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A general framework for fuzzy morphological associative memories

Marcos Eduardo Valle, Peter Sussner*

Department of Applied Mathematics, State University of Campinas, Campinas, SP 13083 859, Brazil

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Abstract

Fuzzy associative memories (FAMs) can be used as a powerful tool for implementing fuzzy rule-based systems. The insight that FAMs are closely related to mathematical morphology (MM) has recently led to the development of new fuzzy morphological associative memories (FMAMs), in particular implicative fuzzy associative memories (IFAMs). As the name FMAM indicates, these models belong to the class of fuzzy morphological neural networks (FMNNs). Thus, each node of an FMAM performs an elementary operation of fuzzy MM. Clarifying several misconceptions about FMAMs that have recently appeared in the literature, we provide a general framework for FMAMs within the class of FMNN. We show that many well-known FAM models fit within this framework and can therefore be classified as FMAMs. Moreover, we employ certain concepts of duality that are defined in the general theory of MM in order to derive a large class of strategies for learning and recall in FMAMs.

Keywords: Fuzzy inference systems; Fuzzy associative memories; Fuzzy mathematical morphology; Fuzzy morphological associative memories; Fuzzy learning by adjunction

1. Introduction

Mathematical morphology (MM) has found broad application in image and signal processing [11,43]. Although MM was initially developed for binary image processing [26,39] and later extended to gray-scale image processing [39,40,45], MM can be conducted very generally in the *complete lattice* setting [16,38].

One of the most important results of the theory of MM states that every mapping from one complete lattice into another can be expressed as a composition of certain *elementary operations* in terms of supremum and infimum operations [2]. Specifically, these elementary operations consist of erosion, dilation, anti-erosion, and anti-dilation. In the last decade, a host of researchers has used elementary operations of MM as aggregation functions of neurons for a new class of artificial neural networks that have become known as *morphological neural networks* [35–37,53,33,48,1]. At the same time, *fuzzy mathematical morphology* (FMM) emerged as another approach for extending binary MM to gray-scale MM [41,5,9,27,10,51]. More information on applications of FMM and other fuzzy techniques in image processing can be found in edited volumes by Nachtegael, Kerre, et al. [18,29,28].

* Corresponding author. Tel.: +55 19 3521 5959.
E-mail addresses: mevalle@ime.unicamp.br (M.E. Valle), sussner@ime.unicamp.br (P. Sussner)
URLs: http://www.ime.unicamp.br/∼mevalle (M.E. Valle), http://www.ime.unicamp.br/∼sussner (P. Sussner).

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FMM is based on the observation that the fuzzy interval [0, 1] and the fuzzy hyperbox $[0, 1]^n$ constitute examples of complete lattices. Therefore, we introduce *fuzzy morphological neural networks* (FMNNs) as the type of artificial neural networks that compute *fuzzy* erosions, dilations, anti-erosions, or anti-dilations from $[0, 1]^n$ to [0, 1] at every node.

The theory and applications of morphological neural networks, especially morphological associative memories (AMs), have experienced a steady and consistent growth in the last few years [19,12,33,46–48,34,13,1]. A FMNN that serves as an AM is called *fuzzy morphological associative memory* (FMAM) [50]. Clearly, FMAMs belong to the class of *fuzzy associative memories* (FAMs) which have been studied extensively since the introduction of Kosko's FAM models [22,23].

Although the lattice operations of maximum and minimum are routinely used in several FAM models [8,22,24], the relationship between FAMs and MM has not been explored thus far. In this paper, we show that FMAMs include many well known models of FAM such as the FAMs of Kosko, Junbo et al., Liu, and Bělohlávek [22–24,17,3]. Furthermore, this paper is the first to provide a general theoretical framework for FMAMs. Specifically, we derive a large sub-class of FMAMs whose neurons perform dilations of the form max-C, i.e., a maximum of fuzzy conjunctions. Then, we develop a learning strategy for general max-C FMAMs based on the concept of adjunction that—besides negation—represents the main concept of duality in MM. Other learning strategies arise by taking the adjoint model or the negation of a given max-C FMAM.

The paper is organized as follows. First, we review some basic concepts of MM, fuzzy set theory, and FMM. Then, we present some general aspects of the FMAM subclass of FMNNs including the types of neurons used in FMAMs and relationships of duality between FMAM models that are based on the concepts of adjunction and negation. In Section 4, we employ these duality relationships in order to derive a novel class of recording schemes for FMAMs. We finish the paper with some concluding remarks and Appendix A that contains the proofs of the theorems and lemmas.

2. Mathematical background

2.1. Complete lattice framework of MM

MM is a theory that is concerned with the processing and analysis of objects using operators and functions based on topological and geometrical concepts [16,43]. During the last few decades, it has acquired a special status within the field of image processing, pattern recognition, and computer vision. Applications of MM include image segmentation and reconstruction [20], feature detection [42], and signal decomposition [7].

The mathematical foundations of MM can be found in lattice theory which is concerned with algebraic structures that arise by imposing some type of ordering on a set [4,16,38]. A partially ordered set X is called a *lattice* if and only if every finite, non-empty subset of X has an infimum and a supremum in X. The infimum of $Y \subseteq X$ is denoted by the symbol $\bigwedge Y$. Alternatively, we write $\bigwedge_{j \in J} y_j$ instead of $\bigwedge Y$ if $Y = \{y_j : j \in J\}$ for some index set J. Similar notations are used to denote the supremum of Y. We speak of a *complete lattice* X if every (finite or infinite) subset has an infimum and a supremum in X. For instance, the interval [0, 1] represents a complete lattice. Moreover, the set of functions from a set U to [0, 1], denoted by $[0, 1]^U$, inherits the complete lattice structure of [0, 1] in terms of the following partial order. For every $\mathbf{x}, \mathbf{y} \in [0, 1]^U$, we have

$$\mathbf{x} \leqslant \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x}(u) \leqslant \mathbf{y}(u) \quad \forall u \in U. \tag{1}$$

From now on, we denote complete lattices by the symbols $\mathbb{L}$ and $\mathbb{M}$.

The elementary operators of MM are erosion, dilation, anti-erosion, and anti-dilation [2,16,40]. These four operators are defined as follows.

An *erosion* is a mapping ε from a complete lattice $\mathbb{L}$ to a complete lattice $\mathbb{M}$ that commutes with the infimum operation. In other words, the operator ε represents an erosion if and only if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\varepsilon\left(\bigwedge Y\right) = \bigwedge_{y \in Y} \varepsilon(y). \tag{2}$$

Similarly, an operator $\delta : \mathbb{L} \to \mathbb{M}$ that commutes with the supremum operation is called a *dilation*. In other words, the operator δ represents a dilation if and only if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\delta\left(\bigvee Y\right) = \bigvee_{y \in Y} \delta(y). \tag{3}$$

An operator $\bar{\varepsilon} : \mathbb{L} \to \mathbb{M}$ is called an *anti-erosion* if and only if the first equality in Eq. (4) holds for every $Y \subseteq \mathbb{L}$ and an operator $\bar{\delta} : \mathbb{L} \to \mathbb{M}$ is called an *anti-dilation* if and only if the second equality in Eq. (4) holds for every subset $Y \subseteq \mathbb{L}$.

$$\bar{\varepsilon}\left(\bigwedge Y\right) = \bigvee_{y \in Y} \bar{\varepsilon}(y) \quad \text{and} \quad \bar{\delta}\left(\bigvee Y\right) = \bigwedge_{y \in Y} \bar{\delta}(y). \tag{4}$$

The operators of erosion, dilation, anti-erosion, and anti-dilation represent the backbone of MM since every mapping Ψ between complete lattices $\mathbb{L}$ and $\mathbb{M}$ can be expressed in terms of supremums and infimums of these four operators [2]. More precisely, every mapping $\Psi : \mathbb{L} \to \mathbb{M}$ can be represented as a supremum of infimums of erosions and anti-dilations. Alternatively, $\Psi : \mathbb{L} \to \mathbb{M}$ can be represented as an infimum of supremums of dilations and anti-erosions.

Two important notions of duality permeate MM: *adjunction* and *negation*. A *negation* on a complete lattice $\mathbb{L}$ is an involutive bijection $v_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$ which reverses the partial ordering [16]. For example, the operator $N_S(x) = 1 - x$ represents a negation on the interval [0, 1].

Suppose that $\mathbb{L}$ and $\mathbb{M}$ are complete lattices equipped with negations $v_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$ and $v_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$, respectively. The following lemma reveals that anti-erosions and anti-dilations can be readily constructed from $v_{\mathbb{L}}$, $v_{\mathbb{M}}$, erosions, and dilations. Since every complete lattice considered in this paper is endowed with a negation, we are mainly concerned with erosions and dilations.

Lemma 1. *Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices with negations $v_{\mathbb{L}}$ and $v_{\mathbb{M}}$, respectively. An operator $\bar{\varepsilon} : \mathbb{L} \to \mathbb{M}$ represents an anti-erosion if and only if $v_{\mathbb{M}} \circ \bar{\varepsilon}$ is an erosion and $\bar{\varepsilon} \circ v_{\mathbb{L}}$ is a dilation. Similarly, an operator $\bar{\delta} : \mathbb{L} \to \mathbb{M}$ represents an anti-dilation if and only if $v_{\mathbb{M}} \circ \bar{\delta}$ is a dilation and $\bar{\delta} \circ v_{\mathbb{L}}$ is an erosion.*

The operators of erosion and dilation can be linked by means of the concept of negation which is defined as follows. Let Ψ be an operator mapping a complete lattice $\mathbb{L}$ into a complete lattice $\mathbb{M}$ and let $v_{\mathbb{L}}$ and $v_{\mathbb{M}}$ be negations on $\mathbb{L}$ and $\mathbb{M}$, respectively. The operator Ψ^v given by

$$\Psi^v(x) = v_{\mathbb{M}}\left(\Psi\left(v_{\mathbb{L}}(x)\right)\right) \quad \forall x \in \mathbb{L} \tag{5}$$

is called the *negation* of Ψ (with respect to $v_{\mathbb{L}}$ and $v_{\mathbb{M}}$).

In view of this definition, the following statement [16] arises as an immediate consequence of Lemma 1.

Corollary 2. *The negation of an erosion is a dilation, and vice versa.*

Many researchers—including Bloch and Maître [5], Sinha and Dougherty [41], as well as Nachtegael and Kerre [27]—consider negation to be the most important notion of duality in MM. Other researchers such as Deng and Heijmans [10], Ronse [38], and Maragos [25] advocate the duality relationship of *adjunction*, which is closely related to the concepts of Galois connection [2] and the residuum of an operator [6,16].

Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices. Consider two arbitrary operators $\delta : \mathbb{L} \to \mathbb{M}$ and $\varepsilon : \mathbb{M} \to \mathbb{L}$. We say that the pair (ε, δ) is an *adjunction* from $\mathbb{L}$ to $\mathbb{M}$ or that ε and δ are *adjoint* if and only if we have

$$\delta(x) \leqslant y \;\Leftrightarrow\; x \leqslant \varepsilon(y) \quad \forall x \in \mathbb{L},\; y \in \mathbb{M}. \tag{6}$$

Adjunction constitutes a concept of duality due to the following proposition [40,16].

Proposition 3. *Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices. Consider mappings $\delta : \mathbb{L} \to \mathbb{M}$ and $\varepsilon : \mathbb{M} \to \mathbb{L}$.*

(i) *If (ε, δ) is an adjunction then δ is a dilation and ε is an erosion.*

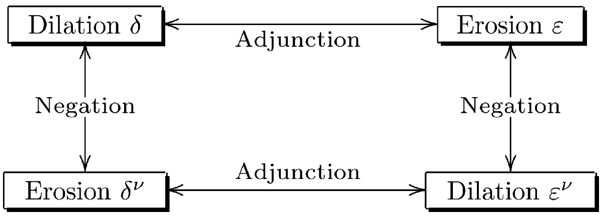

Fig. 1. Scheme to obtain a dilation from an erosion and vice versa.

(ii) *For any dilation δ there is a unique erosion ε such that (ε, δ) is an adjunction. The adjoint erosion is given by*

$$\varepsilon(y) = \bigvee \{x \in \mathbb{L} : \delta(x) \leqslant y\}, \tag{7}$$

for every $y \in \mathbb{M}$.

(iii) *For any erosion ε there is a unique dilation δ such that (ε, δ) is an adjunction. The adjoint dilation is given by*

$$\delta(x) = \bigwedge \{y \in \mathbb{M} : \varepsilon(y) \geqslant x\}, \tag{8}$$

for every $x \in \mathbb{L}$.

The preceding observations clarify that there is a unique erosion that can be associated with a certain dilation, and vice versa, in terms of either negation or adjunction. Furthermore, given an adjunction (ε, δ), the pair $(\delta^\nu, \varepsilon^\nu)$ forms an adjunction [16]. These observations lead to the commutative diagram depicted in Fig. 1. This diagram will be used later to develop new FMAM models.

2.2. *Basic concepts of fuzzy set theory*

In this paper, the class of fuzzy sets in U will be denoted by $\mathcal{F}(U) = [0, 1]^U$. In particular, if $U = \{u_1, \ldots, u_n\}$ is a finite set then $\mathbf{x} \in \mathcal{F}(U)$ will be represented by a column vector $\mathbf{x} = [x_1, \ldots, x_n]^T \in [0, 1]^n$ where $x_j = \mathbf{x}(u_j)$ is the degree of membership of u_j in $\mathbf{x}$, for every $j = 1, \ldots, n$. In the following, we will focus on finite fuzzy sets such as $\mathbf{x} \in [0, 1]^n$ and $\mathbf{y} \in [0, 1]^m$.

2.2.1. *Some basic operations of fuzzy logic*

We define a *fuzzy conjunction* as an increasing mapping $C : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies $C(0, 0) = C(0, 1) = C(1, 0) = 0$ and $C(1, 1) = 1$. Examples of fuzzy conjunction include the following operators:

$$C_M(x, y) = x \wedge y, \tag{9}$$

$$C_P(x, y) = x \cdot y, \tag{10}$$

$$C_L(x, y) = 0 \vee (x + y - 1), \tag{11}$$

$$C_K(x, y) = \begin{cases} 0, & x + y \leqslant 1, \\ x, & x + y > 1. \end{cases} \tag{12}$$

Note that the fuzzy conjunctions C_M, C_P, and C_L are examples of t-norms [21,32]. In contrast, the fuzzy conjunction C_K does not constitute a t-norm since C_K fails to be commutative. Note that C_K slightly differs from the fuzzy conjunction of Kleene and Dienes that was defined by Deng and Heijmans [10].

Lemma 4. *The operators C_M, C_P, C_L, and C_K represent dilations on* $[0, 1]$ *in both arguments*.

A *fuzzy disjunction* is an increasing mapping $D : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies $D(0, 0) = 0$ and $D(0, 1) = D(1, 0) = D(1, 1) = 1$. The following operators are examples of fuzzy disjunctions:

$$D_M(x, y) = x \vee y, \tag{13}$$

$$D_P(x, y) = x + y - x \cdot y, \tag{14}$$

$$D_L(x, y) = 1 \wedge (x + y), \tag{15}$$

$$D_K(x, y) = \begin{cases} 1, & x + y \geqslant 1, \\ x, & x + y < 1. \end{cases} \tag{16}$$

Note that the operators D_M, D_P, and D_L represent t-conorm [21,32]. The following observation is also pertinent.

Lemma 5. *The operators D_M, D_P, D_L, and D_K represent erosions on* [0, 1] *in both arguments.*

An operator $I : [0, 1] \times [0, 1] \to [0, 1]$ that is decreasing in the first argument and that is increasing in the second argument is called a *fuzzy implication* if I extends the usual crisp implication on $\{0, 1\} \times \{0, 1\}$, i.e., $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$. We obtain a *reverse fuzzy implication* $J : [0, 1] \times [0, 1] \to [0, 1]$ by inverting the arguments of a fuzzy implication I, i.e., J is defined as follows for every $x, y \in [0, 1]$.

$$J(x, y) = I(y, x). \tag{17}$$

The values $I(x, y)$ and $J(x, y)$ can be interpreted as the degree of truth of the sentences "x implies y" and "x is implied by y", respectively. Some particular reverse fuzzy implications can be found below:

$$J_M(x, y) = \begin{cases} 1, & y \leqslant x, \\ x, & y > x, \end{cases} \quad \text{(Gödel)}, \tag{18}$$

$$J_P(x, y) = \begin{cases} 1, & y \leqslant x, \\ x/y, & y > x, \end{cases} \quad \text{(Goguen)}, \tag{19}$$

$$J_L(x, y) = 1 \wedge (x - y + 1) \quad \text{(Lukasiewicz)}, \tag{20}$$

$$J_K(x, y) = x \vee (1 - y) \quad \text{(Kleene)}. \tag{21}$$

Finally, a negation on the unit interval [0, 1] is called a *fuzzy negation*. The following unary operators represent examples of fuzzy negations.

$$N_S(x) = 1 - x, \tag{22}$$

$$N_D(x) = \frac{1 - x}{1 + px}, \quad p > -1, \tag{23}$$

$$N_R(x) = \sqrt[p]{1 - x^p}, \quad p \in (0, \infty). \tag{24}$$

Note that a fuzzy negation N on [0, 1] induces a negation $\mathbf{N}$ on $\mathcal{F}(U)$ that is given by applying N pointwise, i.e., $\mathbf{N}(\mathbf{x})(u) = N(\mathbf{x}(u))$. In this paper, we use a bold symbol $\mathbf{N}$ for a negation to indicate that the negation is vector-valued.

2.2.2. Duality relationships between fuzzy operators

We say that a fuzzy conjunction C and a fuzzy disjunction D are *dual operators with respect to a fuzzy negation N* if and only if the following equation holds for every $x, y \in [0, 1]$:

$$C(x, y) = N(D(N(x), N(y))). \tag{25}$$

Note that if C is a dilation in one of the arguments then D is an erosion in the same argument, and vice versa [10].

Lemma 6. *The pairs (C_M, D_M), (C_P, D_P), (C_L, D_L), (C_K, D_K) are dual operators with respect to the standard fuzzy negation $N_S(x) = 1 - x$.*

We also define a duality relationship of negation between a fuzzy disjunction D and a fuzzy implication I. More precisely, we say that the operators I and D are *dual operators with respect to a fuzzy negation N* or that *D is the negation of I* if and only if the following equation holds for every $x, y \in [0, 1]$:

$$I(x, y) = D(N(x), y). \tag{26}$$

Note that Eq. (26) corresponds to the following classical idea of implication: the statement "P implies Q" holds true if and only if the statement "not P or Q" also holds true [30,32].

We say that *a fuzzy conjunction C and a fuzzy implication I form an adjunction* if and only if $C(z,\cdot)$ and $I(z,\cdot)$ form an adjunction for every $z \in [0,1]$ [10]. Equivalently, we have that $C(z,\cdot)$ and $D(N(z),\cdot)$ are adjoint. In this case, the following relation holds for every $x, y, z \in [0,1]$:

$$C(z,x) \leqslant y \quad \Leftrightarrow \quad x \leqslant D(N(z), y). \tag{27}$$

Moreover, by Proposition 3, the operator $C(z,\cdot)$ is a dilation and the operator $D(N(z),\cdot)$ is an erosion.

Similarly, *a fuzzy conjunction C and a reverse fuzzy implication J* are called *adjoint operators* if and only if $C(\cdot,z)$ and $J(\cdot,z)$ form an adjunction for every $z \in [0,1]$. Note that $C(\cdot,z)$ and $J(\cdot,z)$ represent a dilation and an erosion, respectively. Furthermore, C and J satisfy the following relation for every $x, y, z \in [0,1]$:

$$C(x,z) \leqslant y \quad \Leftrightarrow \quad x \leqslant J(y,z). \tag{28}$$

In view of Proposition 3, a reverse fuzzy implication J and a fuzzy conjunction C can be obtained by means of the following equations, respectively:

$$J(x,y) = \bigvee \{z \in [0,1] : C(z,y) \leqslant x\} \quad \text{and} \quad C(x,y) = \bigwedge \{z \in [0,1] : J(z,y) \geqslant x\}. \tag{29}$$

Lemma 7. *The pairs (C_M, J_M), (C_P, J_P), (C_L, J_L), and (C_K, J_K) are examples of adjoint operators.*

2.2.3. Matrix products based on fuzzy logic operations

The fuzzy operations C, D, and J can be combined with the maximum or the minimum operation to yield the following matrix products. We define the max-C *product* of $A \in [0,1]^{m\times k}$ and $B \in [0,1]^{k\times n}$, denoted by $E = A \circ B$, as follows:

$$e_{ij} = \bigvee_{\xi=1}^{k} C(a_{i\xi}, b_{\xi j}) \quad \forall i = 1, \ldots, m \ \ \forall j = 1, \ldots, n. \tag{30}$$

Similarly, the min-D *product* and the min-J *product*, denoted by $G = A \bullet B$ and $H = A \circledast B$, respectively, are given by the following equations:

$$g_{ij} = \bigwedge_{\xi=1}^{k} D(a_{i\xi}, b_{\xi j}) \quad \text{and} \quad h_{ij} = \bigwedge_{\xi=1}^{k} J(a_{i\xi}, b_{\xi j}) \ \ \forall i = 1, \ldots, m \ \ \forall j = 1, \ldots, n. \tag{31}$$

Note that if the operators C and D of a max-C and a min-D products are dual with respect to a fuzzy negation N then Eq. (32) holds for appropriately sized matrices A and B where $N(E)$ denotes the entry-wise negation of a fuzzy matrix E.

$$N(A \circ B) = N(A) \bullet N(B). \tag{32}$$

Subscripts of the product symbols $\circ$, $\bullet$, or $\circledast$ indicate the type of fuzzy operators used in Eqs. (30) and (31). For example, the matrix $E = A \circ_M B$ is given by $e_{ij} = \bigvee_{\xi=1}^{k} C_M(a_{i\xi}, b_{\xi j}) = \bigvee_{\xi=1}^{k} (a_{i\xi} \wedge b_{\xi j})$.

2.3. Some basic concepts of FMM

In the 1960s, MM was introduced by Matheron and Serra for the analysis of binary images [26,39]. In the 1980s, Serra and Sternberg developed successful approaches to extend binary to gray-scale MM [39,45]. The classical and most widely known method for the generalization of binary MM to gray-scale image processing employs the notion of umbra and is due to Sternberg [44,45]. FMM is an extension of binary MM that is based on techniques of fuzzy set theory [27,10,25,28,51].

Various researchers have set out to define approaches towards FMM [27]. Among these definitions are the approaches of De Baets [9], Sinha and Dougherty [41], Bloch and Maître [5], Deng and Heijmans [10], and Maragos [25]. We refer the reader to [51] for an overview as well as a classification scheme of the most important approaches to FMM.

A certain approach to FMM is determined by certain definitions of fuzzy erosion and fuzzy dilation since anti-dilations and anti-erosions can be obtained by means of Lemma 1. We say that a function $\varepsilon_{\mathscr{F}} : \mathcal{F}(U) \to \mathcal{F}(V)$ is a *fuzzy erosion* if and only if $\varepsilon_{\mathscr{F}}$ is an erosion in the sense of Eq. (2). Similarly, an operator $\delta_{\mathscr{F}} : \mathcal{F}(U) \to \mathcal{F}(V)$ is a *fuzzy dilation* if and only if it satisfies Eq. (3).

An erosion, a dilation respectively, is usually associated with a *structuring element* which is used to extract some relevant information on the shape and form of objects. A fuzzy erosion $\varepsilon_{\mathscr{F}} : \mathcal{F}(U) \to \mathcal{F}(V)$, a fuzzy dilation $\delta_{\mathscr{F}} : \mathcal{F}(U) \to \mathcal{F}(V)$, respectively, is generally given by a rule that combines an input fuzzy set $\mathbf{x} \in \mathcal{F}(U)$ with a fuzzy structuring element $\mathbf{w} \in \mathcal{F}(U)$ and generates an output fuzzy set $\mathbf{y} \in \mathcal{F}(V)$. Specific choices of fuzzy dilations can be defined in terms of supremums of fuzzy conjunctions and specific choices of fuzzy erosions can be defined in terms of infimums of fuzzy implications or fuzzy disjunctions. For example, if $U = \{u_1, \ldots, u_n\}$ and if $C(w, \cdot)$ is a dilation for every $w \in [0, 1]$ then we obtain a fuzzy dilation $\mathcal{D}(\cdot, \mathbf{w}) : \mathcal{F}(U) \to [0, 1]$ by defining

$$\mathcal{D}(\mathbf{x}, \mathbf{w}) = \bigvee_{i=1}^{n} C(w_i, x_i). \tag{33}$$

Thus, examples of fuzzy dilations include the operators given by Eq. (33) for the fuzzy conjunctions C_M, C_P, C_L, and C_K. This claim follows from Proposition 8 and Lemma 1.

Similarly, if $I(w, \cdot)$ is an erosion for every $w \in [0, 1]$ and if D is the fuzzy disjunction that is the dual operator of I with respect to a fuzzy negation N then we obtain a fuzzy erosion $\mathcal{E}(\cdot, \mathbf{w}) : \mathcal{F}(U) \to [0, 1]$ by defining

$$\mathcal{E}(\mathbf{x}, \mathbf{w}) = \bigwedge_{i=1}^{n} I(w_i, x_i) = \bigwedge_{i=1}^{n} D(m_i, x_i), \tag{34}$$

where $\mathbf{m} = \mathbf{N}(\mathbf{w})$ and D is the negation of I (cf. Eq. (26)). Thus, $D(m, \cdot)$ is an erosion for every $m \in [0, 1]$. In view of Eqs. (30) and (31), these operators have the following representations:

$$\mathcal{D}(\mathbf{x}, \mathbf{w}) = \mathbf{w}^{\mathrm{T}} \circ \mathbf{x} \quad \text{and} \quad \mathcal{E}(\mathbf{x}, \mathbf{w}) = \mathcal{E}(\mathbf{x}, \mathbf{N}(\mathbf{m})) = \mathbf{m}^{\mathrm{T}} \bullet \mathbf{x}. \tag{35}$$

These two equations constitute the basis of the fuzzy morphological neurons that will be defined in the next section.

Note that if C and D represent a dilation and an erosion in the second argument, then the operators $\mathcal{D}$ and $\mathcal{E}$ given by Eqs. (33) and (34) represent a fuzzy dilation and a fuzzy erosion, respectively. The following proposition, that corresponds to a slight adaptation of Proposition 5.2 of [10], reveals that the converse also holds true.

Proposition 8. *An operator $\mathcal{D}(\cdot, \mathbf{w})$ given by Eq.* (35) *represents a fuzzy dilation for every $\mathbf{w} \in \mathcal{F}(U)$ if and only if $C(w, \cdot)$ is a dilation for every $w \in [0, 1]$.*

Similarly, an operator $\mathcal{E}(\cdot, \mathbf{w}) = \mathcal{E}(\cdot, \mathbf{N}(\mathbf{m}))$ given by Eq. (35) represents a fuzzy erosion for every $\mathbf{m} \in \mathcal{F}(U)$ if and only if $D(m, \cdot)$ is an erosion for every $m \in [0, 1]$.

3. General aspects of FMAM

AMs allow for the storage of pattern associations and the retrieval of the desired output pattern upon presentation of a possibly noisy or incomplete version of an input pattern. Mathematically speaking, the AM design problem can be stated as follows: Given a finite set of desired associations $\{(\mathbf{x}^{\xi}, \mathbf{y}^{\xi}) : \xi = 1, \ldots, k\}$, determine a mapping G such that $G(\mathbf{x}^{\xi}) = \mathbf{y}^{\xi}$ for all $\xi = 1, \ldots, k$. Furthermore, the mapping G should be endowed with a certain tolerance with respect to noise, i.e., $G(\tilde{\mathbf{x}}^{\xi})$ should equal $\mathbf{y}^{\xi}$ for noisy or incomplete versions $\tilde{\mathbf{x}}^{\xi}$ of $\mathbf{x}^{\xi}$.

The set of associations $\{(\mathbf{x}^{\xi}, \mathbf{y}^{\xi}) : \xi = 1, \ldots, k\}$ is called *fundamental memory set* and each association $(\mathbf{x}^{\xi}, \mathbf{y}^{\xi})$ in this set is called a *fundamental memory* [14,15]. In this paper, we often consider the matrix $X = [\mathbf{x}^1, \ldots \mathbf{x}^k]$ whose columns consist of the vectors $\mathbf{x}^{\xi}$ and the matrix $Y = [\mathbf{y}^1, \ldots \mathbf{y}^k]$ whose columns consist of the vectors $\mathbf{y}^{\xi}$.

We speak of an *auto-associative memory* if the fundamental memory set is of the form $\{(\mathbf{x}^{\xi}, \mathbf{x}^{\xi}) : \xi = 1, \ldots, k\}$. The memory is said to be *hetero-associative* if the output $\mathbf{y}^{\xi}$ differs from the input $\mathbf{x}^{\xi}$. One of the most common problem associated with the design of an AM is the creation of false or spurious memories. A *spurious memory* is a memory association that does not belong to the fundamental memory set, i.e., it was unintentionally stored in the memory.

The process of determining G is called *recording phase* and the mapping G is called *associative mapping*. We speak of a *neural associative memory* if the associative mapping G is given by an *artificial neural network* (ANN). In particular, we have a *fuzzy (neural) associative memory* (FAM) if the associative mapping G is given by a fuzzy neural network and the patterns $\mathbf{x}^\xi$ and $\mathbf{y}^\xi$ are fuzzy sets for every $\xi = 1, \ldots, k$.

We speak of a [*fuzzy*] *morphological neural network* if its neurons perform an elementary [fuzzy] morphological operation. The neurons of an ANN of this type are called [*fuzzy*] *morphological neurons*. A FMNN that serves as an AM is called a FMAM.

3.1. Types of neurons used in FAM models

Let us now present the most important types of fuzzy neurons that occur in FMAM models. These models of artificial neurons can be formulated in terms of max-C and min-D matrix products. From now on, the symbol $\mathbf{x} = [x_1, \ldots, x_n]^{\mathrm{T}}$ denotes the fuzzy input vector and y denotes the fuzzy output value. The weights $w_i, m_i \in [0, 1]$ of these fuzzy neurons form vectors $\mathbf{w} = [w_1, \ldots, w_n]^{\mathrm{T}}$ and $\mathbf{m} = [m_1, \ldots, m_n]^{\mathrm{T}}$.

One of the most general classes of fuzzy neurons was introduced by Pedrycz in the early 1990s [31]. We are particularly interested in an *OR-neuron* of the following form, where S is a t-conorm and T is a t-norm.

$$y = \mathop{S}_{j=1}^{n} T(w_j, x_j). \tag{36}$$

Consider the special case that S equals the maximum operation. Moreover, let us substitute the t-norm with the more general operation of fuzzy conjunction. Thus, we obtain a max-C *neuron* of the following form:

$$y = \bigvee_{j=1}^{n} C(w_j, x_j) = \mathbf{w}^{\mathrm{T}} \circ \mathbf{x}. \tag{37}$$

Particular choices of fuzzy conjunctions yield particular max-C neurons. We will indicate the underlying type of fuzzy conjunction by means of a subscript. A similar notation will be applied to describe the min-D neuron that is introduced below. Max-C neurons occur in several FAM models, including the famous FAMs of Kosko [23] and the recent IFAM models [49].

A closer look at Eqs. (37) and (33) reveals that a max-C neuron represents a fuzzy dilation if C commutes with the supremum in the second argument. In fact, by Proposition 8, we have a max-C *morphological neuron* or max-C *dilative neuron* if and only if $C(x, \cdot)$ is a dilation for every $x \in [0, 1]$. Examples of max-C morphological neurons include max-C_M, max-C_P, max-C_L, and max-C_K neurons.

In a similar vein, we slightly adapt Pedrycz's *AND-neuron* [31] in order to obtain the neural model that is given in terms of the following equation:

$$y = \bigwedge_{j=1}^{n} D(m_j, x_j) = \mathbf{m}^{\mathrm{T}} \bullet \mathbf{x}. \tag{38}$$

We refer to neurons of this type as min-D *neurons*. We will show that the FLBAM models [3] and some dual FAM models are equipped with min-D neurons [49]. In view of Proposition 8, we speak of a min-D *morphological neuron* or min-D *erosive neuron* if and only if $D(x, \cdot)$ is an erosion for every $x \in [0, 1]$. Min-D_M, min-D_P, min-D_L, and min-D_K neurons exemplify min-D morphological neurons.

3.2. Max-C *FMAM*

This section shows that many well-known FAM models can be viewed as single layer feedforward ANNs with max-C morphological neurons. These models are given by

$$\mathbf{y} = \mathcal{W}(\mathbf{x}) = W \circ \mathbf{x}, \tag{39}$$

where $W \in [0, 1]^{m \times n}$ represents the synaptic weight matrix and $\mathbf{x} \in [0, 1]^n$ and $\mathbf{y} \in [0, 1]^m$ are the fuzzy input and fuzzy output patterns, respectively. Some exceptions to this rule will be presented in the next section.

Note that Eq. (39) describes an FMAM if and only if the corresponding fuzzy conjunction corresponds to a dilation. In this case, the associative mapping $\mathcal{W}$ represents a dilation from $[0, 1]^n$ into $[0, 1]^m$ and the AM model given by Eq. (39) belongs to the class of max-C FMAMs. Several examples of max-C FMAMs are listed below.

Example 9. Kosko's FAMs, introduced in the early 1990s, constitute one of the earliest attempts to develop neural AM models based on fuzzy set theory [23]. These models employ max-C_M or max-C_P products in Eq. (39). Consequently, they are usually referred to as *max–min FAM* and *max-product FAM*. Note that both max–min and max-product FAMs represent FMAMs.

Let us construct the matrices $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0, 1]^{n\times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0, 1]^{m\times k}$. The weight matrix of the max–min FAM is synthesized by setting $W = Y \circ_M X^{\mathrm{T}}$ and the weight matrix of the max-product FAM is synthesized by setting $W = Y \circ_P X^{\mathrm{T}}$.

Example 10. Chung and Lee generalized Kosko's FAMs by substituting the max–min or the max-product with a more general max-t product [8]. The resulting model, called *generalized FAM* (GFAM), can be described in terms of the following relationship between an input pattern $\mathbf{x}$ and the corresponding output pattern $\mathbf{y}$. Here, the symbol $\circ_T$ denotes the max-C product where C is a t-norm.

$$\mathbf{y} = W \circ_T \mathbf{x} \quad \text{where } W = Y \circ_T X^{\mathrm{T}}. \tag{40}$$

Note that a GFAM performs a dilation at each node (and overall) if and only if the t-norm represents a dilation. In this case, we speak of a morphological or dilative GFAM.

We would like to point out that Chung and Lee were particularly interested in the Lukasiewicz GFAM, i.e., the GFAM based on the max-C_L product [8]. Note that the Lukasiewicz GFAM belongs to the FMAM class.

Example 11. Junbo's FAM and Kosko's max–min FAM share the same network topology and the same type of morphological neurons, namely max-C_M neurons [17]. Consequently, Junbo's FAM computes the output pattern $\mathbf{y}$ according to the rule $\mathbf{y} = W \circ_M \mathbf{x}$ upon presentation of an input pattern $\mathbf{x} \in [0, 1]^n$. Thus, Junbo's model belongs to the FMAM class.

The difference between the max–min FAM and Junbo's FAM lies in the learning rule. Junbo et al. introduced a new learning rule for FAMs that allows for the storage of multiple fuzzy fundamental memories. The synaptic weight matrix is computed as follows:

$$W = Y \circledast_M X^{\mathrm{T}}. \tag{41}$$

Here, the symbol $\circledast_M$ denotes the min-J_M product of Eq. (31). We refer to this learning rule as *Gödel implicative learning* since it employs Gödel's reverse fuzzy implication J_M [49].

Example 12. The max–min FAM with threshold of Liu is a variation of Junbo's FAM that incorporates a threshold (or bias) at the input and output layer [24]. The recall phase of this model is given by the following equation where $W \in [0, 1]^{m\times n}$ is the synaptic weight matrix and $\mathbf{c} \in [0, 1]^n$ and $\mathbf{d} \in [0, 1]^m$ are the threshold vectors:

$$\mathbf{y} = (W \circ_M (\mathbf{x} \vee \mathbf{c})) \vee \mathbf{d}. \tag{42}$$

Note that Eq. (42) boils down to adding bias terms to the max–min FAM. However, due to the monotonicity of max-C_M product, the action of both thresholds $\mathbf{c}$ and $\mathbf{d}$ can be captured in terms of a new threshold $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_m] \in [0, 1]^m$ [49]. Note that adding a threshold $\boldsymbol{\theta}$ amounts to setting $\mathbf{x} = [1, x_1, \ldots, x_n]^{\mathrm{T}}$ and $w_{i0} = \theta_i$ for $i = 1, \ldots, m$. Consequently, the max–min FAM with threshold can be reduced to a FAM model described by Eq. (39) with max-C_M morphological neurons. Thus, the max–min FAM with threshold belongs to the class of FMAMs.

Example 13. We recently introduced a new class of max-C FMAMs called *implicative fuzzy associative memories* (IFAMs) [49,52]. An IFAM model has max-T neurons, where T is a continuous t-norm. In contrast to the GFAM, the IFAM model includes a bias term $\boldsymbol{\theta} \in [0, 1]^m$ and employs a learning rule called *R-implicative fuzzy learning*. This learning rule defines W and $\boldsymbol{\theta}$ as follows:

$$W = Y \circledast_T X^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{\theta} = \bigwedge_{\xi=1}^{k} \mathbf{y}^{\xi}. \tag{43}$$

The reverse fuzzy implication J_T employed in the min-J_T product "$\circledast_T$" is uniquely determined by setting $J_T(y, x) = I_T(x, y)$ where I_T is the fuzzy implication given by the following equation:

$$I_T(x, y) = \bigvee \{z \in [0, 1] : T(z, x) \leqslant y\}. \tag{44}$$

A closer look at Eqs. (44) and (29), reveals that J_T is simply the reverse fuzzy implication that is adjoint to the t-norm T. Particular choices of T, J_T, respectively, lead to particular IFAM models. The name of a particular IFAM model indicates the choice of T and J_T. For example, the Gödel IFAM corresponds to the IFAM model given by $\mathbf{y} = (W \circ_M \mathbf{x}) \vee \boldsymbol{\theta}$, where $W = Y \circledast_M X^{\mathrm{T}}$ and $\boldsymbol{\theta} = \bigwedge_{\xi=1}^{k} \mathbf{y}^{\xi}$. If the columns of the matrices $X \in [0, 1]^{3\times 3}$ and $Y = [0, 1]^{2\times 3}$ given by Eq. (45) represent the patterns $\mathbf{x}^{\xi}$ and $\mathbf{y}^{\xi}$, where $\xi = 1, 2, 3$, then the matrix $W \in [0, 1]^{2\times 3}$ and the vector $\boldsymbol{\theta} \in [0, 1]^2$ of the Gödel IFAM are given by Eq. (46).

$$X = \begin{bmatrix} 0.4 & 0.8 & 0.9 \\ 0.7 & 1.0 & 0.2 \\ 0.6 & 0.5 & 1.0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0.7 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.2 \end{bmatrix}, \tag{45}$$

$$W = \begin{bmatrix} 0.7 & 0.8 & 0.7 \\ 0.2 & 0.6 & 0.2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} = \begin{bmatrix} 0.7 \\ 0.2 \end{bmatrix}. \tag{46}$$

Note that an IFAM model can be described by Eq. (37) setting $\mathbf{x} = [1, x_1, \ldots, x_n]^{\mathrm{T}}$ and $w_{i0} = \theta_i$ for $i = 1, \ldots, m$. Furthermore, a continuous t-norm represents a dilation in [0, 1]. Thus, IFAM models belong to the class of FMAMs.

3.3. Min-D FMAMs, the negations of max-C FMAMs

This section concerns FAM models given by the following equation where $M \in [0, 1]^{m\times n}$ represents the synaptic weight matrix and $\mathbf{x} \in [0, 1]^n$ and $\mathbf{y} \in [0, 1]^m$ are the fuzzy input and fuzzy output patterns, respectively.

$$\mathbf{y} = \mathcal{M}(\mathbf{x}) = M \bullet \mathbf{x}. \tag{47}$$

Note that Eq. (47) describes a single layer feedforward ANN with min-D neurons. In particular, this equation yields an FMAM if and only if the corresponding fuzzy disjunction corresponds to an erosion in the second argument. In this case, the associative mapping $\mathcal{M}$ represents an erosion from $[0, 1]^n$ into $[0, 1]^m$. The resulting model will be called min-*D FMAM*. The following establishes a relationship between min-D and max-C FMAMs.

Recall that a max-C FMAM corresponds to a dilation $\mathcal{W} : [0, 1]^n \to [0, 1]^m$. Thus, the two relationships of duality of MM can be used to formulate new FMAM models. In particular, let $\mathbf{N}(\mathbf{x})$ denote the component-wise fuzzy negation of a vector $\mathbf{x}$. We define the *negation of a* max-*C FAM* $\mathcal{W}$ as the AM model $\mathcal{M}$ that corresponds to the negation of $\mathcal{W}$ with respect to $\mathbf{N}$, i.e., the FAM $\mathcal{M}$ given by the following equation where $\mathbf{x}$ and $\mathbf{y}$ are the input and the recalled patterns, respectively:

$$\mathbf{y} = \mathcal{M}(\mathbf{x}) = \mathcal{W}^{\mathbf{N}}(\mathbf{x}) = \mathbf{N}\left(\mathcal{W}\left[\mathbf{N}(\mathbf{x})\right]\right). \tag{48}$$

The following theorem reveals that the negation of a max-C FMAM represents a min-D FMAM.

Theorem 14. *Let N be a fuzzy negation. If we define a function ϕ_N such that $\phi_N(\mathcal{W}) = \mathcal{W}^N$ for all* max-*C FAMs $\mathcal{W}$ then ϕ_N constitutes a bijection between the set of* max-*C FAMs and the set of* min-*D FAMs. Given an arbitrary* max-*C FAM $\mathcal{W}$ with corresponding weight matrix W, we have that the weight matrix of the* min-*D FAM $\mathcal{M} = \mathcal{W}^{\mathbf{N}}$ is simply given by $M = N(W)$. Moreover, the fuzzy operators C and D are dual with respect to N.*

In particular, if C is a dilation in the second argument then D is an erosion in the second argument. Therefore, restricting ϕ_N to the set of max-*C FMAMs yields a bijection between the sets of* max-*C FMAMs and* min-*D FMAMs. Similarly, if C is a* t-*norm then D is a* t-*conorm and therefore ϕ_N induces a bijection between the set of* max-*T FAMs and the set of* min-*S FAMs.*

Example 15. Consider a max-T FMAM such as a dilative GFAM given by Eq. (40). The negation of the original model computes an output pattern $\mathbf{y}$ according to the rule $\mathbf{y} = M \bullet_S \mathbf{x}$. Here, the symbol $\bullet_S$ denotes the min-D product where D is a t-conorm. We refer to [49] for another example of a class of min-D FMAMs, namely, the *dual IFAMs*.

Example 16. The *fuzzy logical bidirectional associative memory* (FLBAM) [3] employs fuzzy implications I_T given by Eq. (44) where T is a t-norm. Given an input pattern $\mathbf{x}^{(0)} \in [0,1]^n$, the FLBAM generates a sequence $(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})$, $(\mathbf{x}^{(1)}, \mathbf{y}^{(0)})$, $(\mathbf{x}^{(1)}, \mathbf{y}^{(1)})$, $(\mathbf{x}^{(2)}, \mathbf{y}^{(1)}), \ldots$ as follows for $k = 0, 1, \ldots$

$$y_i^{(k)} = \bigwedge_{j=1}^{n} I_T(x_j^{(k)}, m_{ij}) \quad \forall\, i = 1, \ldots, m, \tag{49}$$

$$x_j^{(k+1)} = \bigwedge_{i=1}^{m} I_T(y_i^{(k)}, m_{ij}) \quad \forall\, j = 1, \ldots, n. \tag{50}$$

The synaptic weight matrix M of the FLBAM is synthesized using the rule $M = Y \circ_T X^{\mathrm{T}}$. The following theorem reveals that performing one step of the FLBAM model corresponds to an application of a min-D FMAM to the negation of either $\mathbf{x}^{(k)}$ or $\mathbf{y}^{(k)}$.

Theorem 17. *Let* $\mathbf{x}^{(0)} \in [0,1]^n$ *be an input pattern. The sequence* $(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})$, $(\mathbf{x}^{(1)}, \mathbf{y}^{(0)})$, $(\mathbf{x}^{(1)}, \mathbf{y}^{(1)})$, $(\mathbf{x}^{(2)}, \mathbf{y}^{(1)}), \ldots$ *generated by an FLBAM model is given by*

$$\mathbf{y}^{(k)} = M \bullet \mathbf{N}(\mathbf{x}^{(k)}) \quad \textit{and} \quad \mathbf{x}^{(k+1)} = M^{\mathrm{T}} \bullet \mathbf{N}(\mathbf{y}^{(k)}) \ \textit{for}\ k = 0, 1, \ldots . \tag{51}$$

Furthermore, the neurons of the FLBAM are min-D *erosive* (*morphological*) *neurons.*

3.4. Adjoint FMAM

The duality relationship of adjunction can be used to define adjoint models of a max-C or a min-D FMAM. For example, consider a max-C FMAM $\mathcal{W}$ given by Eq. (39). There exists an erosion $\mathcal{A} : [0,1]^m \to [0,1]^n$ such that $(\mathcal{A}, \mathcal{W})$ forms an adjunction. Furthermore, Eq. (7) implies that the following equation holds for every $\mathbf{y} \in [0,1]^m$.

$$\mathcal{A}(\mathbf{y}) = \bigvee \{\mathbf{x} \in [0,1]^n : \mathcal{W}(\mathbf{x}) \leqslant \mathbf{y}\}, \tag{52}$$

The *adjoint FMAM of* $\mathcal{W}$ is defined as the AM model that corresponds to the mapping $\mathcal{A}$.

In a similar vein, if $\mathcal{M} : [0,1]^n \to [0,1]^m$ is the erosion corresponding to a min-D FMAM given by Eq. (47) then we define the *adjoint FMAM of* $\mathcal{M}$ as the dilation $\mathcal{B} : [0,1]^m \to [0,1]^n$ that forms an adjunction with $\mathcal{M}$. Using Eq. (8), we are able to construct the dilation $\mathcal{B}$ as follows for every $\mathbf{y} \in [0,1]^m$:

$$\mathcal{B}(\mathbf{y}) = \bigwedge \{\mathbf{x} \in [0,1]^n : \mathcal{M}(\mathbf{x}) \geqslant \mathbf{y}\}. \tag{53}$$

The following theorem shows that the adjoint model of a max-C FMAM has min-D neurons whereas the adjoint model of a min-D FMAM has max-C neurons. Thus, these models belong indeed to the FMAM class.

Theorem 18. *Let* $\mathcal{W}$ *denote the* max-C *FMAM* $\mathcal{W}$ *and let* $\mathcal{M}$ it denote the min-D *FMAM* $\mathcal{M}$ *that are defined via Eqs.* (39) *and* (47). *Given an input pattern* $\mathbf{y} \in [0,1]^m$, *the adjoint FMAM of* $\mathcal{W}$ *computes the corresponding output pattern* $\mathbf{x} \in [0,1]^n$ *in terms of the following equation where* $M = N(W)$:

$$\mathbf{x} = \mathcal{A}(\mathbf{y}) = M^{\mathrm{T}} \bullet \mathbf{y}. \tag{54}$$

Similarly, the adjoint FMAM of $\mathcal{M}$ *computes the output pattern* $\mathbf{x}$ (*corresponding to the input pattern* $\mathbf{y}$) *in terms of the following equation where* $W = N(M)$:

$$\mathbf{x} = \mathcal{B}(\mathbf{y}) = W^{\mathrm{T}} \circ \mathbf{y}. \tag{55}$$

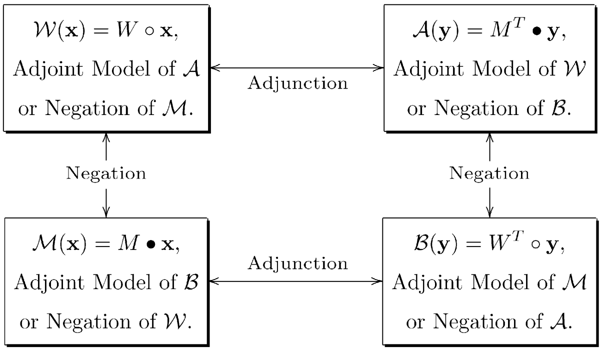

Fig. 2. Relationship scheme between an FMAM model and its dual versions.

We would like to point out that Eq. (55) can also be determined by forming the negation of Eq. (54) since the commutative diagram depicted in Fig. 2 holds true (cf. Fig. 1 and the Proof of Theorem 18). In other words, the operators $\mathcal{W}$ and $\mathcal{M}$ are dual with respect to a negation **N** if and only if the operators $\mathcal{A}$ and $\mathcal{B}$ are also dual with respect to **N**.

4. Recording scheme for FMAMs

In the previous section, we discussed the type of neurons and the network topology of many well-know FAM models. This section introduces a large class of recording schemes for max-C and min-D FMAMs. More precisely, we will show that a general class of recording strategies for max-C FMAMs called *fuzzy learning by adjunction* or *implicative fuzzy learning* can be derived from the duality relationship of adjunction. In particular, we will show that this recording scheme yields the synaptic weight matrix that represents the best approximation from below of Y in terms of the max-C product and that it generalizes the implicative fuzzy learning scheme for IFAMs [49]. Finally, an application of the duality relationship of negation to the synaptic weight matrix of a max-C FMAM leads to a new recording scheme for min-D FMAMs.

4.1. Fuzzy learning by adjunction for max-C *FMAMs*

Let $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$ be a fundamental memory set. For simplicity, let $X \in [0,1]^{n\times k}$ and $Y \in [0,1]^{m\times k}$ denote the matrices whose columns are the vectors $\mathbf{x}^\xi$ and $\mathbf{y}^\xi$, respectively. Consider a max-C FMAM given by Eq. (39) and let $\mathcal{D}_X : [0,1]^{m\times n} \to [0,1]^{m\times k}$ be the operator defined as follows:

$$\mathcal{D}_X(W) = W \circ X. \tag{56}$$

Note that if there exists a synaptic weight matrix $W \in [0,1]^{m\times n}$ such that $Y = \mathcal{D}_X(W)$ then the max-C FMAM produces the desired output $\mathbf{y}^\xi$ upon presentation of the input $\mathbf{x}^\xi$, i.e., the FMAM perfectly recalls the associations $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, for each $\xi = 1, \ldots, k$.

Suppose that $\mathcal{D}_X$ is a fuzzy dilation. By Proposition 6, there exists an unique fuzzy erosion $\mathcal{E}_X : [0,1]^{m\times k} \to [0,1]^{m\times n}$ that forms an adjunction with $\mathcal{D}_X$. The fuzzy erosion $\mathcal{E}_X$ depends on $X \in [0,1]^{n\times k}$ and produces a matrix in $[0,1]^{m\times n}$ for every input $Y \in [0,1]^{m\times k}$. Thus, the following equation can be used to define a synaptic weight matrix for a max-C FMAM:

$$W = \mathcal{E}_X(Y). \tag{57}$$

Before we characterize synaptic weight matrices of this form, we provide necessary and sufficient conditions for the operator $\mathcal{D}_X$ to be a dilation.

Theorem 19. *The operator $\mathcal{D}_X$ given by Eq.* (56) *represents a dilation for every $X \in [0,1]^{n\times k}$ if and only if $C(\cdot, x)$ is a dilation for every $x \in [0,1]$.*

Theorem 19 implies that Eq. (57) can be employed in the recording phase of a max-C FMAM if and only if the fuzzy conjunction C represents a dilation in the first argument.

The following theorem describes the main properties of a synaptic weight matrix W given by Eq. (57). In particular, Theorem 20 implies that W is the best approximation from below of Y in terms of the max-C product.

Theorem 20. *Let $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n\times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{m\times k}$. Consider an adjunction $(\mathcal{D}_X, \mathcal{E}_X)$ where $\mathcal{D}_X$ is given by Eq.* (56). *The synaptic weight matrix $W = \mathcal{E}_X(Y)$ represents the supremum of the set of matrices $V \in [0,1]^{m\times n}$ such that $V \circ X \leqslant Y$, i.e., W satisfies the following equation*:

$$W = \bigvee\{V \in [0,1]^{m\times n} : V \circ X \leqslant Y\}. \tag{58}$$

In particular, if there exists a matrix $V \in [0,1]^{m\times n}$ such that $V \circ X = Y$ then $V \leqslant W$ and $W \circ X = Y$.

A straightforward consequence of Theorem 20 that concerns the auto-associative case is presented in Corollary 21. This corollary guarantees that a max-C FMAM will perfectly store and recall a set of patterns $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ if there exists a fuzzy matrix $I \in [0,1]^{n\times n}$ such that $I \circ X = X$ for every $X \in [0,1]^{n\times k}$. In this case, we say that the max-C product has a left identity matrix I.

Corollary 21. *Consider an adjunction $(\mathcal{D}_X, \mathcal{E}_X)$ where $\mathcal{D}_X$ is given by Eq.* (56) *and let $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n\times k}$. If the* max-$C$ *product has a left identity $I \in [0,1]^{n\times n}$ then the synaptic weight matrix $W = \mathcal{E}_X(X)$ is such that $W \circ \mathbf{x}^\xi = \mathbf{x}^\xi$ for every $\xi = 1, \ldots, k$.*

Note that a max-C product has a left identity, namely the matrix whose entries are given by Kronecker's delta, if $C(1,x) = x$ for every $x \in [0,1]$. In particular, a t-norm satisfies $T(1,x) = x$ for every $x \in [0,1]$. Thus, one can store as many patterns as desired in auto-associative max-T FMAMs using the recording scheme corresponding to Eq. (57).

The following theorem shows that the synaptic weight matrix W given by Eq. (57) can be easily computed by means of a single min-J product.

Theorem 22. *Let $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n\times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{m\times k}$. Consider an operator $\mathcal{D}_X$ given by Eq.* (56) *based on a fuzzy conjunction C that represents a dilation in both arguments. The synaptic weight matrix $W = \mathcal{E}_X(Y)$ is given by the following* min-J *product where J is the reverse fuzzy implication that is adjoint to C.*

$$W = Y \circledast X^{\mathrm{T}}. \tag{59}$$

Recall that we had previously introduced *implicative fuzzy learning*, a recording scheme based on the minimum of fuzzy implications [49,52]. Since a reverse fuzzy implication J corresponds to a fuzzy implication I, Eq. (59) describes implicative fuzzy learning. Therefore, Theorem 22 generalizes the implicative fuzzy learning scheme used in IFAM models to include general max-C FMAMs.

Implicative fuzzy learning has explicitly or implicitly been used in conjunction with several FAM models such as the FAM model of Junbo et al. [17], the max–min FAM with threshold of Liu [24], and the IFAM models [49,52]. The following example concerns an application of the implicative fuzzy learning in an FMAM that is not based on a t-norm.

Example 23. Consider the matrices $X = [\mathbf{x}^1, \ldots, \mathbf{x}^4] \in [0,1]^{5\times 4}$ and $V \in [0,1]^{3\times 5}$ displayed in Eq. (60). In order to verify Theorems 20 and 22, we defined $Y = V \circ_K X$ and computed the matrix $W = Y \circledast_K X^{\mathrm{T}}$ using the implicative fuzzy learning. Here, the symbols "$\circ_K$" and "$\circledast_K$" represent the max-C and the min-J products based on the fuzzy conjunction C_K and reverse fuzzy implication J_K. Recall that C_K does not represent a t-norm but does represent a dilation in the first argument. The matrices $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{3\times 4}$ and $W \in [0,1]^{3\times 5}$ are shown in Eq. (61).

Note that $V \leqslant W$. Moreover, evaluating the max-C_K product of W and X reveals that $Y = W \circ_K X$.

$$X = \begin{bmatrix} 0.35 & 1.00 & 0.82 & 0.25 \\ 0.14 & 0.70 & 0.32 & 0.04 \\ 0.41 & 0.97 & 0.33 & 0.27 \\ 0.13 & 0.17 & 0.64 & 0.33 \\ 0.70 & 0.86 & 0.96 & 0.36 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0.03 & 0.53 & 0.24 & 0.74 & 0.16 \\ 0.28 & 0.44 & 0.70 & 0.73 & 0.87 \\ 0.07 & 0.48 & 0.50 & 0.00 & 0.74 \end{bmatrix}, \tag{60}$$

$$Y = \begin{bmatrix} 0.00 & 0.53 & 0.74 & 0.74 \\ 0.87 & 0.87 & 0.87 & 0.87 \\ 0.74 & 0.74 & 0.74 & 0.74 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0.53 & 0.53 & 0.53 & 0.74 & 0.30 \\ 0.87 & 0.87 & 0.87 & 0.87 & 0.87 \\ 0.74 & 0.74 & 0.74 & 0.74 & 0.74 \end{bmatrix}. \tag{61}$$

4.2. *Synthesis of the weight matrix for a* min-D *FMAM*

A recording scheme for a min-D FMAM can be derived either directly by using fuzzy learning by adjunction or indirectly by forming the negation $\mathcal{M} = \mathcal{W}^{\mathbf{N}}$ of a given max-C FMAM $\mathcal{W}$, i.e., by calculating the negation $M = N(W)$ of the weight matrix W corresponding to a given max-C FMAM $\mathcal{W}$. In this subsection, we adopt the latter, indirect approach. We would like to point out, however, that the two approaches lead to the same synaptic weight matrix (cf. the Proof of Theorem 24).

Consider a min-D FMAM $\mathcal{M}$ and let N be a fuzzy negation. Suppose that the fuzzy disjunction commutes with the infimum operation in the first argument, i.e., $D(\cdot, x)$ is an erosion for every $x \in [0,1]$. By Theorem 14, there exists a max-C FMAM $\mathcal{W}$ such that $\mathcal{W}$ and $\mathcal{M}$ are dual operators with respect to a fuzzy negation N. This max-C FMAM is based on the fuzzy conjunction C that is dual to D with respect to N. Thus, $C(\cdot, x)$ is a dilation for every $x \in [0,1]$ and fuzzy learning by adjunction can be applied to store a set of associations. In particular, given matrices $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n\times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{n\times k}$, we can define $W = N(Y) \circledast N(X)^{\mathrm{T}}$. Note that W is computed in terms of the negations $N(X)$ and $N(Y)$ instead of the matrices X and Y. This follows from the fact that $N(Y) = W \circ N(X)$ if and only if $Y = M \bullet X$, where $M = N(W)$.

Summarizing these observations, we can construct the synaptic weight matrix $M \in [0,1]^{m\times n}$ of a min-D FMAM by means of the following equation:

$$M = N(W) = N(N(Y) \circledast N(X)^{\mathrm{T}}). \tag{62}$$

Here, the min-J product is based on the reverse implication that is adjoint to C, the fuzzy conjunction that is dual to D with respect to N. The following theorem reveals that the matrix M given by Eq. (62) is the best approximation from above of Y in the sense of the min-D product.

Theorem 24. *The synaptic weight matrix M given by Eq.* (62) *represents the infimum of the set of matrices $U \in [0,1]^{m\times n}$ such that $U \bullet X \geqslant Y$, i.e., M satisfies the following equation*:

$$M = \bigwedge \{U \in [0,1]^{m\times n} : U \bullet X \geqslant Y\}. \tag{63}$$

Moreover, if there exists $U \in [0,1]^{m\times n}$ such that $U \bullet X = Y$, then $U \geqslant M$ and $M \bullet X = Y$.

Note that Theorem 24 is the dual of Theorem 20. In fact, every statement concerning a max-C FMAM with synaptic weight given by Eq. (59) induces a corresponding dual statement for the dual min-D FMAM with synaptic weight matrix given by Eq. (62). For example, one can easily show that if there exist a synaptic weight matrix $I \in [0,1]^{n\times n}$ such that $I \bullet X = X$ for every $[0,1]^{n\times k}$, then one can store as many patterns as desired in the auto-associative min-D FMAM.

The recording scheme of Eq. (62) has been applied to store patterns in dual IFAM models [49]. The following theorem shows that the synaptic weight matrix of an FLBAM model can also be obtained by means of the negation of fuzzy learning by adjunction.

Theorem 25. *Consider an FLBAM given by Eq.* (51) *and let N be a fuzzy negation. Let D denote the corresponding fuzzy disjunction. Given matrices $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0, 1]^{n\times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0, 1]^{m\times k}$, we obtain the synaptic weight matrix $M \in [0, 1]^{m\times n}$ of the FLBAM model as follows*:

$$M = N(N(Y)\circledast X^{\mathrm{T}}), \tag{64}$$

where J represents the reverse fuzzy implication that forms an adjunction with the negation of D, i.e., the pair (J, D^N) forms an adjunction.

5. Concluding remarks

This paper successfully relates FAMs and FMM. In particular, we employed concepts of FMM in order to establish a general framework for the recording and recall phases of FAMs. The resulting class of FAMs was called the class of FMAMs.

We have shown that many existing FAM models fit into this framework. More importantly, the results of this paper reveal that the common structure of FMAMs cannot only be used to construct new particular FMAM models but it also provides new insights into the properties of new and existing models.

In the future, we plan to investigate the recall phase of FMAMs. In particular, we will pursue results that characterize the noise tolerance and the fixed points of auto-associative FMAMs that are trained using fuzzy learning by adjunction. Furthermore, we plan to investigate the performance of FMAM models in applications as fuzzy rule-based systems.

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Appendix A. Proof of lemmas and theorems

Proof of Lemma 1. We will only show that an operator $\bar{\varepsilon} : \mathbb{L} \to \mathbb{M}$ represents an anti-erosion if and only if $v_{\mathbb{M}} \circ \bar{\varepsilon}$ is an erosion. The other claims can be demonstrated in a similar fashion.

Suppose that $\bar{\varepsilon}$ represents an anti-erosion and let $Y \subseteq \mathbb{L}$. The following equations reveal that $v_M \circ \bar{\varepsilon}$ is an erosion:

$$(v_{\mathbb{M}} \circ \bar{\varepsilon}) \left(\bigwedge Y\right) = v_{\mathbb{M}} \left(\bar{\varepsilon}\left(\bigwedge Y\right)\right) = v_{\mathbb{M}} \left(\bigvee_{y\in Y} \bar{\varepsilon}(y)\right) = \bigwedge_{y\in Y} v_{\mathbb{M}}(\bar{\varepsilon}(y)) = \bigwedge_{y\in Y} (v_{\mathbb{M}} \circ \bar{\varepsilon})(y). \tag{A.1}$$

Now, suppose that $v_M \circ \bar{\varepsilon}$ is an erosion. The converse follows from the fact that $v_{\mathbb{M}} \circ v_{\mathbb{M}}$ is the identity operator. In fact, the following equations hold true for every $Y \subseteq \mathbb{L}$:

$$\bar{\varepsilon}\left(\bigwedge Y\right) = ((v_{\mathbb{M}} \circ v_{\mathbb{M}}) \circ \bar{\varepsilon}) \left(\bigwedge Y\right) = v_{\mathbb{M}} \circ (v_{\mathbb{M}} \circ \bar{\varepsilon}) \left(\bigwedge Y\right) = v_{\mathbb{M}} \left(\bigwedge_{y\in Y} (v_{\mathbb{M}} \circ \bar{\varepsilon})(y)\right) \tag{A.2}$$

$$= \bigvee_{y\in Y} ((v_{\mathbb{M}} \circ v_{\mathbb{M}})\, \bar{\varepsilon})(y) = \bigvee_{y\in Y} \bar{\varepsilon}(y). \tag{A.3}$$

Thus, $\bar{\varepsilon}$ represents an anti-erosion. □

Proof of Lemma 4. By Lemma 7 and Proposition 3, the operators $C_M(\cdot, y)$, $C_P(\cdot, y)$, $C_L(\cdot, y)$, and $C_K(\cdot, y)$ represent dilations for every $y \in [0, 1]$. Since C_M, C_P, and C_L are commutative operators, these three operators also perform a dilation in second argument.

The following equalities reveal that $\bigvee_{y\in Y} C_K(x,y) = C_K(x, \bigvee Y)$ for every $x \in [0,1]$ and $Y \subseteq [0,1]$. Thus, C_K is a dilation in both arguments.

$$\bigvee_{y\in Y} C_K(x,y) = \bigvee_{y\in Y} \begin{cases} 0, & x+y \leqslant 1 \\ x & \text{otherwise} \end{cases} = \begin{cases} 0, & x+y\leqslant 1 \ \forall\, y \in Y \\ x & \text{otherwise} \end{cases} \tag{A.4}$$

$$= \begin{cases} 0, & \bigvee_{y\in Y}(x+y)\leqslant 1 \\ x & \text{otherwise} \end{cases} = \begin{cases} 0, & x+\bigvee Y \leqslant 1 \\ x & \text{otherwise} \end{cases} = C_K\left(x, \bigvee Y\right). \qquad \square \tag{A.5}$$

Proof of Lemma 5. This lemma is a straightforward consequence of Corollary 2 and Lemmas 4 and 6. □

Proof of Lemma 6. The negations of C_M, C_P, and C_L with respect to the standard fuzzy negation satisfy the following equations:

$$N_S(C_M(N_S(x), N_S(y))) = 1 - [(1-x)\wedge(1-y)] = x \vee y = D_M(x,y), \tag{A.6}$$

$$N_S(C_P(N_S(x), N_S(y))) = 1 - (1-x)\cdot(1-y) = x + y - x\cdot y = D_P(x,y), \tag{A.7}$$

$$N_S(C_L(N_S(x), N_S(y))) = 1 - 0 \vee [(1-x)+(1-y)-1] = 1 \wedge (x+y) = D_L(x,y). \tag{A.8}$$

Similarly, the following equations hold true for the negation of C_K with respect to N_S:

$$N_S(C_M(N_S(x), N_S(y))) = 1 - \begin{cases} 0, & (1-x)+(1-y)\leqslant 1, \\ 1-x & \text{otherwise} \end{cases} \tag{A.9}$$

$$= \begin{cases} 1, & x+y\geqslant 1, \\ x & \text{otherwise,} \end{cases} = D_K(x,y). \qquad \square \tag{A.10}$$

Proof of Lemma 7. The following arguments show that the pairs (C_M, J_M), (C_P, J_P), (C_L, J_L), and (C_K, J_K) are adjoint operators since they satisfy Eq. (29):

$$J_M(x,y) = \bigvee\{z\in[0,1] : C_M(z,y)\leqslant x\} = \bigvee\{z \in [0,1] : z\wedge y \leqslant x\} \tag{A.11}$$

$$= \bigvee\{z\in[0,1] : z\leqslant x \text{ or } y \leqslant x\} = \begin{cases} 1, & y\leqslant x, \\ x, & y > x. \end{cases} \tag{A.12}$$

$$J_P(x,y) = \bigvee\{z\in[0,1] : C_P(z,y)\leqslant x\} = \bigvee\{z\in[0,1] : z\cdot y \leqslant x\} = \begin{cases} 1, & y\leqslant x, \\ x/y, & y > x. \end{cases} \tag{A.13}$$

$$J_L(x,y) = \bigvee\{z\in[0,1] : C_L(z,y)\leqslant x\} = \bigvee\{z\in[0,1] : 0\vee(z+y-1)\leqslant x\} \tag{A.14}$$

$$= \bigvee\{z\in[0,1] : z+y-1\leqslant x\} = \bigvee\{z\in[0,1] : z\leqslant x-y+1\} \tag{A.15}$$

$$= (x-y+1)\wedge 1. \tag{A.16}$$

Note that $x - y + 1 \geqslant 0$ since $x \geqslant 0$ and $1 - y \geqslant 0$ for all $x, y \in [0,1]$. Finally, note that C_K satisfies the following equalities:

$$C_K(x,y) = \bigwedge\{z\in[0,1] : J_K(z,y)\geqslant x\} = \bigwedge\{z\in[0,1] : z\vee(1-y)\geqslant x\} \tag{A.17}$$

$$= \bigwedge\{z\in[0,1] : z\geqslant x \text{ or } 1-y\geqslant x\} = \begin{cases} 0, & 1-y\geqslant x, \\ x, & 1-y < x. \end{cases} \qquad \square \tag{A.18}$$

Proof of Theorem 14. First, recall that the negation $\mathbf{N}$ is obtained by applying N element-wise, i.e., $[\mathbf{N}(\mathbf{z})]_j = N(z_j)$ for every fuzzy vector $\mathbf{z}$ and for all index $j \in \mathcal{J}$. Furthermore, the following equations hold true since a negation N is a bijection that reverses the partial ordering of the lattice [0, 1]:

$$N\left(\bigvee_{j\in\mathcal{J}} z_j\right) = \bigwedge_{j\in\mathcal{J}} N(z_j) \quad \text{and} \quad N\left(\bigwedge_{j\in\mathcal{J}} z_j\right) = \bigvee_{j\in\mathcal{J}} N(z_j). \tag{A.19}$$

It suffices to show that every max-C FAM corresponds to an unique min-D FAM in order to prove that the mapping ϕ_N is injective. Let $\mathcal{W}$ be a max-C FAM with corresponding synaptic weight matrix W. The following equations hold true for every $i = 1, \ldots, m$:

$$[\mathcal{W}^{\mathbf{N}}(\mathbf{x})]_i = [\mathbf{N}(\mathcal{W}(\mathbf{N}(\mathbf{x})))]_i = N([\mathcal{W}(\mathbf{N}(\mathbf{x}))]_i) \tag{A.20}$$

$$= N\left(\bigvee_{j=1}^{n} C(w_{ij}, N(x_j))\right) = \bigwedge_{j=1}^{n} N(C(w_{ij}, N(x_j))) \tag{A.21}$$

$$= \bigwedge_{j=1}^{n} D(N(w_{ij}), x_j) = \bigwedge_{j=1}^{n} D(m_{ij}, x_j), \tag{A.22}$$

where $m_{ij} = N(w_{ij})$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$. In a similar fashion, one can demonstrate that every min-D FAM corresponds to a max-C FAM, i.e., that ϕ_N is surjective. Thus, ϕ_N constitutes a bijection between the sets of max-C and min-D FAMs.

Note that the duality relationship between C and D was used to derive the last equalities in (A.22). Thus, C and D are dual with respect to the fuzzy negation N. In particular, if C is a t-norm, then D is a t-conorm [21,30,32]. Finally, by Corollary 2, C is a dilation if and only if D is an erosion. □

Proof of Theorem 17. Let $D(m, x) = I_T(N(x), m)$ for every $m, x \in [0, 1]$. The operator $D : [0, 1] \times [0, 1] \to [0, 1]$ is a fuzzy disjunction since D is increasing in both arguments and satisfies the following equations:

$$D(0, 0) = I_T(N(0), 0) = I_T(1, 0) = 0, \tag{A.23}$$

$$D(1, 0) = I_T(N(0), 1) = I_T(1, 1) = 1, \tag{A.24}$$

$$D(0, 1) = I_T(N(1), 0) = I_T(0, 0) = 1, \tag{A.25}$$

$$D(1, 1) = I_T(N(1), 1) = I_T(0, 1) = 1. \tag{A.26}$$

Moreover, $D(m, \cdot)$ represents an erosion since the following equalities hold true for every $m \in [0, 1]$ and $X \subseteq [0, 1]$:

$$D\left(m, \bigwedge X\right) = I_T\left(N\left(\bigwedge X\right), m\right) = I_T\left(\bigvee_{x\in X} N(x), m\right) \tag{A.27}$$

$$= \bigvee\left\{z \in [0, 1] : T\left(z, \bigvee_{x\in X} N(x)\right) \leqslant m\right\} \tag{A.28}$$

$$= \bigvee\left\{z \in [0, 1] : \bigvee_{x\in X} T(z, N(x)) \leqslant m\right\} \tag{A.29}$$

$$= \bigvee\{z \in [0, 1] : T(z, N(x)) \leqslant m, \forall\, x \in X\} \tag{A.30}$$

$$= \bigwedge_{x\in X} \left\{\bigvee\{z \in [0, 1] : T(z, N(x)) \leqslant m\}\right\} \tag{A.31}$$

$$= \bigwedge_{x\in X} I_T(N(x), m) = \bigwedge_{x\in X} D(m, x). \tag{A.32}$$

Here, we used the fact that the t-norm T performs a dilation, i.e., $T(z, \bigvee X) = \bigvee_{x\in X} T(z, x)$ for every $X \subseteq [0, 1]$. This claim follows from the adjunction between T and I_T.

We conclude the proof by comparing the min-D product given by Eq. (31) with the following equations:

$$y_i^{(k)} = \bigwedge_{j=1}^{n} I_T(x_j^{(k)}, m_{ij}) = \bigwedge_{j=1}^{n} D(m_{ij}, N(x_j^{(k)})) \quad \forall i = 1, \ldots, m, \tag{A.33}$$

$$x_j^{(k+1)} = \bigwedge_{i=1}^{m} I_T(y_i^{(k)}, m_{ij}) = \bigwedge_{i=1}^{m} D(m_{ij}, N(y_i^{(k)})) \quad \forall j = 1, \ldots, n. \qquad \square \tag{A.34}$$

Proof of Theorem 18. Let $\mathcal{J} = \{1, \ldots, n\}$ and $\mathcal{I} = \{1, \ldots, m\}$. We derive Eq. (54) as follows:

$$\mathbf{x} = \mathcal{A}(\mathbf{y}) = \bigvee\{\mathbf{u} \in [0,1]^n : \mathcal{W}(\mathbf{u}) \leqslant \mathbf{y}\} \tag{A.35}$$

$$= \bigvee\left\{\mathbf{u} \in [0,1]^n : \bigvee_{j=1}^{n} C(w_{ij}, u_j) \leqslant y_i,\ \forall i \in \mathcal{I}\right\} \tag{A.36}$$

$$= \bigvee\{\mathbf{u} \in [0,1]^n : C(w_{ij}, u_j) \leqslant y_i,\ \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}\} \tag{A.37}$$

$$= \bigvee\{\mathbf{u} \in [0,1]^n : u_j \leqslant D(N(w_{ij}), y_i),\ \forall i \in \mathcal{I} \text{ and } \forall j \in \mathcal{J}\} \tag{A.38}$$

$$= \bigvee\left\{\mathbf{u} \in [0,1]^n : u_j \leqslant \bigwedge_{i=1}^{n} D(m_{ij}, y_i),\ \forall j \in \mathcal{J}\right\} \tag{A.39}$$

$$= \bigvee\{\mathbf{u} \in [0,1]^n : \mathbf{u} \leqslant M^{\mathrm{T}} \bullet \mathbf{y}\} = M^{\mathrm{T}} \bullet \mathbf{y}. \tag{A.40}$$

Here, we assumed that C and D forms an adjunction in the second argument, thus the relation given by Eq. (27) holds.

Now, let us prove Eq. (55). First, recall that $\mathbf{N}$ is a negation. Thus, $\mathbf{x} \leqslant \mathbf{y}$ if and only if $\mathbf{N}(\mathbf{x}) \geqslant \mathbf{N}(\mathbf{y})$, for every fuzzy vectors $\mathbf{x}$ and $\mathbf{y}$. The relationships of duality with respect to adjunction and negation yield the following equalities:

$$\mathbf{x} = \mathcal{B}(\mathbf{y}) = \bigwedge\{\mathbf{u} \in [0,1]^n : \mathcal{M}(\mathbf{u}) \geqslant \mathbf{y}\} = \bigwedge\{\mathbf{u} \in [0,1]^n : \mathcal{W}^{\mathbf{N}}(\mathbf{u}) \geqslant \mathbf{y}\} \tag{A.41}$$

$$= \bigwedge\{\mathbf{u} \in [0,1]^n : \mathbf{N}(\mathcal{W}(\mathbf{N}(\mathbf{u}))) \geqslant \mathbf{y}\} = \bigwedge\{\mathbf{u} \in [0,1]^n : \mathcal{W}(\mathbf{N}(\mathbf{u})) \leqslant \mathbf{N}(\mathbf{y})\} \tag{A.42}$$

$$= \bigwedge\{\mathbf{u} \in [0,1]^n : \mathbf{N}(\mathbf{u}) \leqslant \mathcal{A}(\mathbf{N}(\mathbf{y}))\} = \bigwedge\{\mathbf{u} \in [0,1]^n : \mathbf{u} \geqslant \mathbf{N}(\mathcal{A}(\mathbf{N}(\mathbf{y})))\} \tag{A.43}$$

$$= \mathbf{N}(\mathcal{A}(\mathbf{N}(\mathbf{y}))). \tag{A.44}$$

Note that $\mathcal{B}(\mathbf{y}) = \mathbf{N}(\mathcal{A}(\mathbf{N}(\mathbf{y}))) = \mathcal{A}^{\mathbf{N}}(\mathbf{y})$, i.e., $\mathcal{A}$ is the negation of $\mathcal{B}$ if $\mathcal{M}$ is the negation of $\mathcal{W}$. We would like to point out that the converse also holds true.

We conclude the proof of Theorem 18 as follows. For every index $j \in \mathcal{J}$, we have

$$x_j = [\mathbf{N}(\mathcal{A}(\mathbf{N}(\mathbf{z})))]_i = N\left(\bigwedge_{j=1}^{n} D(m_{ij}, N(z_j))\right) \tag{A.45}$$

$$= \bigvee_{j=1}^{n} N(C(m_{ij}, N(z_j))) = \bigvee_{j=1}^{n} C(w_{ij}, z_j), \tag{A.46}$$

where $w_{ij} = N(m_{ij})$ for every $i \in \mathcal{I}$ and for every $j \in \mathcal{J}$. $\square$

Proof of Theorem 19. Suppose that $C(\cdot, x)$ represents a dilation for every $x \in [0,1]$ and let $X \in [0,1]^{n \times k}$. For an arbitrary subset of fuzzy matrices $S \subseteq [0,1]^{m \times n}$, we will use the symbol $[S]_{ij}$ to denote the set $\{w \in [0,1] : w = w_{ij}, W \in S\}$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, m$. Thus, the ijth entry of $\bigvee S$ equals the supremum of $[S]_{ij}$, i.e., $[\bigvee S]_{ij} = \bigvee [S]_{ij}$. The following equalities reveal that $y_i^\xi = [\mathcal{D}_X(\bigvee S)]_{i\xi} = \bigvee_{W \in S} [\mathcal{D}_X(W)]_{i\xi}$ for every $i = 1, \ldots, m$ and $\xi = 1, \ldots, k$.

$$\left[\mathcal{D}_X\left(\bigvee S\right)\right]_{i\xi} = \left[\left(\bigvee S\right) \circ X\right]_{i\xi} = \bigvee_{j=1}^{n} C\left(\left[\bigvee S\right]_{ij}, x_{j\xi}\right) \tag{A.47}$$

$$= \bigvee_{j=1}^{n} C\left(\bigvee [S]_{ij}, x_{j\xi}\right) = \bigvee_{j=1}^{n}\left[\bigvee_{w \in [S]_{ij}} C(w, x_{j\xi})\right] \tag{A.48}$$

$$= \bigvee_{j=1}^{n}\left[\bigvee_{W \in S} C(w_{ij}, x_{j\xi})\right] = \bigvee_{W \in S}\left[\bigvee_{j=1}^{n} C(w_{ij}, x_{j\xi})\right] \tag{A.49}$$

$$= \bigvee_{W \in S} [W \circ X]_{i\xi} = \bigvee_{W \in S} [\mathcal{D}_X(W)]_{i\xi}. \tag{A.50}$$

Thus, the operator $\mathcal{D}_X$ commutes with the supremum, i.e., $\mathcal{D}_X$ represents a dilation.

Suppose that $\mathcal{D}_X$ is a dilation for every $X \in [0,1]^{m \times n}$. Given $x \in [0,1]$ and $S \subseteq [0,1]$, we will show that $C(\bigvee S, x) = \bigvee_{s \in S} C(s, x)$. Consider constant fuzzy matrices X and W_s with entries $x_{j\xi} = x$ and $w_{ij} = s$, for every $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\xi = 1, \ldots, k$, and $s \in S$. Thus, the following equations hold for every $i = 1, \ldots, m$ and $\xi = 1, \ldots, \xi$:

$$[\mathcal{D}_X(W_s, X)]_{i\xi} = \bigvee_{j=1}^{n} C([W_s]_{ij}, x_{j\xi}) = C(s, x). \tag{A.51}$$

Moreover, since $\mathcal{D}_X$ represents a dilation, we have

$$C\left(\bigvee S, x\right) = \left[D_X\left(\bigvee\{W_s : s \in S\}, X\right)\right]_{i\xi} = \left[\bigvee_{s \in S} D_X(W_s, X)\right]_{i\xi} \tag{A.52}$$

$$= \bigvee_{s \in S} [D_X(W_s, X)]_{i\xi} = \bigvee_{s \in S} C(s, x). \tag{A.53}$$

Thus, $C(\cdot, x)$ represents a dilation for every $x \in [0,1]$. □

Proof of Theorem 20. The proof follows directly from Proposition 6. □

Proof of Theorem 22. Let $\mathcal{I} = \{1, \ldots, m\}$, $\mathcal{J} = \{1, \ldots, n\}$, and $\mathcal{K} = \{1, \ldots, k\}$. By Theorem 20 and the relation given by Eq. (28), we conclude that the following equalities hold true.

$$W = \bigvee\{V \in [0,1]^{m \times n} : V \circ X \leqslant Y\} \tag{A.54}$$

$$= \bigvee\left\{V \in [0,1]^{m \times n} : \bigvee_{j=1}^{n} C(v_{ij}, x_j^\xi) \leqslant y_i^\xi,\ \forall i \in \mathcal{I},\ \forall \xi \in \mathcal{K}\right\} \tag{A.55}$$

$$= \bigvee\{V \in [0,1]^{m \times n} : C(v_{ij}, x_j^\xi) \leqslant y_i^\xi,\ \forall i \in \mathcal{I},\ \forall j \in \mathcal{J},\ \forall \xi \in \mathcal{K}\} \tag{A.56}$$

$$= \bigvee\{V \in [0,1]^{m \times n} : v_{ij} \leqslant J(y_i^\xi, x_j^\xi),\ \forall i \in \mathcal{I},\ \forall j \in \mathcal{J},\ \forall \xi \in \mathcal{K}\} \tag{A.57}$$

$$= \bigvee\left\{V \in [0,1]^{m \times n} : v_{ij} \leqslant \bigwedge_{\xi=1}^{k} J(y_i^\xi, x_j^\xi),\ \forall i \in \mathcal{I},\ \forall j \in \mathcal{J}\right\} \tag{A.58}$$

$$= \bigvee\{V \in [0,1]^{m \times n} : V \leqslant Y \circledast X^{\mathrm{T}}\} = Y \circledast X^{\mathrm{T}}. \tag{A.59}$$

Here, x_j^ξ and y_i^ξ correspond to $x_{j\xi}$ and $y_{i\xi}$, for every $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $\xi \in \mathcal{K}$, respectively. □

Proof of Theorem 24. Let $E_X : [0,1]^{m\times n} \to [0,1]^{m\times k}$ be the erosion defined as follows for every $M \in [0,1]^{m\times n}$:

$$E_X(M) = M \bullet X. \tag{A.60}$$

By Proposition 3, there exists an unique dilation $D_X : [0,1]^{m\times k} \to [0,1]^{m\times n}$ such that (E_X, D_X) forms an adjunction. We will show that $M = D_X(Y)$. Thus, by Proposition 3, Eq. (63) holds true.

Recall that (E_X, D_X) forms an adjunction if and only if (D_X^N, E_X^N) forms an adjunction [16]. Here, the symbols D_X^N and E_X^N denote the negations of D_X and E_X. On one hand, the operator $E_X^N : [0,1]^{m\times n} \to [0,1]^{m\times k}$ satisfies the following equations where $\mathcal{D}_{N(X)}$ is given by Eq. (56) except that $N(X)$ replaces X.

$$E_X^N(M) = N(N(M) \bullet X) = M \circ N(X) = \mathcal{D}_{N(X)}(M). \tag{A.61}$$

By Theorem 22, the adjoint of $\mathcal{D}_{N(X)}$ is the erosion given by

$$\mathcal{E}_{N(X)}(Y) = Y \circledast N(X)^{\mathrm{T}}. \tag{A.62}$$

On the other hand, the adjoint operator is unique. Therefore, $D_X^N(Y) = \mathcal{E}_{N(X)}(Y)$ and the following equations holds true:

$$D_X(Y) = N(D_X^N(N(Y))) = N(N(Y) \circledast N(X)^{\mathrm{T}}) = M. \qquad \square \tag{A.63}$$

Proof of Theorem 25. Recall that the synaptic weight matrix of an FLBAM is given by $M = Y \circ_T X^{\mathrm{T}}$, where "$\circ_T$" denotes a max-$T$ product based on a t-norm T that performs a dilation. The following equalities show that the ijth element of M can be expressed in terms of a reverse fuzzy implication J, for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$. These equalities were derived using the commutativity of the t-norm (which represents a particular fuzzy conjunction) and the relations given by Eqs. (25), (26), and (17), respectively.

$$m_{ij} = \bigvee_{\xi=1}^{k} T(y_i^\xi, x_j^\xi) = \bigvee_{\xi=1}^{k} T(x_j^\xi, y_i^\xi) \tag{A.64}$$

$$= \bigvee_{\xi=1}^{k} N(S(N(x_j^\xi), N(y_i^\xi))) = \bigvee_{\xi=1}^{k} N(I(x_j^\xi, N(y_i^\xi))) \tag{A.65}$$

$$= \bigvee_{\xi=1}^{k} N(J(N(y_i^\xi), x_j^\xi)) = N\left(\bigwedge_{\xi=1}^{k} J(N(y_i^\xi), x_j^\xi)\right). \tag{A.66}$$

Hence, the synaptic weight matrix of an FLBAM model satisfies Eq. (64). Note that the reverse fuzzy implication J and the t-norm T are such that $T(x, y) = T(y, x) = N(J(N(y), x))$. Thus, the following equation holds true for every $x, y \in [0,1]$:

$$J(y, x) = N(T(x, N(y))). \tag{A.67}$$

Let C denote the negation of the fuzzy disjunction that appears in Eq. (51). The following sequence of inequalities reveals that C and J satisfy the relation given in Eq. (28) for every $x, y, w \in [0,1]$. Thus, C and J are adjoint operators.

$$w \leqslant J(y, x) \Leftrightarrow w \leqslant N(T(x, N(y))) \Leftrightarrow T(x, N(y)) \leqslant N(w) \tag{A.68}$$

$$\Leftrightarrow N(y) \leqslant I_T(x, N(w)) \Leftrightarrow N(y) \leqslant D(N(w), N(x)) \tag{A.69}$$

$$\Leftrightarrow N(D(N(w), N(x))) \leqslant y \Leftrightarrow C(w, x) \leqslant y. \tag{A.70}$$

Here, we used the facts that T and I_T are adjoint operators (cf. Eq. (44)) and that I_T and D satisfy $D(m, x) = I_T(N(x), m)$ for every $m, x \in [0,1]$. $\square$

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