Auto-Associative Memories Based on Complete Inf-Semilattices and Conditionally Complete L-Groups

Presented by Majid Ali
Supervised by Prof.Dr. Peter Sussner

Mathematical Imaging and Computational Intelligence Group
Department of Applied Mathematics, IMECC
University of Campinas

References

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Organization of this talk

1. Introduction On Complete Inf-semilattices
2. MAMs in cisls based on Conditionally Complete *l*-Groups
3. Characterizations of fixed points
4. Further Research Issues

Some Basic Definitions

- Let $\mathcal{L}$ be a partially ordered set (poset) and let $\{x, y\}$, X, $\{x_j : j \in J\} \subseteq \mathcal{L}$.
  - $x \wedge y$, $\bigwedge X$, $\bigwedge_{j \in J} x_j$ denote the infimum of these sets.
  - $x \vee y$, $\bigvee X$, $\bigvee_{j \in J} x_j$ denote the supremum of these sets.
- A poset $\mathcal{L}$ is an inf-semilattice if $\exists\, x \wedge y \in \mathcal{L}\ \forall x, y \in \mathcal{L}$. If, in addition, $\exists\, x \vee y \in \mathcal{L}\ \forall\, x, y \in \mathcal{L}$ then $\mathcal{L}$ is a lattice.
- An inf-semilattice $\mathcal{L}$ is complete if $\exists \bigwedge X \in \mathcal{L}\ \forall\ \emptyset \neq X \subseteq \mathcal{L}$. A lattice $\mathcal{L}$ is complete if $\exists \bigwedge X, \bigvee X \in \mathcal{L}$ for all $X \subseteq \mathcal{L}$.
- A lattice $\mathcal{L}$ is conditionally complete if $\bigwedge X$ and $\bigvee X$ exist for all bounded $X \subseteq \mathcal{L}$.
- If $\mathcal{L}$ is a (complete) lattice or an inf-semilattice then $\mathcal{L}^n$ is also a (complete) lattice or an inf-semilattice, respectively.

Erosions and Openings on Inf-semilattice

An operator $\mathcal{E}$ in Inf-semilattice $\mathcal{L}$ is an erosion $\Leftrightarrow$ for non empty collection $\{x_i\} \subseteq \mathcal{L}$:

$$\mathcal{E}\left(\bigwedge_i x_i\right) = \bigwedge_i \mathcal{E}(x_i)$$

Propositions [1]

- Erosion in Inf- semilattices are increasing, i.e $X \leq Y \Rightarrow \mathcal{E}(X) \leq \mathcal{E}(Y)$.
- If $\{\mathcal{E}_i\}$ is a non- empty collection of erosions on a complete inf-semilattice $\mathcal{L}$, then the operator $\mathcal{E}$ defined by $\mathcal{E}(x) \triangleq \bigwedge_i \mathcal{E}_i(x)$ for all $x \in \mathcal{L}$ is also an erosion.

Opening

An operator γ on inf-semilattice $\mathcal{L}$ is an algebraic opening $\Leftrightarrow$ it is idempotent ($\gamma\gamma = \gamma$), increasing, and anti-extensive ($\gamma(x) \leq x$) .

Definition

In a complete inf-semilattice $\mathcal{L}$, the morphological openings $\gamma_{\mathcal{E}}$ associated to an erosion $\mathcal{E}$ is defined for any $x \in \mathcal{L}$ by

$$\gamma_{\epsilon}(x) \triangleq \bigwedge \{y \in \mathcal{L} : \mathcal{E}(x) \leq \mathcal{E}(y)\}$$

Since this set is non-empty, so the infimum of a non-empty set is always exist and unique in complete Inf-semilattice.

Proposition

1. For any erosion $\mathcal{E}$ in a complete inf-semilattice $\mathcal{L}$, $\mathcal{E}\gamma_{\mathcal{E}} = \mathcal{E}$.
2. The morphological opening in an complete inf-semilattice is an algebraic opening.

Examples of Complete Inf-Semilattice

Difference Semilattices:

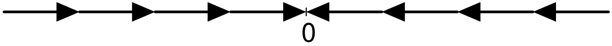
A Set $\mathcal{L}$ is called difference semilattice;

- It is composed of functions $f : E \to R$, where E is an Euclidean space and $R = \mathbb{R}$ *or* $\mathbb{Z}$.
- It is associated with the partial ordering $\preceq$, i.e $\forall \quad f, g$ given by

$$f \preceq g \Leftrightarrow \forall x \begin{cases} g(x) \geq f(x) \geq 0 & \text{if } g(x) \geq 0 \\ g(x) \leq f(x) \leq 0 & \text{if } g(x) < 0 \end{cases}$$

Geometrically

It is the concatenation of two chains $(\mathbb{R}_-, \geq)$ and $(\mathbb{R}_+, \leq)$.



The least element 0 is the function $0(x) = 0$.

Examples of Complete Inf-Semilattice

- Consider a complex plane $\mathbb{C}$ as an (infinite) union of chains $\mathbb{C}_\alpha = \{ re^{i\alpha} | r \geq 0 \}$ ordered by the magnitude of the modulus. Thus, given two elements $w, z \in \mathbb{C}$, we have

$$w \preceq z \Leftrightarrow \begin{cases} arg w = arg z \\ |w| \leq |z| \end{cases}$$

- The family of all finite subsets of an infinite set E provided with the set inclusion as partial ordering.

Infimum and Supremum operation on CISL

Infimum

For any f and g in $\mathcal{L}$, the $\curlywedge$ is given by

$$\left(f \curlywedge g\right)(x) = \begin{cases} min\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \geq 0 \\ max\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

supremum

For any f and g in $\mathcal{L}$, the $\curlyvee$ is given by

$$\left(f \curlyvee g\right)(x) = \begin{cases} max\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \geq 0 \\ min\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \leq 0 \\ Non\, existent, & \text{otherwise} \end{cases}$$

Definitions

let $(\mathcal{L}, \leq)$ be a lattice. An element $r \in \mathcal{L}$ is called reference element if for every two elements $x, y \in \mathcal{L}$ we have

$$x \wedge r = y \wedge r \text{ and } x \vee r = y \vee r \Longleftrightarrow x = y.$$

Let $\mathcal{L}$ be a lattice and $r \in \mathcal{L}$ a fixed element. Define the binary relation $\preceq_r$ on $\mathcal{L} \times \mathcal{L}$ by

$$x \preceq_r y \; If \begin{cases} r \wedge y \leq r \wedge x \\ r \vee y \geq r \vee x \end{cases}$$

Examples of Complete Inf-Semilattice

Reference Semilattices

Another example of complete inf semilattice, called reference semilattice, consists of real functions. A reference semilattice $\mathcal{L}$, the partial ordering $\preceq_r$ is defined by

$$f \preceq_r g \Leftrightarrow \forall x \begin{cases} g(x) \geq f(x) \geq r(x), & \text{if } g(x) \geq r(x) \\ g(x) \leq f(x) \leq r(x), & \text{if } g(x) < r(x) \end{cases}$$

Infimum

For any f and g in $\mathcal{L}$, the $\curlywedge$ is given by

$$\left(f \curlywedge g\right)(x) = \begin{cases} min\left\{f(x), g(x)\right\} & \text{if } f(x), g(x) \geq r(x) \\ max\left\{f(x), g(x)\right\} & \text{if } f(x), g(x) \leq r(x) \\ r(x) & \text{otherwise} \end{cases}$$

Gemotrically representation

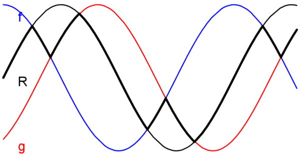


Figure: The black line repersent Infimum.

supremum

For any f and g in $\mathcal{L}$, the $\curlyvee$ is given by

$$\left(f \curlyvee g\right)(x) = \begin{cases} max\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \geq r(x) \\ min\left\{f(x), g(x)\right\} & \text{if } f(x),\ g(x) \leq r(x) \\ Non\, existent, & \text{otherwise} \end{cases}$$

Adjunctions on Complete infsemilattice

An operator $\mathcal{E} : \mathcal{L} \to \mathcal{M}$, where both $\mathcal{L}$ and $\mathcal{M}$ are cisl's is an erosion if

$$\mathcal{E}(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} \mathcal{E}(x_i) \,.$$

for all nonempty collection $\{x_i\}$.
The set $\mathcal{M}[\mathcal{E}]$ defined by

$$\mathcal{M}[\mathcal{E}] = \{y \in \mathcal{M} \,|\exists x \in \mathcal{L} : y \preceq \mathcal{E}(x)\}$$

Thus dilation $\delta = \Delta(\mathcal{E})$ is defined by

$$\delta(y) = \bigwedge \{x \in \mathcal{L} \,| y \preceq \mathcal{E}(x)\}\,, y \in \mathcal{M}[\mathcal{E}].$$

Then the pair $(\mathcal{E}, \delta)$ is an adjunction on from $\mathcal{L}$ to $\mathcal{M}[\mathcal{E}]$.

Proposition

Let $\mathcal{L}, \mathcal{M}$ be cisl's and $\mathcal{N}$ a poset. Assume that $\mathcal{E}_1 : \mathcal{L} \to \mathcal{M}$ and $\mathcal{E}_2 : \mathcal{M} \to \mathcal{N}$ are erosions, and that $\mathcal{E} = \mathcal{E}_2\mathcal{E}_1$. Then $\mathcal{E}$ is an erosion from $\mathcal{L}$ into $\mathcal{N}$ and

1. $\mathcal{N}[\mathcal{E}] \subseteq \mathcal{N}[\mathcal{E}_2]$;
2. $\Delta(\mathcal{E}_2)$ maps $\mathcal{N}[\mathcal{E}]$ into $\mathcal{M}[\mathcal{E}_1]$;
3. $\Delta(\mathcal{E}_1)\Delta(\mathcal{E}_2) = \Delta(\mathcal{E})$ on $\mathcal{N}[\mathcal{E}]$.

Proof.

We write $\delta_i = \Delta(\mathcal{E}_i)$ for $i = 1, 2$ and $\delta = \Delta(\mathcal{E})$.

1. $z \in \mathcal{N}[\mathcal{E}] \Rightarrow z \preceq \mathcal{E}_2\mathcal{E}_1(x)$ for some $x \in \mathcal{L}$. $\Rightarrow z \in \mathcal{N}[\mathcal{E}_2]$.
2. $z \in \mathcal{N}[\mathcal{E}]$ means $z \preceq \mathcal{E}_2\mathcal{E}_1(x)$ for some $x \in \mathcal{L}$, Furthermore

$$\delta_2(z) = \bigwedge \{y \in \mathcal{M} \,|\, z \preceq \mathcal{E}_2(y)\},$$

we derive that $\delta_2(z) \preceq \mathcal{E}_1(x) \Rightarrow \delta_2(z) \in \mathcal{M}[\mathcal{E}_1]$.

1. For $x \in \mathcal{L}$ and $z \in \mathcal{N}[\mathcal{E}]$ we have,

$$z \preceq \mathcal{E}_2\mathcal{E}_1(x) \quad \Leftrightarrow \delta_2(z) \preceq \mathcal{E}_1(x)\ [since\, z \in \mathcal{N}[\mathcal{E}_2]\, by\, (i)]$$
$$\Leftrightarrow \delta_1\delta_2(z) \preceq x\ [since \delta_2(z) \in \mathcal{M}[\mathcal{E}_1]\, by\ (ii)]$$

We used that $(\mathcal{E}_2, \delta_2)$ forms an adjunction between $\mathcal{M}$ and $\mathcal{N}[\mathcal{E}_2]$, and that $(\mathcal{E}_1, \delta_1)$ is an adjunction between $\mathcal{L}$ and $\mathcal{M}[\mathcal{E}_1]$. On the other hand,

$$z \preceq \mathcal{E}_2\mathcal{E}_1(x) = \mathcal{E}(x) \Leftrightarrow \delta(z) \preceq x.$$

This gives that $\delta = \delta_1\delta_2$ on $\mathcal{N}[\mathcal{E}]$.

Proposition

If $(\mathcal{E}, \delta_1)$ and $(\mathcal{E}, \delta_2)$ are adjunctions between $\mathcal{L}$ and $\mathcal{M}[\mathcal{E}]$, then $\delta_1 = \delta_2$

Proof For all $x \in \mathcal{M}[\mathcal{E}]$

$\delta_1(x) \preceq \delta_1(x) \Leftrightarrow x \preceq \mathcal{E}\delta_1(x) \Leftrightarrow \delta_2(x) \preceq \delta_1(x)$.Similarly $\delta_1(x) \preceq \delta_2(x)$.

Example

Let $\mathcal{L} = \mathcal{M} = \mathcal{N} = [-3, 3]$ and define $\mathcal{E}_1 = \mathcal{E}_2$ as in Fig. We have $\mathcal{M}[\mathcal{E}_1] = [-2, 2]$ and $\mathcal{N}[\mathcal{E}] = [-1, 1]$.

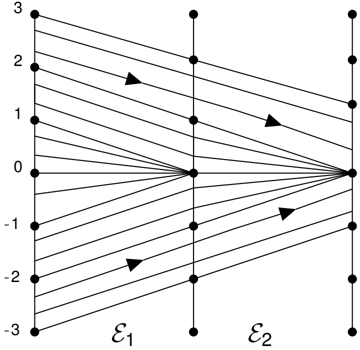


Figure: Composition of two erosions.

Invariance properties on CISL

consider the cisl $\mathcal{T} = \mathbb{R}_0$ with partial ordering $\preceq_0$. Define the family of mappings $\rho_\nu, \nu \in \mathbb{R}$ on $\mathbb{R}_0$.

$$\rho_\nu(t) = \begin{cases} t+\nu & \text{if} t, t+\nu > 0 \\ t-\nu & \text{if} t, t-\nu < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

The family ρ_ν satisfying the following properties

1. $\rho_0 = id$
2. $\rho_\omega \rho_\nu = \rho_{\nu+\omega}$ if $\nu, \omega \geq 0$
3. $\rho_{-\omega}\rho_{-\nu} = \rho_{-\nu-\omega}$ if $\nu, \omega \geq 0$
4. $\rho_{-\omega}\rho_\nu = \rho_{\nu-\omega}$ if $\nu, \geq \omega \geq 0$

Vertical Translation

Denote $\mathcal{F}_r(E,\mathcal{T})$ is the set of all functions $x : E \rightarrow \mathcal{T}$ provided with cisl ordering $\preceq_r$; here $r : E \rightarrow \mathcal{T}$. where $\mathcal{T} = \mathbb{R}_0$ *or* $\mathbb{Z}_0$.
We define $\rho_\nu : \mathcal{F}_r \rightarrow \mathcal{F}_r$ by point wise application $\rho_\nu(x)(p) = \rho_\nu\left(x(p)\right)$

$$\rho_\nu\left(x(p)\right) = \begin{cases} x(p) + \nu & \text{if} x(p), x(p) + \nu > r(p) \\ x(p) - \nu & \text{if} x(p), x(p) - \nu < r(p) \\ r(p) & \text{otherwise.} \end{cases}$$

- The conditions $x(p) > r(p)$ and $x(P) < r(p)$ within the definition of ρ_ν are not closed under the reference cisl.
- The out put values corresponding to the above conditions do not necessarily converge to $r(p)$. As $x(p) \rightarrow r(p)$

Vertical Translation Geometrically

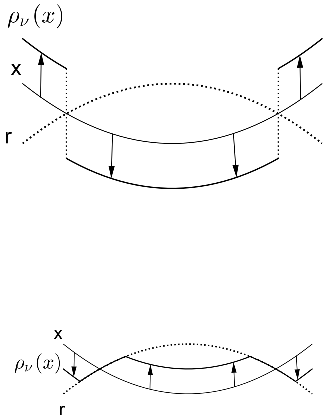


Figure: Vertical translation for $\nu > 0$ and $\nu < 0$.

Morphology on Semilattices

Vertical Translation

To obtain an erosion also for continuous, we define translation in this form

$$\rho_\nu^\epsilon(x)(p) = \begin{cases} [x \curlywedge \rho_\nu(x)](p) & \text{if } r(p) - \epsilon \prec x(p) \prec r(p) + \epsilon \\ \rho_\nu(x(p)) & \text{Otherwise} \end{cases}$$

where $\curlywedge$ represent the infimum of reference semilattice, and ϵ is some positive constant.

Proposition

The "translation" $\rho_{\nu}^{\epsilon}(x)$ is an erosion on $\mathcal{F}_r$.

Proof

Checking all possible situations leads us to the following logic table

Conditions	(1) $[\rho_{\nu}^{\epsilon}(\curlywedge_{i\in I} x_i)](p)$	(2) $[\curlywedge_{i\in I} \rho_{\nu}^{\epsilon}(x_i)](p)$	(1)=(2)
$\curlywedge_{i\in I} x_i(p) \geq r(p) + \epsilon$	$\rho_{\nu}(\curlywedge_i x_i(p))$	$\curlywedge_{i\in I} \rho_{\nu}(x_i(p))$	true
$\curlywedge_i x_i(p) \leq r(p) - \epsilon$	$\rho_{\nu}(\curlywedge_i x_i(p))$	$\curlywedge_i \rho_{\nu}(x_i(p))$	true
$r(p) + \epsilon > \curlywedge_i x_i(p) > r(p)$	$\rho_{\nu}^{\epsilon}(\curlywedge_i x_i(p))$	$\curlywedge_i \rho_{\nu}^{\epsilon}(x_i(p))$	true
$r(p) - \epsilon < \curlywedge_i x_i(p) < r(p)$	$\rho_{\nu}^{\epsilon}(\curlywedge_i x_i(p))$	$\curlywedge_i \rho_{\nu}^{\epsilon}(x_i(p))$	true
$\curlywedge_i x_i(p) = r(p)$	r(p)	r(p)	true

Lemmas

1. For any p and positive ϵ, $X \curlywedge \rho_\nu^\epsilon(X) = X \curlywedge \rho_\nu(X)$.
2. The operator $X \curlywedge \rho_\nu(X)$ is an erosion, both in discrete and continuous cases.

Proof

The operator $X \curlywedge \rho_\nu^\epsilon(X)$ is an erosion, since it consists of the infimum of two erosions. The identity operator and ρ_ν^ϵ. Therefore, according to Proposition[1], $X \curlywedge \rho_\nu(X)$ is also an erosion.

Lattice Ordered Group

- A lattice that also represents a group such that every group translation $x \mapsto a + x + b$ is isotone is called an *l*-group.
- An *l*-group $\mathbb{F}$ such that $\mathbb{F}$ is a conditionally complete lattice is called a conditionally complete l-group.
- A complete lattice $\mathbb{G}$ such that $\mathbb{F} = \mathbb{G} \setminus \{\bigvee \mathbb{G}, \bigwedge \mathbb{G}\}$ forms an *l*-group is called a complete l-group extension.

Definitions

Let $\mathbb{F}$ be a conditionally complete *l*-group.

1. If $\mathbb{F}^+ = \{x \in \mathbb{F} : 0 \leq x\}$ then $(\mathbb{F}^+, \leq)$ is a cisl.
2. The positive and negative parts of $x \in \mathbb{F}$ are resp. $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$
3. These expression are equivalent. (i) $x \wedge y = 0$ (ii) $x + y = x \vee y$ (iii) $x = (x - y)^+$ and $y = (x - y)^-$.

Constructing the cisls $\mathbb{F}_0 = (\mathbb{F}, \preceq)$ and $\mathbb{F}_r = (\mathbb{F}, \preceq_r)$

Definitions

1. A pair of operators ψ^+, ψ^- is called disjointness-preserving if $x \wedge y = 0$ implies that $\psi^+(x) \wedge \psi^-(y) = 0 \; \forall x, y \in \mathbb{F}^+$.
2. $r \in \mathbb{F}$ is called a reference element if $\forall x, y \in \mathbb{F}$:
$$(x - r)^+ = (y - r)^+ \text{ and } (x - r)^- = (y - r)^- \; \Leftrightarrow \; x = y \,.$$
3. Let "$\preceq$" and "$\preceq_r$" be defined as follows $\forall \; x, y, r \in \mathbb{F}$:
$x \preceq y \; \Leftrightarrow \; x^+ \leq y^+$ and $x^- \leq y^-$,
$x \preceq_r y \; \Leftrightarrow \; (x - r)^+ \leq (y - r)^+$ and $(x - r)^- \leq (y - r)^-$.
4. Define a new operator ψ on $(\mathbb{F}, \preceq)$ is given by
$$\psi(x) = \psi^+(x^+) - \psi^-(x^-).$$

The cisls $\mathbb{F}_0 = (\mathbb{F}, \preceq)$ and $\mathbb{F}_r = (\mathbb{F}, \preceq_r)$

Facts

$(\mathbb{F}^+, \leq)$, $\mathbb{F}_0 = (\mathbb{F}, \preceq)$ are cisls and $\curlywedge_{i \in I} x_i = \bigwedge_{i \in I} x_i^+ - \bigwedge_{i \in I} x_i^-$.

Commutative Diagram

$x \mapsto x + r$ and $x \mapsto x - r$ represent cisl isomorphisms. We have

$$\begin{array}{ccc} \mathbb{F}_r & \xrightarrow{\psi} & \mathbb{F}_r \\ {\scriptstyle x \longmapsto x - r}\downarrow & & \uparrow{\scriptstyle x \longmapsto x + r} \\ \mathbb{F}_0 & \xrightarrow[\psi_0]{} & \mathbb{F}_0 \end{array}$$

proposition

If ψ^+, ψ^- are disjointness preserving, then the following holds.

1. If ψ^+, ψ^- are increasing on $(\mathbb{F}^+, \leq)$, then ψ is increasing on $(\mathbb{F}, \preceq)$.
2. If $\psi^+ = \psi^-$ then ψ is self dual, i.e, $\psi(-x) = -\psi(x)$.
3. If ψ^+, ψ^- are anti-extensive on $\mathbb{F}^+$, then ψ is anti-extensive on $(\mathbb{F}, \preceq)$.
4. If ψ^+, ψ^- are idempotent then ψ is also idempotent.

Max Product, Min Product, and Conjugate

Let $\mathbb{F}$ be a conditional complete l-group. Let $A \in \mathbb{F}^{m \times n}$ e $B \in \mathbb{F}^{n \times p}$.

- $C = A \boxdot_{\vee} B$ - max product of A and B: $c_{ij} = \bigvee_{k=1}^{n}(a_{ik} + b_{kj})$.
- $D = A \boxdot_{\wedge} B$ - min product of A and B: $d_{ij} = \bigwedge_{k=1}^{n}(a_{ik} + b_{kj})$
- A^* - conjugate of A: $A^* = -A^T$

Max product, Min product with reference function

Max product, Min product

Let $\mathbb{G}$ be a complete $\mathbb{L}$-group extension. Let $A \in \mathbb{G}^{m\times p}$ and $B \in \mathbb{G}^{p\times n}$.

1. The max-product of A and B is given by

$$C = A \boxdot_r B \Longleftrightarrow c_{ij} = \bigvee_{\xi=1}^{p} {}_r \left(a_{i\xi} + b_{\xi j}\right).$$

2. The min-product of A and B is given by

$$C = A \boxtimes_r B \Longleftrightarrow c_{ij} = \bigwedge_{\xi=1}^{p} {}_r \left(a_{i\xi} \acute{+} b_{\xi j}\right).$$

In this case we assume that $\mathbb{G} = \mathbb{R} \cup \{-\infty, +\infty\}$

Autoassociative Lattice Memories

Autoassociative Memories

Given a set $\{x^1 ... x^k\}$, an AM is a mapping $\mathcal{A}$ such that $\mathcal{M}\left(x^\xi\right) = x^\xi$. Furthermore, $\mathcal{M}\left(\widetilde{x}^\xi\right) = x^\xi$ for noise or incomplete version $\widetilde{x}^\xi$ of x^ξ .

Characteristics

1. They exhibit optimal absolute storage capacity.
2. They exhibit one step convergence when employed with feedback.

A MAM on the cisl $(\mathbb{F}^n, \preceq)$

Notations

Let $\mathbb{F}$ be a conditionally complete l-group and $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{F}^n$. Let $X^+ = [(\mathbf{x}^1)^+, \ldots, (\mathbf{x}^k)^+, (\mathbf{x}^1)^-, \ldots, (\mathbf{x}^k)^-] \in \mathbb{F}^{n \times 2k}$ (the ξth column is $(\mathbf{x}^\xi)^+$ and the $(\xi + k)$th column is $(\mathbf{x}^\xi)^-$).

Theorem

If M_{XX}^+ denotes the matrix $M_{X^+X^+} \in \mathbb{F}^{n \times n}$ then an anti-extensive and disjointness-preserving erosion on the cisl $(\mathbb{F}^n)^+$ is given by

$$\mathcal{M}_{XX}^+(\mathbf{x}) = M_{XX}^+ \boxtimes \mathbf{x} \;\forall\, \mathbf{x} \in (\mathbb{F}^n)^+ .$$

The erosion $\mathcal{M}_{XX}^+$ on the cisl $((\mathbb{F}^n)^+, \leq)$ yields an anti-extensive erosion $\mathcal{M}_{XX}$ on the cisl $(\mathbb{F}^n, \preceq)$ that is given as follows:

$$\mathcal{M}_{XX}(\mathbf{x}) = M_{XX}^+ \boxtimes \mathbf{x}^+ - M_{XX}^+ \boxtimes \mathbf{x}^- \;\forall\, \mathbf{x} \in \mathbb{F}^n .$$

Definition of fixed point

A vector $\mathbf{x} \in \mathbb{F}^n$ is called a fixed point of $\mathcal{M}_{XX} \Longleftrightarrow \mathcal{M}_{XX}(\mathbf{x}) = \mathbf{x}$. Denote the set of finite fixed points of $\mathcal{M}_{XX}$ by using the symbols $F(\mathcal{M}_{XX})$. The following Corollary represent that the absolute storage capacity of $\mathcal{M}_{XX}$ is unlimited.

Corollary

Let $X^+ \in \mathbb{F}^{n\times 2k}$. The set $F(\mathcal{M}_{XX})$ consist of all

$$\mathbf{y} = \mathcal{M}_{XX}(\mathbf{x}) = M_{XX}^+ \boxtimes \mathbf{x}^+ - M_{XX}^+ \boxtimes \mathbf{x}^-$$

such that $\mathbf{x} \in \mathbb{F}^n$. This Implies that $\mathbf{y} \preceq \mathbf{x}$

A MAM on the cisl $(\mathbb{F}^n, \preceq)$

Lemma

Suppose $X \in \mathbb{F}^{n\times k}$. The matrix M_{XX}^{+} has a zero diagonal and non negative entries. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ are fixed points of $\mathcal{M}_{XX}$, then $-\mathbf{x}$, $\mathbf{x} \curlywedge \mathbf{y}$, $\mathbf{x} \curlyvee \mathbf{y}$ (same side of reference) are fixed points of $\mathcal{M}_{XX}$.

proof

$$
\begin{aligned}
\mathcal{M}_{XX}(-\mathbf{x}) &= M_{XX}^{+} \boxtimes (-\mathbf{x})^{+} - M_{XX}^{+} \boxtimes (-\mathbf{x})^{-} \\
&= M_{XX}^{+} \boxtimes \mathbf{x}^{-} - M_{XX}^{+} \boxtimes \mathbf{x}^{+} \\
&= \mathbf{x}^{-} - \mathbf{x}^{+} = -\left(\mathbf{x}^{+} - \mathbf{x}^{-}\right) = -\mathbf{x}
\end{aligned}
$$

.

Characterizations of fixed points

$$
\begin{aligned}
\mathcal{M}_{XX}(\mathbf{x} \curlywedge \mathbf{y}) &= M_{XX}^{+} \boxtimes (\mathbf{x} \wedge \mathbf{y})^{+} - M_{XX}^{+} \boxtimes (\mathbf{x} \wedge \mathbf{y})^{-} \\
&= M_{XX}^{+} \boxtimes \left(\mathbf{x}^{+} \wedge \mathbf{y}^{+}\right) - M_{XX}^{+} \boxtimes \left(\mathbf{x}^{-} \vee \mathbf{y}^{-}\right) \\
&= \left(M_{XX}^{+} \boxtimes \mathbf{x}^{+}\right) \wedge \left(M_{XX}^{+} \boxtimes \mathbf{y}^{+}\right) - \left(M_{XX}^{+} \boxtimes \mathbf{x}^{-}\right) \vee \left(M_{XX}^{+} \boxtimes \mathbf{y}^{-}\right. \\
&= \mathbf{x}^{+} \wedge \mathbf{y}^{+} - \mathbf{x}^{-} \vee \mathbf{y}^{-} = (\mathbf{x} \wedge \mathbf{y})^{+} - (\mathbf{x} \wedge \mathbf{y})^{-} = \mathbf{x} \curlywedge \mathbf{y} = \mathbf{z}
\end{aligned}
$$

If the component of **x**, and **y** have same side with reference zero. Then the supremum is exist. Otherwise the supremum does not exist.

$$
\begin{aligned}
\mathcal{M}_{XX}(\mathbf{x} \curlyvee \mathbf{y}) &= M_{XX}^{+} \boxtimes (\mathbf{x} \vee \mathbf{y})^{+} - M_{XX}^{+} \boxtimes (\mathbf{x} \vee \mathbf{y})^{-} \\
&= M_{XX}^{+} \boxtimes \left(\mathbf{x}^{+} \vee \mathbf{y}^{+}\right) - M_{XX}^{+} \boxtimes \left(\mathbf{x}^{-} \wedge \mathbf{y}^{-}\right) \\
&= \left(M_{XX}^{+} \boxtimes \mathbf{x}^{+}\right) \vee \left(M_{XX}^{+} \boxtimes \mathbf{y}^{+}\right) - \left(M_{XX}^{+} \boxtimes \mathbf{x}^{-}\right) \wedge \left(M_{XX}^{+} \boxtimes \mathbf{y}^{-}\right. \\
&= \mathbf{x}^{+} \vee \mathbf{y}^{+} - \left(\mathbf{x}^{-} \wedge \mathbf{y}^{-}\right) = (\mathbf{x} \vee \mathbf{y})^{+} - (\mathbf{x} \vee \mathbf{y})^{-} = \mathbf{x} \curlyvee \mathbf{y}
\end{aligned}
$$

Lemma

If $\mathbf{x}_1$ and $\mathbf{x}_2 \in (\mathbb{F}^n)^+$ are disjoint fixed points of M_{XX}^+,then $\mathbf{x}_1 - \mathbf{x}_2 \in \mathbb{F}^n$ are also fixed points of $\mathcal{M}_{XX}$.

Proof

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2 \in (\mathbb{F}^n)^+$ are disjoint fixed points of M_{XX}^+.

$$\begin{aligned}\mathcal{M}_{XX}(\mathbf{x}_1 - \mathbf{x}_2) &= M_{XX}^+ \boxtimes (\mathbf{x}_1 - \mathbf{x}_2)^+ - M_{XX}^+ \boxtimes (\mathbf{x}_1 - \mathbf{x}_2)^- \\ &= M_{XX}^+ \boxtimes \mathbf{x}_1 - M_{XX}^+ \boxtimes \mathbf{x}_2 = \mathbf{x}_1 - \mathbf{x}_2\end{aligned}$$

Characterizations of fixed points

Theorem

1. If $\mathbf{x}$ is a fixed point translated by positive constant a, then $\mathcal{M}_{XX}(a+\mathbf{x}) \longrightarrow \mathcal{M}_{XX}(\mathbf{x}) = \mathbf{x}$.
2. If $\mathbf{x}$ is a fixed point translated by negative constant a, then $\mathcal{M}_{XX}(a+\mathbf{x}) = (a+\mathbf{x})$.

Proof

1. For $a > 0$

$$\begin{aligned} \mathcal{M}_{XX}(a+\mathbf{x}) &= M_{XX}^{+} \boxtimes (a+\mathbf{x})^{+} - M_{XX}^{+} \boxtimes (a+\mathbf{x})^{-} \\ &\cong \mathbf{x}^{+} - \mathbf{x}^{-} = \mathbf{x}. \end{aligned}$$

since $(\mathbf{x})^{+} \leq (a+\mathbf{x})^{+} \Longleftrightarrow (x_j)^{+} \leq (a+x_j)^{+} \, \forall j = 1:n$

$$\wedge_{j=1}^{n}\left(m_{ij}+(x_j)^+\right)\leq\wedge_{j=1}^{n}\left(m_{ij}+(x_j+a)^+\right)$$

$$\bigwedge_{j=1}^{n}\left(m_{ij}+(x_j+a)^+\right)=\begin{cases}0=(x_k)^+ & \text{if}(a+x_k)^+=0\\ (x_s)^+ & \text{if}(a+x_s)^+=(x_s)^+\\ m_{tj}=(x_t)^+ & \text{if}(a+x_t)^+>(x_t)^+\text{where}t\neq j\end{cases}$$

$$(x_i)^+=\bigwedge_{j=1}^{n}\left(m_{ij}+(x_j)^+\right)\leq\bigwedge_{j=1}^{n}\left(m_{ij}+(x_j+a)^+\right)\cong(x_i)^+$$

$$\Longleftrightarrow M_{XX}^{\pm}\boxtimes(a+\mathbf{x})^+\cong\mathbf{x}^+.$$

Similarly $\quad M_{XX}^{\pm}\boxtimes(a+\mathbf{x})^-\cong\mathbf{x}^-.$

(2) If $a<0$

$$\begin{aligned}\mathcal{M}_{XX}(a+\mathbf{x}) &= M_{XX}^{+}\boxtimes(a+\mathbf{x})^+-M_{XX}^{+}\boxtimes(a+\mathbf{x})^-\\ &= (a+\mathbf{x})^+-(a+\mathbf{x})^-=(a+\mathbf{x}).\end{aligned}$$

Since $(\mathbf{x})^+ = M^+_{XX} \boxed{\wedge} (\mathbf{x})^+ \geq (a + \mathbf{x})^+ \geq M^+_{XX} \boxed{\wedge} (a + \mathbf{x})^+$.
The pattern $M^+_{XX} \boxed{\wedge} (a + \mathbf{x})^+$ is a fixed point of M^+_{XX}. Also $M^+_{XX} \boxed{\wedge} (a + \mathbf{x})^+$ is the greatest fixed point, which is less than or equal to $(a + \mathbf{x})^+$, so they must be $M^+_{XX} \boxed{\wedge} (a + \mathbf{x})^+ = (a + \mathbf{x})^+$.
Similarly $M^+_{XX} \boxed{\wedge} (a + \mathbf{x})^- = (a + \mathbf{x})^-$.

Example

$$X^{+} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 5 & 4 \\ 0 & 6 & 6 & 0 \\ 3 & 4 & 0 & 0 \\ 7 & 1 & 0 & 0 \end{bmatrix} \quad M_{XX}^{+} = \begin{bmatrix} 0 & 1 & 0 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 2 & 2 \\ 4 & 5 & 0 & 4 & 5 & 5 \\ 6 & 6 & 6 & 0 & 6 & 6 \\ 4 & 4 & 4 & 3 & 0 & 3 \\ 7 & 5 & 7 & 7 & 4 & 0 \end{bmatrix}$$

Since $\mathbf{x} = [-3\ -2\ -4\ 6\ 4\ 1]^t$ is a fixed point of $\mathcal{M}_{XX}$. Here $\mathcal{M}_{XX}(1+\mathbf{x}) = \mathbf{x}$ and $\mathcal{M}_{XX}(-1+\mathbf{x}) = -1+\mathbf{x}$

$$1 + \begin{bmatrix} -3 \\ -2 \\ -4 \\ 6 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ -5 \\ 7 \\ 5 \\ 1 \end{bmatrix}, -1 + \begin{bmatrix} -3 \\ -2 \\ -4 \\ 6 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -3 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

An AM with Varying Reference Element

Notations

Let $\rho : \mathbb{F}^n \to \mathbb{F}^n$ be a "reference function". Consider $X_\rho^+ \in \mathbb{F}^{n \times 2k}$ given by $[(\mathbf{x}^1 - \rho(\mathbf{x}^1))^+, \ldots, (\mathbf{x}^k - \rho(\mathbf{x}^k))^+, (\mathbf{x}^1 - \rho(\mathbf{x}^1))^-, \ldots, (\mathbf{x}^k - \rho(\mathbf{x}^k))^-]$.

Theorem

If M_{XX}^ρ denotes the matrix $M_{X_\rho^+ X_\rho^+} \in \mathbb{F}^{n \times n}$ then we define:

$$\mathcal{M}_\rho(\mathbf{x}) = M_{XX}^\rho \boxtimes (\mathbf{x} - \rho(\mathbf{x}))^+ - M_{XX}^\rho \boxtimes (\mathbf{x} - \rho(\mathbf{x}))^- + \rho(\mathbf{x})\,,$$

For all $X \in \mathbb{F}^{n \times k}, \mathbf{x} \in \mathbb{F}^n$ we have:

$$\mathcal{M}_\rho(\mathbf{x}^\xi) = \mathbf{x}^\xi \, \forall \xi = 1, \ldots, k\,,$$

Image recalled by the Median filter and $\mathcal{M}_\rho$

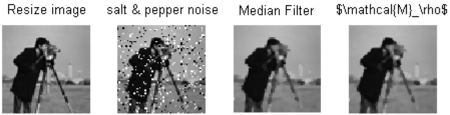


Figure:

Further Research Issues

- To choose of ρ in application of $\mathcal{M}_\rho$
- To Show any minimax combination of input pattern is a fixed point.
- To produced a new Auto associative memory model for Commutative complete lattice ordered double Monoid.