Exemplo: Identifique a ~~cônica~~ dada por

$$\frac{7}{2}x^2+3xy-\frac{1}{2}y^2+\sqrt{10}\,x+\sqrt{10}\,y=0$$

Passo 1

(a) Eq. na forma matricial:

$$(x\ y)\underbrace{\begin{pmatrix} \frac{7}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}}_{A}\begin{pmatrix} x \\ y\end{pmatrix}+(\sqrt{10}\ \sqrt{10})\begin{pmatrix} x \\ y\end{pmatrix}=0$$

Note que $\det(A) < 0$, i.e. é hipe... ou par de retas

(b) Cálculo dos autovalores λ_1, λ_2:

$$0=\det\begin{pmatrix} \frac{7}{2}-\lambda & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2}-\lambda\end{pmatrix}=\left(\frac{7}{2}-\lambda\right)\left(\frac{1}{2}+\lambda\right)(-1)-\frac{9}{4}$$

$$=\left(\frac{7}{4}-\frac{1}{2}\lambda+\frac{7}{2}\lambda-\lambda^2\right)(-1)-\frac{9}{4}$$

$$=\lambda^2-3\lambda-\frac{7}{4}-\frac{9}{4}=\lambda^2-3\lambda-4=$$

$$=(\lambda-4)(\lambda+1)$$

$$\therefore\ \lambda_1=4,\ \lambda_2=-1$$

(c) Autovetores para λ_1, λ_2: Resolver $(A-\lambda_1 I)V=0$

(i) $\lambda_1=4$:

$$\begin{pmatrix} \frac{7}{2}-4 & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2}-4\end{pmatrix}\rightsquigarrow\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2}\end{pmatrix}\rightsquigarrow\begin{pmatrix} -1 & 3 \\ 3 & -9\end{pmatrix}$$

$$\rightsquigarrow\begin{pmatrix} -1 & 3 \\ 1 & -3\end{pmatrix}\rightsquigarrow\begin{array}{cc} & \text{variável livre} \\ & \downarrow \end{array}\begin{pmatrix} -1 & 3 \\ 0 & 0\end{pmatrix}\quad x=3y\quad S=\left\{\begin{pmatrix} 3y \\ y\end{pmatrix}\,\middle|\, y\in\mathbb{R}\right\}$$

$$=\left\{\alpha\begin{pmatrix} 3 \\ 1\end{pmatrix}\,\middle|\,\alpha\in\mathbb{R}\right\}$$

Seja $V_1 = \begin{pmatrix}3\\1\end{pmatrix}$ e $\bar{V}_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$

(ii) $\lambda_2 = -1$

$$\begin{pmatrix}\frac{7}{2}-\lambda_2 & \frac{3}{2}\\ \frac{3}{2} & -\frac{1}{2}-\lambda_2\end{pmatrix} \rightsquigarrow \frac{1}{2}\begin{pmatrix}9 & 3\\ 3 & 1\end{pmatrix} \rightsquigarrow \begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix}3 & 1\\ 0 & 0\end{pmatrix} \quad y = -3x \quad S = \left\{\begin{pmatrix}x\\-3x\end{pmatrix} \mid x \in \mathbb{R}\right\}$$

$$= \left\{\alpha\begin{pmatrix}1\\-3\end{pmatrix} \mid \alpha \in \mathbb{R}\right\}$$

Seja $V_2 = \begin{pmatrix}1\\-3\end{pmatrix}$ e $\bar{V}_2 = \frac{V_2}{\|V_2\|} = \frac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$

$\bar{\mathcal{B}} = \{\bar{V}_1, \bar{V}_2\}$ é base ortonormal de autovetores de A

(d) Encontre a matriz de rotação que representa a matriz da mudança de base

Seja $\mathcal{B} = \{E_1, E_2\}$ e Temos $I_{\mathcal{B}}^{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix}3 & 1\\ 1 & -3\end{pmatrix}$

Daí $\det\left(I_{\mathcal{B}}^{\bar{\mathcal{B}}}\right) = \frac{1}{10}\cdot -10 = -1$, i.e. não dá uma matriz de rotação, mas note que escolhendo como autovetor de $\lambda_2 =$

$$V_2 = -\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ e } \bar{V}_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

obtemos $\bar{\mathcal{B}} = \left\{ \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$

e assim

$$I_{\mathcal{B}}^{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

Com $\det(I_{\mathcal{B}}^{\bar{\mathcal{B}}}) = 1$. Portanto

$$R = I_{\mathcal{B}}^{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \text{ é uma}$$

matriz de rotação.

(e) Eq. no sistema rodado $\bar{x}-\bar{y}$:

$$(\bar{x}\ \bar{y})_{\bar{\mathcal{B}}}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} + \sqrt{10}(1\ 1)\, I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} =$$

$$\Leftrightarrow (\bar{x}\ \bar{y})_{\bar{\mathcal{B}}}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} + (1\ 1)\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} = ($$

$$\Leftrightarrow 4\bar{x}^2 - \bar{y}^2 + (4\ \ 2)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} = 0$$

$$\Leftrightarrow 4\bar{x}^2 - \bar{y}^2 + 4\bar{x} + 2\bar{y} = 0$$

Passo 2: Completa Quadrados para Determinar a Translação, i.e. para colocar-l na forma canônica

$$4\bar{x}^2 + 4\bar{x} = 4(\bar{x}^2 + \bar{x})$$

$$\bar{x}^2 + \bar{x} = \bar{x}^2 + 2\cdot\tfrac{1}{2}\bar{x} + \tfrac{1}{4} - \tfrac{1}{4}$$

$$= \left(\bar{x} + \tfrac{1}{2}\right)^2 - \tfrac{1}{4}$$

$$\Rightarrow 4\bar{x}^2 + 4\bar{x} = 4\left(\bar{x} + \tfrac{1}{2}\right)^2 - 1$$

$$-\bar{y}^2 + 2\bar{y} = -(\bar{y}^2 - 2\bar{y})$$

$$\bar{y}^2 - 2\bar{y} = \bar{y}^2 - 2\cdot 1\bar{y} + 1 - 1 = (\bar{y} - 1)^2 - 1$$

$$\Rightarrow -\bar{y}^2 + 2\bar{y} = -(\bar{y} - 1)^2 + 1$$

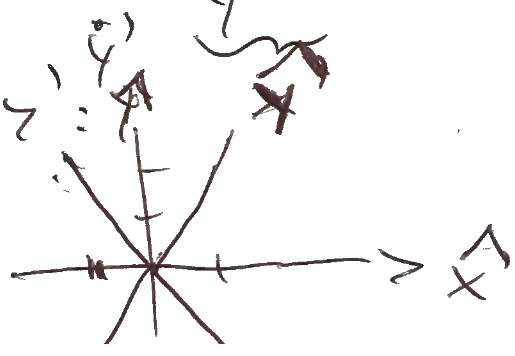
Portanto, obtemos

$$4\left(\bar{x} + \tfrac{1}{2}\right)^2 \cancel{-1} - (\bar{y} - 1)^2 \cancel{+1} = 0$$

$$\Leftrightarrow 4\left(\bar{x} + \tfrac{1}{2}\right)^2 = (\bar{y} - 1)^2$$

$$\Rightarrow \underbrace{\bar{y} - 1}_{y'} = \pm 2\underbrace{\left(\bar{x} + \tfrac{1}{2}\right)}_{x'}$$ par de retas

Em x', y':

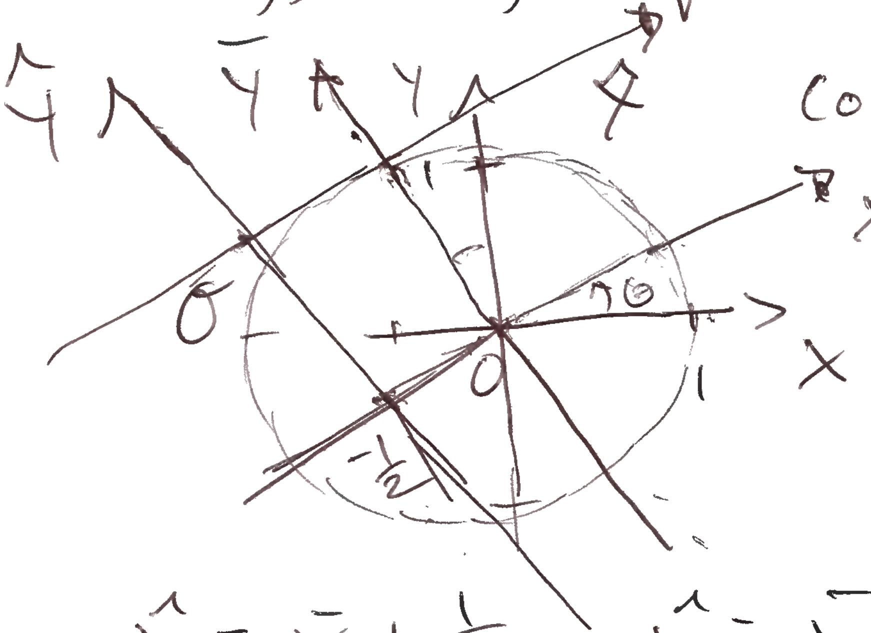


Note que as coordenadas de

$\begin{pmatrix}1\\0\end{pmatrix}_{\bar{\mathcal{B}}}$ e $\begin{pmatrix}0\\1\end{pmatrix}_{\bar{\mathcal{B}}}$ em $\mathcal{B}=\{E_1, E_2\}$ são

$$I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix}1\\0\end{pmatrix}_{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix}3 & -1\\ 1 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}_{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}$$

e $$I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix}0\\1\end{pmatrix}_{\bar{\mathcal{B}}} = \frac{1}{\sqrt{10}}\begin{pmatrix}-1\\3\end{pmatrix}$$



$$\cos\theta = \left\langle E_1, \frac{1}{\sqrt{10}}\begin{pmatrix}3\\1\end{pmatrix}\right\rangle = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \theta \approx 0{,}3218 \approx 18{,}435$$

$$\hat{x} = \bar{x} + \frac{1}{2}, \quad \hat{y} = \bar{y} - 1$$

$(\hat{x}=0 \Leftrightarrow \bar{x} = -\frac{1}{2})$ $\quad$ $(\hat{y}=0 \Leftrightarrow \bar{y} = 1)$

Exercício Identifique $\mathcal{C}$ dado por

$$\frac{7}{2}x^2 + 3xy - \frac{1}{2}y^2 - \sqrt{10}\,x + 7\sqrt{10}\,y = 0$$