Sparse Distribution representation

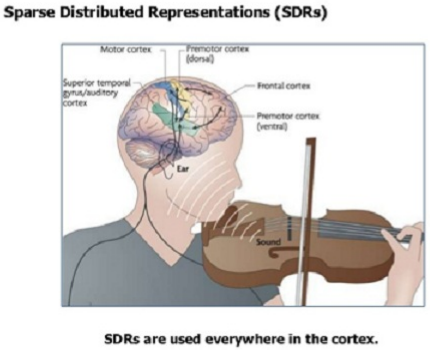

Figure: 1

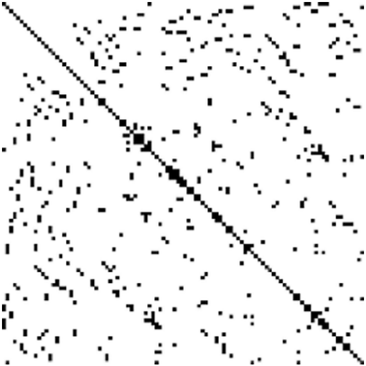

Figure: 2

Sparsely Connected Autoassociative Lattice Memories

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Organization of this talk

1. Introduction
2. Sparsely Connected Autoassociative Lattice Memories
3. The Relationship between SCALMs and Gray-Scale AMMs
4. Application of SCALMs for the Reconstruction of Color Images

Some definitions on Morphological Operators

Erosion and Dilation

let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices and $\epsilon, \delta : \mathbb{L} \to \mathbb{M}$:

1. ϵ is called an (*algebraic*) erosion if $\forall$ J, $\forall$ $x_j \in \mathbb{L}$:

$$\epsilon(\bigwedge_{j \in J} x_j) = \bigwedge_{j \in J} \epsilon(x_j)\,.$$

2. δ is called an (*algebraic*) dilation if $\forall$, J, $\forall$ $x_j \in \mathbb{L}$:

$$\delta(\bigvee_{j \in J} x_j) = \bigvee_{j \in J} \delta(x_j)\,.$$

Some definitions on Morphological Operators

Adjunctions

Consider $\epsilon : \mathbb{L} \to \mathbb{M}$ and $\delta : \mathbb{M} \to \mathbb{L}$ where $\mathbb{L}$ and $\mathbb{M}$ are complete lattices.

1. The pair (ϵ, δ) is called an adjunction (*from* $\mathbb{L}$ *to* $\mathbb{M}$) iff :

$$\delta(x) \leq y \Leftrightarrow x \leq \epsilon(y) \forall x \in \mathbb{L}, y \in \mathbb{M}.$$

   In this case, ϵ and δ are said to be adjoint.

2. If ϵ and δ are adjoint then ϵ is an erosion and δ is a dilation.
3. Let (ϵ, δ) be an adjunction then the following relation hold;

$$\epsilon(x) = \bigvee \{y \in \mathbb{M} : \delta(y) \leq x\}$$

$$\delta(y) = \bigwedge \{x \in \mathbb{L} : y \leq \epsilon(x)\}$$

Sparsely Connected Autoassociative Lattice Memories

Autoassociative Memories

Given a set $\{x^1 ... x^k\}$, an AM is a mapping $\mathcal{A}$ such that $\mathcal{A}\left(x^\xi\right) = x^\xi$. Furthermore, $\mathcal{A}\left(\widetilde{x}^\xi\right) = x^\xi$ for noise or incomplete version $\widetilde{x}^\xi$ of x^ξ .

Characteristics

1. They exhibit optimal absolute storage capacity.
2. They exhibit one step convergence when employed with feedback.

Sparsely Connected Autoassociative Lattice Memories

Definition

Given a fundamental memory set $\{x^1, ..., x^p\} \subseteq \mathbb{V}^n$, the partial order define on $\mathbb{V}$ is used in the set $S \subseteq N \times N$ where $N = \{1, 2, .., n\}$.

$$S = \left\{(i,j) : x_i^\xi \leq x_j^\xi, \forall \xi = 1, 2, ..., p\right\} \quad (1)$$

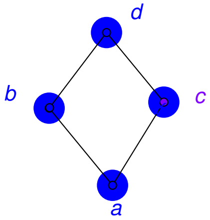
Supremum and infimum operations on SCALM

Let $\mathcal{M}$ and $\mathcal{W}$ be the mapping on $\chi = \mathbb{V}^n$ defined as follows: for $x \in \chi$

$$[\mathcal{M}(x)]_i = \bigwedge \{x_j : (i,j) \in S\} \; \forall i \in N \quad (2)$$

$$[\mathcal{W}(x)]_i = \bigvee \{x_j : (j,i) \in S\} \; \forall i \in N \quad (3)$$

Example of the SCALM



Hasse diagram

$\mathbb{V} = \{a, b, c, d\}$ represent complete lattice with $b \vee c = d$ and $b \wedge c = a$.

Example

Fundamental memory $x^1 = [d, b, c, c], x^2 = [d, c, a, b]$ and $x^3 = [b, a, c, d] \in \mathbb{V}^4$

$S = \{(1, 1), (2, 1), (2, 2), (3, 3), (3, 4), (4, 4)\}$.

Input pattern $x = [c, b, c, a]$.

Output patterns :

$\mathcal{M}(x) = [x_1, x_1 \wedge x_2, x_3 \wedge x_4, x_4] = [c, a, a, a]$

$\mathcal{W}(x) = [x_1 \vee x_2, x_2, x_3, x_3 \vee x_4] = [d, b, c, c]$.

Properties of the SCALM

1. SCALMs exhibit optimal absolute storage capacity.
2. They exhibit one step convergence when employed with feedback.
3. They are correspond to single layer feedforward neural network.
4. Computational point view, the number of synaptic junctions of $\mathcal{M}$ and $\mathcal{W}$ usually decreases (considerably) as the number of fundamental memories increase.
5. The pattern recalled by $\mathcal{W}$ represents the smallest fixed point of the model that is greater than or equal to the input pattern.
6. They are require less computational efforts.
7. They have large number of spurious memories.

Erosion, Dilation and Adjunction on SCALMs

Erosion and Dilation

Given a fundamental memory set $\left\{x^1, ..., x^p\right\}$, the SCALMs $\mathcal{M}$ and $\mathcal{W}$ given by $(1), (2)$ and (3) respectively.

1. An erosion is defined by for all subset $X \subseteq \chi$":

$$\mathcal{M}\left(\bigwedge X\right) = \bigwedge_{x \in X} \mathcal{M}(x).$$

2. A dilation is given by for all subset $X \subseteq \chi$":

$$\mathcal{W}\left(\bigvee X\right) = \bigvee_{x \in X} \mathcal{W}(x).$$

Adjunction on SCALMs

Adjunction

Consider the pair $(\mathcal{M}, \mathcal{W})$ is an adjunction on $\mathcal{X}$, i.e the following relation hold for all $x, y \in \mathcal{X}$.

$$\mathcal{W}(y) \leq_{\mathcal{X}} x \Leftrightarrow y \leq_{\mathcal{X}} \mathcal{M}(x).$$

- The relation between $\mathcal{M}$ and $\mathcal{W}$ as follows for every input pattern $x \in \mathcal{X}$.

$$\mathcal{M}(x) = \bigvee \{y \in \chi : \mathcal{W}(y) \leq_{\mathcal{X}} x\},$$

$$\mathcal{W}(x) = \bigwedge \{y \in \chi : x \leq_{\mathcal{X}} \mathcal{M}(y)\},$$

The Opening and Closing on SCALMs

The SCALMs $\mathcal{M}$ and $\mathcal{W}$ constitute an opening and closing respectively.

Opening $\mathcal{M}$

- $\mathcal{M}$ is increasing i.e $(x \leq_{\mathcal{X}} y \Rightarrow \mathcal{M}(x) \leq_{\mathcal{X}} \mathcal{M}(y))$.
- $\mathcal{M}$ is idempotent i.e $\mathcal{M}^2 = \mathcal{M}$.
- $\mathcal{M}$ is anti-extensive i.e $(\mathcal{M}(x) \leq_{\mathcal{X}} x \, \forall x \in \mathcal{X})$.

Closing $\mathcal{W}$

- $\mathcal{W}$ are increasing and idempotent.
- $\mathcal{W}$ is extensive i.e $x \leq_{\mathcal{X}} \mathcal{W}(x) \, \forall x \in \mathcal{X}$.

Invariance Domain of $\mathcal{M}$ and $\mathcal{W}$

Invariance Domain

Invariance domain is the collection of all fixed points of ψ, i.e

$$Inv\,(\psi) = \{x \in \chi : \psi(x) = x\}\,.$$

Inv($\mathcal{M}$) is sup-closed.

Mathematically as, for every pattern $x \in \chi$.

$$\mathcal{M}\,(x) = \bigvee \{y \in Inv(\mathcal{M}) : y \leq_{\chi} x\}$$

Inv($\mathcal{W}$) is inf-closed.

Mathematically as,

$$\mathcal{W}\,(x) = \bigwedge \{y \in Inv(\mathcal{W}) : x \leq_{\chi} y\}$$

Lattice based operations from Minimax algebra

Max product, Min product

Let $\mathbb{V}$ be a complete $\mathbb{L}$-group extension. Let $A \in \mathbb{V}^{m \times p}$ and $B \in \mathbb{V}^{p \times n}$.

1. The max-product of A and B is given by

$$C = A \boxdot\!\!\!\!\!\vee\, B \iff c_{ij} = \bigvee_{\xi=1}^{p} \left(a_{i\xi} + b_{\xi j}\right).$$

2. The min-product of A and B is given by

$$C = A \boxdot\!\!\!\!\!\wedge\, B \iff c_{ij} = \bigwedge_{\xi=1}^{p} \left(a_{i\xi} \acute{+} b_{\xi j}\right).$$

In this case we assume that $\mathbb{V} = \mathbb{R} \cup \{-\infty, +\infty\}$

Gray-scale AMMs

Definitions of W_{XX} and $M_{XX} \in \mathbb{V}^{n \times n}$

For $X = [\mathbf{x}^1, \ldots, \mathbf{x}^p] \in \mathbb{V}^{n \times p}$

$$[W_{XX}]_{ij} = \bigwedge_{\xi=1}^{p} \left(x_i^\xi \acute{+} (x_j^\xi)^* \right) . \quad (4)$$

$$[M_{XX}]_{ij} = \bigvee_{\xi=1}^{p} \left(x_i^\xi + (x_j^\xi)^* \right) . \quad (5)$$

Gray-scale AMMs $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$

Given $x \in \chi$, the outputs of $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$ are resp. calculated in terms of a dilation and an erosion:

$$\mathcal{M}_{XX}(x) = M_{XX} \boxtimes x \ , \ \mathcal{W}_{XX}(x) = W_{XX} \boxtimes x \, .$$

Relationship between the gray-scal AMMs and the SCALMs

Theorem

Given a set $\{x^1, ..., x^p\}$, the SCALMs $\mathcal{M}$ and $\mathcal{W}$ given by (1), (2) and (3). So there exist unique synaptic weight matrices M and $W \in \mathbb{V}^{n\times n}$ such that

$$\mathcal{M}(x) = M \boxed{\wedge} x \ , \ \mathcal{W}(x) = W \boxed{\vee} x \, .$$

for any input pattern $x \in \chi$.

These two matrices can be obtained from synaptic weight matrices M_{XX} and W_{XX} given by 5 and 4 as for all $i, j \in \mathbb{N}$:

$$m_{ij} = \mathcal{T}_+ \left([M_{XX}]_{ij} \right) \quad w_{ij} = \mathcal{T}_- \left([W_{XX}]_{ij} \right) .$$

Relationship between the gray-scal AMMs and the SCALMs

where $\mathcal{T}_+ : \mathbb{V} \to \mathbb{V}$ and $\mathcal{T}_- : \mathbb{V} \to \mathbb{V}$ are threshold operators given by

$$\mathcal{T}_+(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathcal{T}_-(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Its means that SCALMs $\mathcal{M}$ and $\mathcal{W}$ can be obtained from the gray-scale AMMs $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$ by thresholding their synaptic weight matrices.

Consequences of the previous Theorem

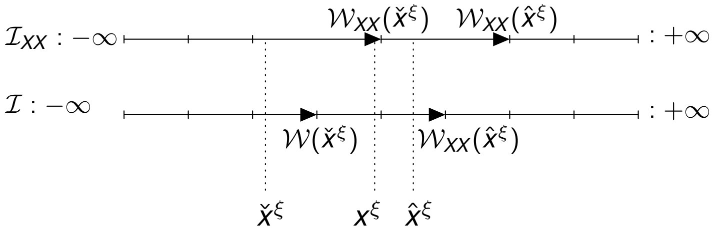
1. $\mathcal{M}_{XX}(x) \leq \mathcal{M}(x)$ and $\mathcal{W}(x) \leq \mathcal{W}_{XX}(x)$.
2. $\text{Inv}(\mathcal{M}_{XX}) \subseteq \mathcal{I}$ and $\text{Inv}(\mathcal{W}_{XX}) \subseteq \mathcal{I}$.
3. Invariance domains $\mathcal{I}_{XX} \subseteq \mathcal{I}$.
   The Invariance domain $\mathcal{I}$ of the SCALMs include all the fixed points of the gray-scale AMMs $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$.

Advantage of SCALMs

1. Less Computational effort and consumed much less memory space.

Noise tolerance of grayScale MAMs and SCALMs

Geometrically



Original Color Images

Consider the following images of size 384 $\times$ 256.

SCALMs defined on the RGB color models

Ordering schemes

- Marginal RGB Ordering. $(\leq^M_{RGB})$

$$u \leq^M_{RGB} \mu \Leftrightarrow \begin{cases} u_r \leq \mu_r, & \\ u_g \leq \mu_g, & \text{and} \\ u_b \leq \mu_b, & . \end{cases}$$

- Lexicographical RGB Ordering. $(\leq^L_{RGB})$

$$u \leq^L_{RGB} \mu \Leftrightarrow \begin{cases} u_r < \mu_r, & \text{or} \\ u_r = \mu_r, & \text{and } u_g < \mu_g, \text{ or} \\ u_r = \mu_r, & u_g = \mu_g, \text{and } u_b \leq \mu_b. \end{cases}$$

SCALMs defined on the RGB color models

- Reduce the excessive dependence of the first component by creating equivalence groups.

Ordering schemes

- α- modulus lexicographical RGB ordering. $(\leq_{RGB}^{\alpha})$

$$u \leq_{RGB}^{\alpha} \mu \Leftrightarrow \begin{cases} \left\lfloor \frac{u_r}{\alpha} \right\rfloor < \left\lfloor \frac{\mu_r}{\alpha} \right\rfloor, & \text{or} \\ \left\lfloor \frac{u_r}{\alpha} \right\rfloor = \left\lfloor \frac{\mu_r}{\alpha} \right\rfloor, & \text{and } u_g < \mu_g, \text{ or} \\ \left\lfloor \frac{u_r}{\alpha} \right\rfloor = \left\lfloor \frac{\mu_r}{\alpha} \right\rfloor, & u_g = \mu_g, \text{and } u_b < \mu_b \text{ or} \\ \left\lfloor \frac{u_r}{\alpha} \right\rfloor = \left\lfloor \frac{\mu_r}{\alpha} \right\rfloor, & u_g = \mu_g, u_b = \mu_b, \text{and } u_r \leq \mu_r. \end{cases}$$

Note: $\alpha \in (0, 1]$ and $u = (u_r, u_g, u_b), \mu = (\mu_r, \mu_g, \mu_b) \in \mathbb{V}_{RGB}$.

Lexicographical Ordering on the Karhunen-Loeve Color System

Karhunen-Loeve Transform (KLT)

let $u_1, ..., u_m$ denote the color values of RGB images.

- column vector mean $m \in \mathbb{R}^3$ and covariance matrix $C \in \mathbb{R}^{3\times3}$.

$$m = \frac{1}{n}\sum_{i=1}^{n} u_i \text{ and } C = \frac{1}{n}\left(\sum_{i=1}^{n} u_i u_i^T\right) - mm^T.$$

- Given a $u \in \mathbb{V}_{RGB}$, the corresponding element $v \in \mathbb{V}_{KLT}$ is given by

$$v = Q(u - m) \iff u = Q^T v + m \text{ by inverse transform.}$$

where $Q = [q_1, q_2, q_3]^T$ denotes orthogonal matrix.

Lexicographical Ordering on the Karhunen-Loeve Color System

Lexicographical order on KLT system

Given two points u and $\mu \in \mathbb{V}_{RGB}$ and the corresponding points $v = (v_1, v_2, v_3)$ and $\nu = (\nu_1, \nu_2, \nu_3)$ on the KLT system.

- Define $u \leq^L_{KLT} \mu \Leftrightarrow v \leq^L_{KLT} \nu$ in the lexicographical ordering i,e

$$u \leq^L_{KLT} \mu \Leftrightarrow \begin{cases} v_1 < \nu_1, & \text{or} \\ v_1 = \nu_1, & \text{and } \; v_2 < \nu_2, \text{ or} \\ v_1 = \nu_1, & v_2 = \nu_2, \text{and } \; v_3 \leq \nu_3. \end{cases}$$

Noisy Images

We introduced the following types of noise:

1. Impulsive noise with probabilities $p_n = 0.1, p_r = p_g = p_b = 0.25$.
2. Gaussian noise (mean 0 and variance 0.01).

Figure: Images ‘parrots‘ and ‘caps‘ corrupted by impulsive and Gaussian noise

Experiment of SCALMs for the reconstruction of color images

Image recalled by the SCALMs $\mathcal{W}^{M}_{RGB}$ (first row) and $\mathcal{M}^{M}_{RGB}$ (Second row).

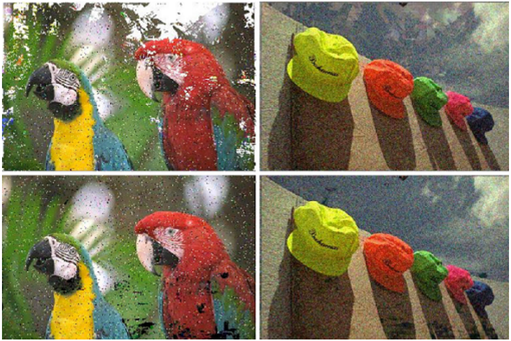


Figure: 3. $\mathcal{W}^{M}_{RGB}$ and $\mathcal{M}^{M}_{RGB}$ polluted with white and black colors.

Image recalled by the SCALMs $\mathcal{W}^{L}_{RGB}$ (first row) and $\mathcal{M}^{L}_{RGB}$ Second row.

Figure: 4. $\mathcal{W}^{L}_{RGB}$ and $\mathcal{M}^{L}_{RGB}$ have been contaminated with white and black, and caps has polluted with red and cyan.

Image recalled by the SCALMs $\mathcal{W}^{\alpha}_{RGB}$ (first row) and $\mathcal{M}^{\alpha}_{RGB}$ Second row.

Figure: 5. Very similar Lexicographical order. Here $\alpha = 20/255$

Image recalled by the SCALMs $\mathcal{W}^L_{KLT}$ (first row) and $\mathcal{M}^L_{KLT}$ Second row.

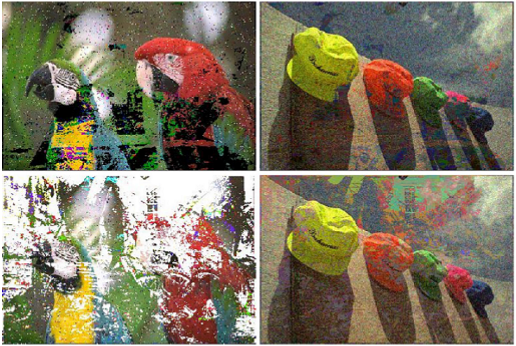


Figure: 6. $\mathcal{W}^L_{KLT}$ and $\mathcal{M}^L_{KLT}$ have been contaminated with black and white color.