On Integer Programming Approaches for Morphological Template Decomposition Problems in Computer Vision

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Abstract

In morphological image processing and analysis, a template or structuring element is applied to an image. Often savings in computation time and a better fit to the given computer architecture can be achieved by using the technique of template decomposition. Researchers have written a multitude of papers on finding such decompositions for special classes of templates. Justifying recent integer programming approaches to the morphological template decomposition problem in its general form, this paper proves the NP-completeness of this problem.

Key Words — Computer vision, mathematical morphology, template decomposition, complexity, integer programming.

Introduction

Morphological image processing is based on the theory of mathematical morphology, which grew out of the belief that many image processing operations can be expressed in terms of a few elementary operations [13, 18, 19, 9]. The operations of dilation and erosion form the mathematical basis for all morphological operations. The theory of image algebra incorporates all known image processing techniques in a translucent manner [17, 16, 4, 3]. In this setting, many image processing operations — including all morphological operations — are represented as convolutions of the image with a template, also known as structuring element. Computer scientists distinguish between linear convolutions and morphological convolutions. Morphological convolutions either consist of certain combinations

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of maximum and addition operations or certain combinations of minimum and addition operations. The decomposition of a template in image algebra consists of the representation of a convolution as a sequence of other convolutions, which involve a certain number of operations. In other words, a template is expressed in terms of other templates having certain supports.

Decomposition of templates is a helpful technique for increasing the computational efficiency of a multitude of image processing operations, either by decreasing the number of operations required or by organizing the computations into forms more compatible with specialized image processing architectures [6, 5, 11, 12]. Often these architectures call for the decomposition of arbitrary rectangular templates into 3x3 templates. Excellent savings in computation time can be achieved, if a templates decomposes into a small number of horizontal and vertical strip templates. A template is called separable if it can be expressed as the outer product of a column and a row template.

Decompositions in linear image processing include Fast Fourier Transforms, Laplacian and Gaussian masks. Decompositions in the nonlinear or morphological domain are thus far only known for a small class of special templates such as separable templates, convex dome shaped templates, and binary templates having certain supports [22, 10, 21, 14, 15]. Recently, researchers have attempted to solve a wide range of decomposition problems for gray-scale morphological templates through formulation as integer programming problems [7]. The most general of these integer programming approaches is concerned with solving arbitrary morphological template decomposition problems [20].

In this paper, we show that the general morphological template decomposition problem is NP-complete. This result justifies the use of integer programming for solving arbitrary instances of morphological template decomposition problems. This paper is organized as follows:

First, we introduce the reader to notations from the theory of image algebra. Then we briefly review the integer programming approaches for solving morphological template decomposition problems, which we introduced in a previous paper. Finally, we prove the main complexity results of this paper.

Image Algebra Background

The mathematical theory of image algebra provides a mathematical background for all computer vision and image processing applications [17, 16]. In

this section, we define some fundamental image algebra operands and operations, which simplify the discussion of the main results of this paper. Our attention focuses on morphological operations. Mathematical morphology, which is only concerned with morphological image processing operations, can be embedded into image algebra [1, 2]. Before we proceed, we introduce some commonly used notations.

Notations. Throughout this paper, the set of integers is denoted by $\mathbb{Z}$, and the set of real numbers is denoted by $\mathbb{R}$. We express the maximum of two numbers a and b by $a \vee b$. Similarly, the expression $a \wedge b$ stands for the minimum of a and b.

Point Sets and Value Sets. A *point set* $\mathbf{X}$ is a subset of a topological space. In digital image processing point sets are often assumed to be rectangular discrete arrays. Throughout this paper, we assume that $\mathbf{X}$ is a subset of $\mathbb{Z}^2$.

A *value set* — often denoted by $\mathbb{F}$ — is a homogeneous algebra such as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$, $\mathbb{R}_{+\infty}$ or $\mathbb{C}$. The algebraic system associated with $\mathbb{R}_{-\infty}$ will be the lattice ordered semi-group $(\mathbb{R}_{-\infty}, \vee, +)$ with the extended arithmetic and logic operations defined as follows:

$$\begin{aligned} a + (-\infty) = (-\infty) + a = -\infty \quad & \forall\, a \in \mathbb{R}_{-\infty}\,, \\ a \vee (-\infty) = (-\infty) \vee a = a \quad & \forall\, a \in \mathbb{R}_{-\infty}\,. \end{aligned}$$

The *dual* of this system is the lattice ordered semi-group $(\mathbb{R}_{+\infty}, \wedge, +)$.

Images and Templates. The theory of image algebra views images as functions and templates are considered functions whose values are images. In particular, an $\mathbb{F}$–valued *image* $\mathbf{a}$ over the point set $\mathbf{X}$ is a function $\mathbf{a} : \mathbf{X} \to \mathbb{F}$ (i.e. $\mathbf{a} \in \mathbb{F}^{\mathbf{X}}$), while an $\mathbb{F}$–valued *template* $\mathbf{t}$ on $\mathbf{X}$ is a function $\mathbf{t} : \mathbf{X} \to \mathbb{F}^{\mathbf{X}}$ (i.e. $\mathbf{t} \in \left(\mathbb{F}^{\mathbf{X}}\right)^{\mathbf{X}}$). For notational convenience, we define $\mathbf{t_y}$ as $\mathbf{t}(\mathbf{y})$ for all $\mathbf{y} \in \mathbf{X}$. Note that the *image* $\mathbf{t_y}$ has representation

$$\mathbf{t_y} = \{(\mathbf{x}, \mathbf{t_y}(\mathbf{x})) : \mathbf{x} \in \mathbf{X}\},$$

where the pixel values $\mathbf{t_y}(\mathbf{x})$ at location $\mathbf{x}$ of this image are called *template weights* at point $\mathbf{y}$. A template $\mathbf{t} \in \left(\mathbb{F}^{\mathbf{X}}\right)^{\mathbf{X}}$ is called *translation invariant* if and only if

$$\mathbf{t_{y+z}}(\mathbf{x} + \mathbf{z}) = \mathbf{t_y}(\mathbf{x}) \quad \forall\, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X} = \mathbb{Z}^2,$$

whenever $\mathbf{y}+\mathbf{z}$ and $\mathbf{x}+\mathbf{z}$ are elements of $\mathbf{X}$. The *support* of a template$\mathbf{t} \in \left(\mathbb{F}^{\mathbf{X}}\right)^{\mathbf{X}}$ at a point $\mathbf{y}$ is denoted by $S(\mathbf{t_y})$ and defined as follows:

$$S(\mathbf{t_y}) = \{\mathbf{x} \in \mathbf{X} \,:\, \mathbf{t_y}(\mathbf{x}) \neq \ 0\} \ ,$$

where 0 denotes the zero element of the algebraic structure $\mathbb{F}$.

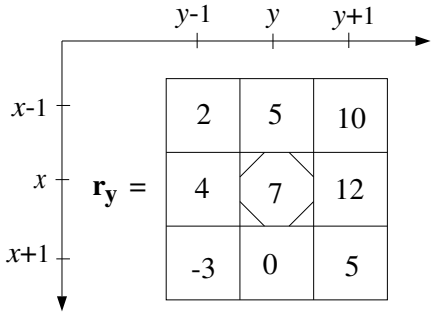
Since we are concerned with morphological operations on images, the value set $\mathbb{F}$ of interest will be he real numbers with the symbol $-\infty$ appended, i. e. $\mathbb{F} = \mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$. From now on, we are going to restrict ourselves to this value set. Note that the element $-\infty$ acts as a null element in the system $(\mathbb{R}_{-\infty}, \vee, +)$ if we view the operation $+$ as multiplication and the operation $\vee$ as addition.

Example. Let $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ be the translation invariant template which is determined at each point $\mathbf{y} = (x, y) \in \mathbf{X}$ by the following function values of $\mathbf{x} \in \mathbf{X}$:

$$\mathbf{r_y}(\mathbf{x}) = \begin{cases} 7 & if \ \ \mathbf{x} = \mathbf{y} \\ 4 & if \ \ \mathbf{x} = \mathbf{y} - (0,\,1) \\ 12 & if \ \ \mathbf{x} = \mathbf{y} + (0,\,1) \\ -2 + \mathbf{r_y}(\mathbf{x} + (1,0)) & if \ x_1 = x - 1 \ \ and \ \ y - 1 \leq x_2 \leq y + 1 \\ -7 + \mathbf{r_y}(\mathbf{x} - (1,0)) & if \ \ x_1 = x + 1 \ and \ \ y - 1 \leq x_2 \leq y + 1 \\ -\infty & else \ . \end{cases}$$

We can visualize the rectangular template $\mathbf{r}$ as shown in Figure 1.

Figure 1. The support of the template $\mathbf{r}$ at point $\mathbf{y}$. The hashed cell indicates the location of the target point $\mathbf{y} = (x, y)$.



Rectangular Templates. A translation invariant template $\mathbf{r}$ is called *rectangular*, if $S(\mathbf{r_y})$ forms a rectangular discrete array. We speak in particular of an $m \times n$ template, if $S(\mathbf{r_y})$ has size $m \times n$.

We introduce the following convenient notation to denote the weights of an $m \times n$ template $\mathbf{r}$:

$$r_{ij} = r_{\mathbf{y}}((\tilde{x_1} + i - 1, \tilde{x_2} + j - 1))$$

where $\tilde{\mathbf{x}} = (\tilde{x_1}, \tilde{x_2})$ satisfies the conditions

$$\begin{gathered} \tilde{x_1} \leq x_1, \ \tilde{x_2} \leq x_2 \\ \forall\, \mathbf{x} = (x_1, x_2) \in S(\mathbf{r_y}) \ . \end{gathered}$$

For example, the 3×3 template in Figure 1 has the weight $r_{23} = 12$.

If $\mathbf{s}$ is an $m \times 1$ template, we abbreviate s_{i1} by s_i, and if $\mathbf{t}$ is an $1 \times n$ template, we abbreviate t_{ij} by t_j. The template $\mathbf{s}$ is said to be a *column template* of length m, and the template $\mathbf{t}$ is said to be a *row template* of length n.

Basic Operations. The basic operations of addition and maximum on $\mathbb{R}_{-\infty}$ induce pixelwise operations on $\mathbb{R}_{-\infty}$–valued images and templates. For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and any $\mathbf{s}, \mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$, we set

$$\begin{gathered} (\mathbf{a} + \mathbf{b})(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}), \, \forall\, \mathbf{x} \in \mathbf{X} \ , \\ (\mathbf{a} \vee \mathbf{b})(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \vee \mathbf{b}(\mathbf{x}), \, \forall\, \mathbf{x} \in \mathbf{X} \ , \\ (\mathbf{s} + \mathbf{t})_{\mathbf{y}} = \mathbf{s}_{\mathbf{y}} + \mathbf{t}_{\mathbf{y}}, \, \forall\, \mathbf{y} \in \mathbf{X} \ , \\ (\mathbf{s} \vee \mathbf{t})_{\mathbf{y}} = \mathbf{s}_{\mathbf{y}} \vee \mathbf{t}_{\mathbf{y}}, \, \forall\, \mathbf{y} \in \mathbf{X} \ . \end{gathered}$$

Additive Maximum Operations. Forming the *additive maximum* ("$\boxtimes$") of an image $\mathbf{a} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ results in the image $\mathbf{a} \boxtimes \mathbf{t} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$, which is determined by the following function values.

$$(\mathbf{a} \boxtimes \mathbf{t})(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x}) \ .$$

Clearly, each template $\mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ defines a function

$$\begin{aligned} f_{\mathbf{t}} \; : \; \mathbb{R}^{\mathbf{X}}_{-\infty} &\to \mathbb{R}^{\mathbf{X}}_{-\infty} \\ \mathbf{a} \quad &\rightarrowtail \mathbf{a} \boxtriangledown \mathbf{t} \; . \end{aligned}$$

The *additive maximum* of a template $\mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ and a template $\mathbf{s} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ is defined as the template $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ which determines $f_{\mathbf{s}} \circ f_{\mathbf{t}}$, the composition of $f_{\mathbf{t}}$ followed by $f_{\mathbf{s}}$. Specifically,

$$(\mathbf{s} \boxtriangledown \mathbf{t})_{\mathbf{y}}(\mathbf{z}) = \bigvee_{\mathbf{x} \in \mathbf{X}} (\mathbf{t}_{\mathbf{x}}(\mathbf{z}) + \mathbf{s}_{\mathbf{y}}(\mathbf{x})) \quad \forall \, \mathbf{y}, \; \mathbf{z} \in \mathbf{X} \; .$$

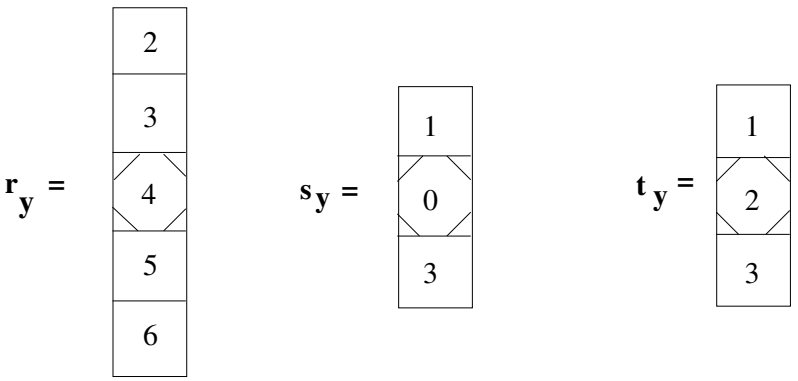
These relationships induce the associative and distributive laws given later. The "additive maximum" operation represents an example of a morphological convolution. The morphological convolution "additive minimum" is defined similarly and is denoted by the symbol "$\boxtriangle$".

In correspondence to the morphological convolutions defined above, Ritter also defines a linear convolution operation $\oplus$ for templates $\mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}\right)^{\mathbf{X}}$ and $\mathbf{s} \in \left(\mathbb{R}^{\mathbf{X}}\right)^{\mathbf{X}}$:

$$(\mathbf{s} \oplus \mathbf{t})_{\mathbf{y}}(\mathbf{z}) = \sum_{\mathbf{x} \in \mathbf{X}} (\mathbf{t}_{\mathbf{x}}(\mathbf{z}) \cdot \mathbf{s}_{\mathbf{y}}(\mathbf{x})) \quad \forall \, \mathbf{y}, \; \mathbf{z} \in \mathbf{X} \; .$$

Example. The following column templates $\mathbf{r}, \mathbf{s}, \mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ satisfy $\mathbf{r} = \mathbf{s} \boxtriangledown \mathbf{t}$.

Figure 2. The template $\mathbf{r}$ constitutes the additive maximum of the templates $\mathbf{s}$ and the template $\mathbf{t}$.



Some Properties of Image and Template Operations. The following associative and distributive laws hold for an arbitrary image $\mathbf{a} \in \mathbb{R}_{-\infty}^{\mathbf{X}}$ and arbitrary templates $\mathbf{s}, \mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$:

$$\mathbf{a} \boxtriangledown (\mathbf{s} \boxtriangledown \mathbf{t}) = (\mathbf{a} \boxtriangledown \mathbf{s}) \boxtriangledown \mathbf{t} \ ,$$
$$\mathbf{a} \boxtriangledown (\mathbf{s} \vee \mathbf{t}) = (\mathbf{a} \boxtriangledown \mathbf{s}) \vee (\mathbf{a} \boxtriangledown \mathbf{t}) \ .$$

These results establish the importance of template decomposition.

Decompositions of Templates. A sequence of templates $\left(\mathbf{t}^1, \ldots, \mathbf{t}^{k_n}\right)$ in $\left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ together with a strictly increasing sequence of natural numbers $(k_1, \ldots, k_n)$ is called a *weak morphological decomposition* (with respect to the operation "$\boxtriangledown$") of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ if the template $\mathbf{t}$ can be represented as follows:

$$\mathbf{t} = \left(\mathbf{t}^1 \boxtriangledown \ldots \boxtriangledown \mathbf{t}^{k_1}\right) \vee \left(\mathbf{t}^{k_1+1} \boxtriangledown \ldots \boxtriangledown \mathbf{t}^{k_2}\right) \vee \ldots \vee \left(\mathbf{t}^{k_{n-1}+1} \boxtriangledown \ldots \boxtriangledown \mathbf{t}^{k_n}\right) \ .$$

Similarly, a *weak linear decomposition* of a template $\mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}\right)^{\mathbf{X}}$ is given by a sequence of templates $\left(\mathbf{t}^1, \ldots, \mathbf{t}^{k_n}\right)$ in $\left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ together with a strictly increasing sequence of integers $(k_1, \ldots, k_n)$ such that

$$\mathbf{t} = \left(\mathbf{t}^1 \oplus \ldots \oplus \mathbf{t}^{k_1}\right) + \left(\mathbf{t}^{k_1+1} \oplus \ldots \oplus \mathbf{t}^{k_2}\right) + \ldots + \left(\mathbf{t}^{k_{n-1}+1} \oplus \ldots \oplus \mathbf{t}^{k_n}\right) \ .$$

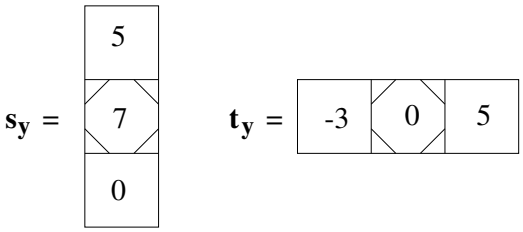
In the case where $n = 1$, we speak of a *strong decomposition* of the template $\mathbf{t}$. For example, a strong morphological decomposition of a template $\mathbf{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ has the following form:

$$\mathbf{t} = \mathbf{t}^1 \boxtriangledown \mathbf{t}^2 \boxtriangledown \ldots \boxtriangledown \mathbf{t}^k \ .$$

If the template $\mathbf{t}$ strongly decomposes into a column template $\mathbf{t}^1$ and a row template $\mathbf{t}^2$, it is referred to as a *separable template*.

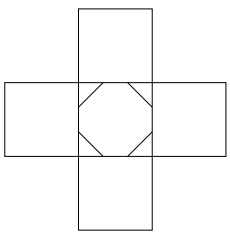
Example. The template $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ given in Figure 1 represents a separable template since this template decomposes into a vertical strip template $\mathbf{s} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ and a horizontal strip template $\mathbf{t} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$.

Figure 3. Pictorial representation of a column template $\mathbf{s}$ and a row template $\mathbf{t}$.



Relevance of Template Decompositions. Realizing the convolution of an image $\mathbf{a}$ with a template $\mathbf{t}$ on a computer is an extremely complex task, since each pixel value at location $\mathbf{y}$ depends on all pixel values of $\mathbf{a}$ within the neighborhood given by $\mathbf{t_y}$. Decomposing the template $\mathbf{t}$ often helps to reduce the number of computations involved in forming an image-template product by providing alternative ways to compute this convolution. For example, the application of a separable template to an image is equivalent to the sequential applications of a column template and a row template. Hence, a quadratic problem reduces to a linear problem. Computing the Fourier Transform of an image represents a linear convolution. The speedup of this computation achieved by the Fast Fourier Transform relies heavily on linear template decomposition.

Template decompositions have also been successfully used for the purpose of achieving a better fit to the computer architecture. For example, if each processing element has direct neighbors in N, S, W, and E directions, then the user of image processing software benefits from template decompositions into invariant templates having supports of the following form:



Researchers have shown that convex, dome-shaped morphological templates decompose into von Neumann templates [11]. A host of researchers has investigated the problem of finding morphological template decomposition into 3×3 templates.

Morphological Template Decomposition Problem. Let $\mathrm{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ be an invariant template, $(k_1, k_2, \ldots, k_n)$ an strictly increasing sequence of natural numbers and $\left(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^{k_n}\right)$ a sequence of subsets of $\mathbf{X}$. Is there a sequence of invariant templates $\mathrm{t}^1, \mathrm{t}^2, \ldots, \mathrm{t}^{k_n} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ such that the following conditions are satisfied?

1. For all $i = 1, \ldots, k_n$ the set $\mathbf{X}^i$ contains the support of $\mathrm{t}_{\mathrm{z}}^i$.
2. The template t factorizes as follows into a product of the templates $\mathrm{t}^1, \mathrm{t}^2, \ldots, \mathrm{t}^{k_n}$:

$$\mathrm{t} = \left(\mathrm{t}^1 \boxtimes \ldots \boxtimes \mathrm{t}^{k_1}\right) \vee \ldots \vee \left(\mathrm{t}^{k_{n-1}+1} \boxtimes \ldots \boxtimes \mathrm{t}^{k_n}\right) .$$

Determine the templates $\mathrm{t}^1, \mathrm{t}^2, \ldots, \mathrm{t}^{k_n}$ in this case.

In this paper, we are particularly interested in the following subproblem, which we shall call the *separable decomposition problem* for rectangular morphological templates. Let a rectangular template $\mathrm{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and an integer k be given. Can the template r be decomposed into column templates $\mathrm{s}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and row templates $\mathrm{t}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ such that

$$\mathrm{r} = \bigvee_{i=1}^{k} \mathrm{s}^i \boxtimes \mathrm{t}^i .$$

Determine the templates s^l and t^l in this case for all $l = 1, \ldots, k$.

A rectangular template $\mathrm{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ which can be expressed in the form $\mathrm{r} = \bigvee_{i=1}^{k} \mathrm{s}^i \boxtimes \mathrm{t}^i$ is said to have *rank* $\leq k$. The rank of the template $\mathrm{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ is exactly k, if k is the smallest integer such that $\mathrm{r} = \bigvee_{i=1}^{k} \mathrm{s}^i \boxtimes \mathrm{t}^i$.

Integer Programming Approaches to Morphological Template Decomposition Problems

The decomposition of morphological templates is an ongoing research topic, which many researchers investigate. Most researchers have concentrated on decomposing particular types of binary or gray-scale morphological templates. Recently, researchers have provided integer programming solutions to the morphological template decomposition problem; some are focusing on decomposing morphological templates into 3×3 templates [7]; others demonstrate how to solve decomposition problems of arbitrary invariant morphological templates [20]. In this section, we review the latter integer programming approach. In this setting, we consider the problem of morphological template decomposition as a subproblem of morphological template approximation. The definition of morphological template approximation requires a notion of distance between two templates.

Metrics on Invariant Templates with Finite Support. If $\mathrm{s} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and $\mathrm{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ are invariant templates with the same finite support S at some target point $\mathrm{y} \in \mathbf{X}$, then the following equations provide metrics on the set of all invariant templates with that particular finite support:

$$d_1(\mathrm{s}, \mathrm{t}) = \sum_{\mathrm{x} \in \mathbf{S}} |\mathrm{s}_{\mathrm{y}}(\mathrm{x}) - \mathrm{t}_{\mathrm{y}}(\mathrm{x})|$$
$$d_2(\mathrm{s}, \mathrm{t}) = \bigvee_{\mathrm{x} \in \mathbf{S}} |\mathrm{s}_{\mathrm{y}}(\mathrm{x}) - \mathrm{t}_{\mathrm{y}}(\mathrm{x})| \; .$$

In particular, the following equivalences hold:

$$d_i(\mathrm{s}, \mathrm{t}) = 0 \Leftrightarrow \mathrm{s} = \mathrm{t} \; .$$

Since the templates s and t are invariant, these definitions do not depend on the choice of the target point $\mathrm{y} \in \mathbf{X}$. For the rest of this section, we use the symbol d in order to denote either one of the template metrics d_1 or d_2.

Morphological Template Approximation Problem. Suppose $\mathrm{t} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ is an invariant template, $(k_1, k_2, \ldots, k_n)$ an strictly increasing sequence of natural numbers and $\left(\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^{k_n}\right)$ a sequence of subsets of $\mathbf{X}$. Define $\mathbf{T}$ to be the

set of all sequences $\left(\mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^{k_n}\right)$ which have the following properties:

$$\mathbf{s}^i \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$$
$$S\left(\mathbf{s}^i_{\mathbf{z}}\right) \subseteq \mathbf{X}^i$$
$$\forall\, i = 1, \ldots, k_n \ .$$

Find a sequence of templates $\left(\mathbf{t}^1, \mathbf{t}^2, \ldots, \mathbf{t}^{k_n}\right) \in \mathbf{T}$ which satisfies the inequality

$$d\Big(\left[\mathbf{t}^1 \boxtimes \ldots \boxtimes \mathbf{t}^{k_1}\right] \vee \ldots \vee \left[\mathbf{t}^{k_{n-1}+1} \boxtimes \ \ldots \boxtimes \mathbf{t}^{k_n}\right], \mathbf{t}\Big)$$
$$\leq d\Big(\left[\mathbf{s}^1 \boxtimes \ldots \boxtimes \mathbf{s}^{k_1}\right] \vee \ldots \vee \left[\mathbf{s}^{k_{n-1}+1} \boxtimes \ \ldots \boxtimes \mathbf{s}^{k_n}\right], \mathbf{t}\Big)$$
$$\forall \left(\mathbf{s}^1, \ldots, \mathbf{s}^{k_n}\right) \in \mathbf{T} \ .$$

Note that a solution to the morphological template approximation problem for a fixed, but arbitrary instance yields a solution to the morphological template decomposition problem for the same instance.

Integer Programming Formulations. In a previous paper, we indicated how to solve arbitrary instances of the morphological template approximation problem by using an integer programming approach [20]. In particular, we presented an integer programming solution to the following approximation problem:

Given a $m \times n$ rectangular template $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ and an arbitrary integer k, minimize

$$d_1\left(\bigvee_{i=1}^{k} \mathbf{s}^i \boxtimes \mathbf{t}^i,\, \mathbf{r}\right) \ ,$$

where the templates $\mathbf{s}^i \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ are $m \times 1$ column templates and the templates $\mathbf{t}^i \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ are $1 \times n$ row templates for all $i = 1, \ldots, k$.

$$Min \ \sum_{i=1}^{n} \sum_{j=1}^{m} \left(q_{ij} + \sum_{l=1}^{k} p_{ij}^{l}\right)$$

$$
\begin{gathered}
subject\ to \\
p_{ij}^l \geq r_{ij} - s_i^l - t_j^l - M * f_{ij}^l \\
q_{ij} \geq s_i^l + t_j^l - r_{ij} \\
p_{ij}^l \geq 0 \\
q_{ij} \geq 0 \\
f_{ij}^l = 0 \ \ or\ 1 \\
\sum_{l=1}^{k} f_{ij}^l = k - 1 \\
\forall\ i = 1, \ldots, n,\ j = 1, \ldots, m,\ l = 1, \ldots, k\ .
\end{gathered}
$$

The constant M should be at least as large as $\bigvee\limits_{i,\,j,\,l} \left| a_{ij} - \left(s_i^l + t_j^l \right) \right|$. Note that the above integer programming formulation generalizes the problem of finding a separable decomposition of $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}} \right)^{\mathbf{X}}$ of the form $\mathbf{r} = \bigvee\limits_{i=1}^{k} \mathbf{s}^i \boxtimes \mathbf{t}^i$. The next section will show that this problem is NP-complete.

Complexity Results

In this section, we are going to prove the NP-completeness results which we addressed in the previous sections. In particular, we show that the separable decomposition problem for a rectangular morphological template is NP-complete. However, an instance of this problem is solvable in polynomial time if the parameter $k \leq 2$. In fact, a decomposition of the form $\mathbf{r} = \mathbf{s} \boxtimes \mathbf{t}$, where $\mathbf{s}$ is a column template and $\mathbf{t}$ is a row template, can easily be found for any rectangular template $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}} \right)^{\mathbf{X}}$ [11]. In an upcoming paper, we will construct a separable decomposition of the form $\mathbf{r} = \bigvee\limits_{i=1}^{k} \mathbf{s}^i \boxtimes \mathbf{t}^i$ for any rectangular morphological template of rank ≤ 2.

Theorem. The following problem Π is NP–complete:

INSTANCE: Let $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}} \right)^{\mathbf{X}}$ be a rectangular $m \times n$ template and k be a positive integer.

QUESTION: Is there a collection of column templates $\mathbf{s}^l \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ of length m and row templates $\mathbf{t}^l \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ of length n, $l = 1, \ldots, k$, such that

$$\mathbf{r} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtimes \mathbf{t}^l\right) \; ?$$

Proof.

(1) $\Pi \in$ NP:

For any guess of templates $\tilde{\mathbf{s}^l} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ and $\tilde{\mathbf{t}^l} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$, the checking stage can be performed in polynomial time:

1. Forming $\bigvee_{l=1}^{k} \left(\tilde{\mathbf{s}^l} \boxtimes \tilde{\mathbf{t}^l}\right)$ costs $k(mn)+(k-1)(mn) = (2k-1)mn$ operations.
2. Another mn operations are needed for comparing the result with $\mathbf{r}$.

(2) Choice of an NP–Complete Problem Π':

INSTANCE: Collection $\mathcal{C}$ of nonempty subsets of a finite set A and a positive integer $k \leq |\mathcal{C}|$.

QUESTION: Is there a collection $\mathcal{B}$ of subsets of A with $|\mathcal{B}| = k$ such that , for each $C \in \mathcal{C}$, there is a subcollection of $\mathcal{B}$ whose union is exactly C.

The following argumentation shows that Π' is indeed NP–complete:
Since Π' represents a subproblem of SET BASIS, which is known to be NP-complete, it is contained in NP. The problem of SET BASIS only differs from Π' by allowing the empty set as an element of $\mathcal{C}$ [8]. The problem of SET BASIS can be partitioned into the problem Π' and a similar problem Π''. Instances of Π'' consist of those collections $\mathcal{C}$ of subsets of A which include $\varnothing$, and a positive integer $k \leq |\mathcal{C}|$. The problems Π' and Π'' are polynomially equivalent. Thus, if either one would be contained in P, the problem of SETBASIS would belong to P.

(3) Construction of a Transformation f from Π' to Π:

Let an instance $(\mathcal{C}, k)$ of Π' be given, where $\mathcal{C}$ is a collection of subsets of a finite set A and k is a positive integer with $k \leq |\mathcal{C}|$. The cardinality of $\mathcal{C}$ is bounded from above by $2^{|A|}$, the cardinality of the power set of A. Hence, $\mathcal{C}$ as well as A are finite sets and they are of the form $\mathcal{C} = \{C_1, \ldots, C_m\}$ and $A = \{a_1, \ldots, a_n\}$ for some $m, n \in \mathbb{N}$.

We now construct an instance $(\mathbf{t}, k)$ of Π which is satisfiable if and only if the instance $(\mathcal{C}, k)$ of $\Pi^{'}$ is satisfiable. The parameter k remains unchanged and

we define an $m \times n$ template $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ by setting

$$r_{ij} = \begin{cases} 1 & if\ a_j \in C_i \\ 0 & else\ . \end{cases}$$

"$\Leftarrow$": Let $(\mathcal{C}, k)$ be satisfiable, where C is a collection of subsets of $A = \{a_1, \ldots, a_n\}$ and k is a positive integer with $k \leq |\mathcal{C}|$. Hence, there exists a collection $\mathcal{B} = \{B_1, \ldots, B_k\}$ of subsets of $A = \{a_1, \ldots, a_n\}$ such that, for each $C_i \in \mathcal{C}$, where $i = 1, \ldots, m$, there is a subcollection $\mathcal{B}_i \subseteq \mathcal{B}$ whose union equals C_i.

We are now going to show that the instance $(\mathbf{t}, k)$ of Π is satisfiable by constructing $m \times 1$ templates $\mathbf{s}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and $1 \times n$ templates $\mathbf{t}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ for all $l = 1, \ldots, k$, such that $\mathbf{r} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)$.

We define the row templates $\mathbf{t}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ for all $l = 1, \ldots, k$ as follows:

$$t_j^l = \begin{cases} 1 & if\ a_j \in B_l \\ 0 & else \end{cases} \quad \forall\, j = 1, \ldots, n\ .$$

Introducing the notation $L_i = \{l \in \{1, \ldots, k\} : B_l \in \mathcal{B}_i\}$ for all $i = 1, \ldots, m$, we define the column row templates $\mathbf{s}^l \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ for all $l = 1, \ldots, k$ by setting

$$s_i^l = \begin{cases} 0 & if\ l \in L_i \\ -1 & else \end{cases} \quad \forall\, i = 1, \ldots, m\ .$$

Note that the invariant templates $\mathbf{r} \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ and $\bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right) \in \left(\mathbb{R}_{-\infty}^{\mathbf{X}}\right)^{\mathbf{X}}$ have the same support at each $\mathbf{y} \in \mathbf{X}$. In order to show that $\mathbf{r} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)$, we first prove that

$$\{(i,j) \in M \times N\ :\ r_{ij} = 1\} = \left\{(i,j) \in M \times N\ :\ \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij} = 1\right\}\ ,$$

where M denotes from now on the set $\{1, \ldots, m\}$ and N denotes from now on the set $\{1, \ldots, n\}$. Furthermore, we set $K = \{1, \ldots, k\}$.

The above equation follows from the sequence of equivalences below, where $i \in M$ and $j \in N$ are arbitrary indices.

$$\begin{aligned}
&\bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij} = 1 \\
&\Leftrightarrow \bigvee_{l=1}^{k} \left(s_i^l + t_j^l\right) = 1 \\
&\Leftrightarrow \exists\, l \in K \;:\; t_j^l = 1 \;\; and \;\; s_i^l = 0 \\
&\Leftrightarrow \exists\, l \in L_i \;:\; j \in B_l \\
&\Leftrightarrow j \in B_l \; for \; some \; l \in L_i \\
&\Leftrightarrow j \in C_i \\
&\Leftrightarrow r_{ij} = 1 \; .
\end{aligned}$$

We now conclude the proof of the desired equality $\mathbf{r} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)$ by showing that the invariant template $\bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)$ — like the invariant template $\mathbf{r}$ — only adopts the values 0 and 1 at each location $\mathbf{y} \in \mathbf{X}$. In other words, it suffices to show that $\bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij} \in \{0, 1\}$ for all $i \in M$ and for all $j \in N$. Recall that the sets C_i are not empty for all $i \in M$.

$$\begin{aligned}
&\forall\, i \in M \;\; \exists\, j_i \in N \;:\; 1 = r_{ij_i} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij_i} = 1 \\
&\Rightarrow \forall\, i \in M \;\; \exists\, j_i \in N,\; l_i \in K \;:\; s_i^{l_i} + t_{j_i}^{l_i} = 1 \\
&\Rightarrow \forall\, i \in M \;\; \exists\, j_i \in N,\; l_i \in K \;:\; s_i^{l_i} = 0 \\
&\Rightarrow \forall\, i \in M \;\; \exists \;\; l_i \in K \;:\; s_i^{l_i} + t_j^{l_i} \geq 0 \quad \forall\, j \in N \\
&\Rightarrow \forall\, i \in M \;:\; \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij} \geq 0 \quad \forall\, j \in N \\
&\Rightarrow \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtriangledown \mathbf{t}^l\right)_{ij} \geq 0 \quad \forall\, i \in M,\; \forall\, j \in N \; .
\end{aligned}$$

"⇒": Let $(\mathbf{r}, k)$ be a satisfiable instance of Π, which means that there exist $m \times 1$ templates $\mathbf{s}^l \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$ and $\mathbf{t}^l \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$, $l = 1, \ldots, k$, such that $\mathbf{r} = \bigvee_{l=1}^{k} \left(\mathbf{s}^l \boxtimes \mathbf{t}^l\right)$. We need to show that the corresponding instance instance $(\mathcal{C}, k)$ of Π' is satisfiable. We will construct a collection $\mathcal{B}$ of k subsets of A such that, for each $C_i \in \mathcal{C}$, $i = 1, \ldots, m$, there is a subcollection $\mathcal{B}_i \subseteq \mathcal{B}$ whose union is exactly C_i.

For all $l \in K$, we define B_l as the set of all $a_{j'} \in A$ such that $t^l_{j'}$ is maximal among all t^l_j, where $j, j' \in N$, and we denote the collection of these sets by $\mathcal{B}$, i.e.

$$B_l = \left\{ a_{j'} \in A \,:\, j' \in N \; and \; t^l_{j'} \geq t^l_j \;\; \forall\, j \in N \right\} \quad \forall\, l \in K$$
$$\mathcal{B} = \{B_1, \ldots, B_k\} \,.$$

If $L_i^{'} \subseteq K$, $i = 1, \ldots, m$, denotes the set of all $l \in K$, such that $s^l_i + t^l_j = 1$ for some $j \in N$, then we define

$$\mathcal{B}_i = \left\{ B_l \,:\, l \in L_i^{'} \right\} .$$

By the definition of $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$, each set C_i equals $\{j \in N \,:\, r_{ij} = 1\}$. Therefore, we obtain the following sequence of equations for all $i = 1, \ldots, m$:

$$\begin{aligned}
C_i &= \left\{ j \in N \,:\, \bigvee_{l=1}^{k} \left(s^l_i + t^l_j \right) = 1 \right\} \\
&= \left\{ j \in N \,:\, s^l_i + t^l_j = 1 \;\; for\; some \;\; l \in K \right\} \\
&= \left\{ j \in N \,:\, s^l_i + t^l_j = 1 \;\; for\; some \;\; l \in L_i \right\} \\
&= \left\{ a_j \in B_l \,:\, s^l_i + t^l_j = 1 \;\; for\; some \;\; l \in L_i \right\} \\
&= \{ a_j \in B_l \;\; for\; some \;\; l \in L_i \} \\
&= \bigcup_{l \in L_i} B_l \\
&= \bigcup \mathcal{B}_i \,.
\end{aligned}$$

(4) Proof that f is a Polynomial Transformation:

By the construction in (3), f maps an instance $(\mathcal{C}, k)$ of Π', where $C = \{C_1, \ldots, C_m\}$ is a collection of m subsets of $A = \{a_1, \ldots, a_n\}$, to an instance $(\mathbf{r}, k)$ of Π. Since the parameter k does not change, the only operations involved in the computation of f are the ones arising from the computation of $\mathbf{r} \in \left(\mathbb{R}^{\mathbf{X}}_{-\infty}\right)^{\mathbf{X}}$.

For each weight r_{ij} of $\mathbf{r}$, where $i \in M$ and $j \in N$, the index j has to be tested if it belongs to the subset C_i of $A = \{a_1, \ldots, a_n\}$. This test involves less than n operations. Thus, the computation of f can be performed in less than mn^2 operations which implies that f is a polynomial transformation.

Remark. It seems reasonable to believe, that the NP-completeness of the following problem can be proven in a similar fashion: does an arbitrary invariant morphological template strongly decompose into a sequence of 3×3 templates?

Finally, we provide a few easily derivable, but important corollaries to the above theorem.

Corollary 1. The problem of finding the rank of a rectangular morphological template is NP-complete.

Corollary 2. The separable decomposition problem for rectangular morphological templates is NP-complete.

Corollary 3. The general morphological template decomposition problem is NP-complete.

Concluding Remarks. This paper attempts to provide a solid theoretical background for the complexity of all morphological template decomposition problems. We have shown that a particular subproblem Π of the general morphological template decomposition problem is NP-complete. Hence, global optimization approaches to solve the morphological template decomposition problem have to employ integer programming techniques. Attempts to exploit special structures of certain morphological template decomposition problems will continue to play a predominant role in future efforts to efficiently solve morphological template decomposition problems.

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