sum_{i=1}^{m} \boldsymbol{x}_i b_{i,k} \\
&= [b_{1,k}\boldsymbol{A}\ b_{2.k}\boldsymbol{A}\ \cdots\ b_{m,k}\boldsymbol{A}] \underbrace{\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix}}_{\mathrm{vec}(\boldsymbol{X})} \\
&= (\underbrace{[b_{1,k}, b_{2,k}, \ldots, b_{m,k}]}_{\boldsymbol{b}_k^T} \otimes \boldsymbol{A}])\mathrm{vec}(\boldsymbol{X})
\end{aligned}
$$

Stacking the columns together

$$
\begin{aligned}
\mathrm{vec}(\boldsymbol{AXB}) &= \begin{bmatrix} (\boldsymbol{AXB})_{:,1} \\ (\boldsymbol{AXB})_{:,2} \\ \vdots \\ (\boldsymbol{AXB})_{:,n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1^T \otimes \boldsymbol{A} \\ \boldsymbol{b}_2^T \otimes \boldsymbol{A} \\ \vdots \\ \boldsymbol{b}_n^T \otimes \boldsymbol{A} \end{bmatrix} \mathrm{vec}(\boldsymbol{X}) \\
&= (\boldsymbol{B}^T \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{X}).
\end{aligned}
$$

$\square$

Corollary: $\mathrm{vec}(\boldsymbol{AB}) = (\boldsymbol{I} \otimes \boldsymbol{A})\ \mathrm{vec}(\boldsymbol{B}) = (\boldsymbol{B}^T \otimes \boldsymbol{I})\ \mathrm{vec}(\boldsymbol{A})$.

Some Properties of the vec Operator

$$\mathrm{tr}(\boldsymbol{ABC}) = \mathrm{vec}(\boldsymbol{A}^T)^T\,(\boldsymbol{I}\otimes\boldsymbol{B})\,\mathrm{vec}(\boldsymbol{C}),$$

and its corollary

$$\mathrm{tr}(\boldsymbol{AB}) = \mathrm{vec}(\boldsymbol{A}^T)^T\,\mathrm{vec}(\boldsymbol{B}).$$

Also

$$\begin{aligned}
\mathrm{vec}(\boldsymbol{a}\boldsymbol{a}^T) &= \boldsymbol{a}\otimes\boldsymbol{a}, \\
\mathrm{tr}(\boldsymbol{A}^T\boldsymbol{BC}\boldsymbol{D}^T) &= \mathrm{vec}(\boldsymbol{A})^T(\boldsymbol{D}\otimes\boldsymbol{B})\,\mathrm{vec}(\boldsymbol{C}).
\end{aligned}$$