Chapter 32

Fuzzy Associative Memories and Their Relationship to Mathematical Morphology

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32.1 Introduction

Fuzzy associative memories (FAMs) belong to the class of *fuzzy neural networks* (FNNs). A FNN is an *artificial neural network* (ANN) whose input patterns, output patterns, and/or connection weights are fuzzy-valued [19, 11].

Research on FAM models originated in the early 1990's with the advent of Kosko's FAM [35, 37]. Like many other associative memory models, Kosko's FAM consists of a single-layer feedforward FNN that stores the fuzzy rule "If x is X_k then y is Y_k" using a fuzzy Hebbian learning rule in terms of max-min or max-product compositions for the synthesis of its weight matrix W.

Despite successful applications of Kosko's FAMs to problems such as backing up a truck and trailer [35], target tracking [37], and voice cell control in ATM networks [44], Kosko's FAM suffers from an extremely low storage capacity of one rule per FAM matrix. Therefore, Kosko's overall fuzzy system comprises several FAM matrices. Given a fuzzy input, the FAM matrices generate fuzzy outputs which are then combined to yield the final result. To overcome the original FAMs severe limitations in storage capacity, several researchers have developed improved FAM versions that are capable of storing multiple pairs of fuzzy patterns [30, 9, 14, 38, 12]. For example, Chung and Lee generalized Kosko's model by proposing a max-t composition for the synthesis of a FAM matrix. Chung and Lee showed that all fuzzy rules can be perfectly recalled by means of a single FAM matrix using max-t composition provided that the input patterns satisfy certain orthogonality conditions [14]. Junbo *et al.* had previously presented an improved learning algorithm for Kosko's max-min FAM model [29, 30]. Liu modified the Junbo's FAM *et al.* by adding a threshold activation function to each node of the network [38].

We recently established *implicative fuzzy associative memories* (IFAMs) [73, 71], a class of associative memories that grew out of *morphological associative memories* (MAMs) [51, 70, 64]. One particular IFAM model can be viewed as an improved version of Liu's FAM [71]. MAMs belong to the class of morphological neural networks (MNNs) [50, 54]. This class of artificial neural networks is called *morphological* because each node performs a morphological operation [57, 58, 55, 28]. Theory and applications of binary and gray-scale MAMs have been developed since late 1990's [51, 64, 65, 70]. For example, one can store as many patterns as desired in an auto-associative MAM [51, 63, 65]. In particular, for binary patterns of length n, the binary auto-associative MAM exhibits an absolute storage capacity of 2^n which either equals or slightly exceeds the storage capacity of the *quantum associative memory* of Ventura and Martinez [74]. Applications of MAMs include face localization, robot vision, hyper-spectral image analysis, and some general classification problems [48, 22, 70, 67, 68].

This article demonstrates that the IFAM model as well as all other FAM models that we mentioned above can be embedded into the general class of *fuzzy morphological associative memories* (FMAMs). *Fuzzy logical bidirectional associative memories* (FLBAMs), which were introduced by Bělohlávek [7], can also be considered a subclass of FMAMs. Although a general framework for FMAMs has yet to appear in the literature, we believe that the class of FMAMs should be firmly rooted in fuzzy mathematical morphology and thus each node of an FMAM should execute a fuzzy morphological operation [17, 66, 69]. In general, the input, output, and synaptic weights of FMAMs are fuzzy valued. Recall that fuzzy sets represent special cases of information granules. Thus, FMAMs can be considered special cases of *granular associative memories*, a broad class of AMs which has yet to be investigated.

The chapter is organized as follows. First, we present some background information and motivation for our research. After providing some general concepts of neural associative memories, fuzzy set theory, and mathematical morphology, we discuss the types of artificial neurons that occur in FAM models. Section 32.5 provides an overview of Kosko's FAM and its generalizations, including the FAM model of Chung and Lee. In Section 32.6, we review variations of Kosko's max-min FAM, in particular the models of Junbo *et al.* and Liu in conjunction with their respective learning strategies. In Section 32.7, we present the most important results on IFAMs and FLBAMs. Section 32.8 compares the performances of different FAM models by means of an example concerning the storage capacity and noise tolerance. Furthermore, an application to a problem of prediction is presented. We conclude the article with some suggestions for further research concerning fuzzy and granular MAM models.

32.2 Some Background Information and Motivation

32.2.1 Associative Memories

Associative memories (AMs) allow for the storage of pattern associations and the retrieval of the desired output pattern upon presentation of a possibly noisy or incomplete version of an input pattern. Mathematically speaking, the associative memory design problem can be stated as follows: Given a finite set of desired associations $\left\{\left(\mathbf{x}^\xi, \mathbf{y}^\xi\right) : \xi = 1, \ldots, k\right\}$, determine a mapping G such that $G(\mathbf{x}^\xi) = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$. Furthermore, the mapping G should be endowed with a certain tolerance with respect to noise, i.e. $G(\tilde{\mathbf{x}}^\xi)$ should equal $\mathbf{y}^\xi$ for noisy or incomplete versions $\tilde{\mathbf{x}}^\xi$ of $\mathbf{x}^\xi$. In the context of *Granular Computing* (GC), the input and the output patterns are information granules [6].

The set of associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$ is called *fundamental memory set* and each association $(\mathbf{x}^\xi, \mathbf{y}^\xi)$ in this set is called a *fundamental memory* [25]. We speak of an *autoassociative memory* when the fundamental memory set is of the form $\{(\mathbf{x}^\xi, \mathbf{x}^\xi) : \xi = 1, \ldots, k\}$. The memory is said to be *heteroassociative* if the output $\mathbf{y}^\xi$ is different from the input $\mathbf{x}^\xi$. One of the most common problem associated with the design of an AM is the creation of false or spurious memories. A *spurious memory* is a memory association that does not belong to the fundamental memory set, i.e. it was unintentionally stored in the memory.

The process of determining G is called *recording phase* and the mapping G is called *associative mapping*. We speak of a *neural associative memory* when the associative mapping G is described by an artificial neural network. In particular, we have a *fuzzy (neural) associative memory* (FAM) if the associative mapping G is given by a fuzzy neural network and the patterns $\mathbf{x}^\xi$ and $\mathbf{y}^\xi$ are fuzzy sets for every $\xi = 1, \ldots, k$.

32.2.2 Morphological Neural Networks

In this paper, we are mainly concerned with fuzzy associative memories. As we shall point out during the course of this paper, many models of fuzzy associative memories can be classified as *fuzzy morphological associative memories* (FMAMs) which in turn belong to the class of *morphological neural networks* (MNNs) [50, 67]. The name "morphological neural networks" was coined because MNNs perform operations of mathematical morphology at every node.

Many models of morphological neural networks are implicitly rooted in the mathematical structure $(\mathbb{R}_{\pm\infty}, \vee, \wedge, +, +')$ which represents a *bounded lattice ordered group* (blog) [15, 16, 20, 50, 51, 64, 4, 3, 70]. The symbols “$\vee$” and “$\wedge$" represent the maximum and the minimum operation. The operations “$+$” and “$+'$” act like the usual sum operation and are identical on $\mathbb{R}_{\pm\infty}$ with the following exceptions:

$$(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty \quad \text{and} \quad (-\infty) +' (+\infty) = (+\infty) +' (-\infty) = +\infty \,. \tag{32.1}$$

In practice, the inputs, outputs, and synaptic weights of a MNN have values in $\mathbb{R}$ where the operations "$+$" and "$+'$" coincide.

In most cases, models of morphological neural networks, including MAMs, are defined in terms of certain matrix products known as the max product and the min product. Specifically, for an $m \times p$ matrix A and a $p \times n$ matrix B with entries from $\mathbb{R}_{\pm\infty}$, the matrix $C = A \boxdot B$, also called the *max product* of A and B, and the matrix $D = A \boxtimes B$, also called the *min product* of A and B, are defined by

$$c_{ij} = \bigvee_{k=1}^{p} (a_{ik} + b_{kj}) \quad \text{and} \quad d_{ij} = \bigwedge_{k=1}^{p} (a_{ik} +' b_{kj}) \,. \tag{32.2}$$

Let us consider an arbitrary neuron in a MNN defined on the blog $(\mathbb{R}_{\pm\infty}, \vee, \wedge, +, +')$. Suppose that the inputs are given by a vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and let $\mathbf{w} = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ denote the vector of corresponding synaptic strengths. The accumulative effect of the inputs and the synaptic weights in a simple morphological neuron is given by either one of the following equations:

$$\tau(\mathbf{x}) = \mathbf{w}^T \boxdot \mathbf{x} = \bigvee_{i=1}^{n} (w_i + x_i) \quad \text{or} \quad \tau(\mathbf{x}) = \mathbf{w}^T \boxtimes \mathbf{x} = \bigwedge_{i=1}^{n} (w_i +' x_i) \,. \tag{32.3}$$

Since the Equations in 32.3 are non-linear, researchers in the area of MNNs generally refrain from using a possibly non-linear activation function. It should be mentioned that Koch and Poggio make a strong case for multiplying with synapses [33], i.e. for $w_i \cdot x_i$ instead of $w_i + x_i$ or $w_i +' x_i$ as written in the Equations in 32.3. However, multiplication could have been used just as well in these equations because the blog $(\mathbb{R}_{\pm\infty}, \vee, \wedge, +, +')$ is isomorphic to the blog $([0, \infty], \vee, \wedge, \cdot, \cdot')$ under the isomorphism $\phi(x) = e^x$ (we use the conventions $e^{-\infty} = 0$ and $e^{\infty} = \infty$). Here, the multiplications "$\cdot$" and "$\cdot'$" generally behave as one would expect with the following exceptions:

$$0 \cdot \infty = \infty \cdot 0 = 0 \quad \text{and} \quad 0 \cdot' \infty = \infty \cdot' 0 = +\infty \,. \tag{32.4}$$

Note that in the multiplicative blog $([0, \infty], \vee, \wedge, \cdot, \cdot')$, the Equations 32.3 become respectively

$$\tau(\mathbf{x}) = \bigvee_{i=1}^{n} (w_i \cdot x_i) \qquad \text{and} \quad \tau(\mathbf{x}) = \bigwedge_{i=1}^{n} (w_i \cdot' x_i) \;. \tag{32.5}$$

Despite the facts that weights are generally considered to be positive quantities and that morphological neural networks can also be developed in the multiplicative blog $([0, \infty], \vee, \wedge, \cdot, \cdot')$, computational reasons have generally led researchers to work in the additive blog $(\mathbb{R}_{\pm\infty}, \vee, \wedge, +, +')$ [71]. In fact, it is sufficient to consider the blog $(\mathbb{Z}_{\pm\infty}, \vee, \wedge, +, +')$. Moreover, the Equations in 32.3 are closely linked to the operations of gray-scale dilation and erosion in classical mathematical morphology [61, 62]. These equations can also be interpreted as non-linear operations (image-template products) in the mathematical structure of image algebra [52, 53]. In fact, existing formulations of traditional neural network models in image algebra induced researchers such as Ritter, Davidson, Gader, and Sussner to formulate models of morphological neural networks [16, 20, 50, 51, 64, 70].

Thus, the motivation for establishing MNNs can be found in mathematics instead of biology. Nevertheless, recent research results by Yu, Giese, and Poggio have revealed that the maximum operation that lies at the core of morphological neurons is neurobiologically plausible [76]. In addition to its potential involvement in a variety of cortical processes [21, 49, 23], the maximum operation can be implemented by simple, neurophysiologically plausible circuits. Previously, prominent neuroscientists such as Gurney, Segev, and Shepherd had already shown that simple logical functions can be modeled by local interactions in dendritic trees [24, 56, 59].

For fuzzy inputs $\mathbf{x} \in [0, 1]^n$ and fuzzy weights $\mathbf{w} \in [0, 1]^n$, the identity on the left-hand side of the first Equation in 32.5 describes a fuzzy morphological neuron because it corresponds to an operation of dilation in fuzzy mathematical morphology [43, 66]. Note that the operation of multiplication represents a special case of fuzzy conjunction. At this point, we prefer not to go into the details of fuzzy morphological neural networks, in particular FMAMs. We would only like to point out that the lattice ordering of fuzzy sets has been paramount to the development of FMAMs. Thus, the lattice ordering of other information granules may turn out to be useful for the development of other granular associative memories.

32.2.3 Information Granules and Their Inherent Lattice Ordering

Granular computing is based on the observation that we are only able to process the incoming flow of information by means of a process of abstraction which involves representing information in the form of aggregates or information granules [77, 78, 46, 6, 75]. Thus, granulation of information occurs in everyday life whenever we form collections of entities that are arranged together due to their similarity, functional adjacency, indistinguishability, coherency or alike.

These considerations indicate that set theory serves as a suitable conceptual and algorithmic framework for granular computing. Since a given class of sets is equipped with a partial ordering given by set inclusion we believe that granular computing is closely related to lattice theory. More formally speaking, information granules include fuzzy sets, rough sets, intervals, shadowed sets, and probabilistic sets. Observe that all of these classes of constructs are endowed with an inherent lattice ordering.

In this paper, we focus our attention on the class of fuzzy sets $[0,1]^{\mathbf{X}}$, i.e. the set of functions from a universe $\mathbf{X}$ to $[0,1]$, because we are not aware of any significant research results concerning other classes of information granules in the context of associative memories. However, we believe that their inherent lattice structure will provide for the means to establish associative memories that store associations of other types of information granules.

Thus, this paper is concerned with fuzzy associative memories. More precisely, we describe a relationship between fuzzy associative memories and mathematical morphology that is ultimately due to the complete lattice structure of $[0,1]^{\mathbf{X}}$.

32.3 Relevant Concepts of Fuzzy Set Theory and Mathematical Morphology

32.3.1 The Complete Lattice Framework of Mathematical Morphology

In this article, we will establish a relationship between FAMs and mathematical morphology that is due to the fact that the neurons of most FAM models perform morphological operations.

Mathematical morphology (MM) is a theory which is concerned with the processing and analysis of objects using operators and functions based on topological and geometrical concepts [28, 61]. This theory was introduced by Matheron and Serra in the early 1960's as a tool for the analysis of binary images [42, 57]. During the last decades, it has acquired a special status within the field of image processing, pattern recognition, and computer vision. Applications of MM include image segmentation and reconstruction [32], feature detection [60], and signal decomposition [10].

The most general mathematical framework in which MM can be conducted is given by complete lattices [55, 28]. A *complete lattice* is defined as a partially ordered set $\mathbb{L}$ in which every (finite or infinite) subset has an infimum and a supremum in $\mathbb{L}$ [8]. For any $Y \subseteq \mathbb{L}$, the infimum of Y is denoted by the symbol $\bigwedge Y$. Alternatively, we write $\bigwedge_{j \in J} y_j$ instead of $\bigwedge Y$ if $Y = \{y_j : j \in J\}$ for some index set J. Similar notations are used to denote the supremum of Y. The interval $[0,1]$ represents an example of a complete lattice. The class of *fuzzy sets* $[0,1]^{\mathbf{X}}$, i.e. the set of functions from a universe $\mathbf{X}$ to $[0,1]$, inherits the complete lattice structure of the unit interval $[0,1]$.

The two basic operators of MM are *erosion* and *dilation* [58, 28]. An *erosion* is a mapping ε from a complete lattice $\mathbb{L}$ to a complete lattice $\mathbb{M}$ that commutes with the infimum operation. In other words, the operator ε represents an erosion if and only if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\varepsilon\left(\bigwedge Y\right) = \bigwedge_{y \in Y} \varepsilon(y)\,. \tag{32.6}$$

Similarly, an operator $\delta : \mathbb{L} \rightarrow \mathbb{M}$ that commutes with the supremum operation is called a *dilation*.

In other words, the operator δ represents a dilation if and only if the following equality holds for every subset $Y \subseteq \mathbb{L}$:

$$\delta\left(\bigvee Y\right) = \bigvee_{y \in Y} \delta(y)\,. \tag{32.7}$$

Apart from erosions and dilations, we will also consider the elementary operators anti-erosion and anti-dilation that are defined as follows [5, 28]. An operator $\bar{\varepsilon}$ is called an anti-erosion if and only if the first Equality in 32.8 holds for every $Y \subseteq \mathbb{L}$ and an operator $\bar{\delta}$ is called an anti-dilation if and only if the second Equality in 32.8 holds for every subset $Y \subseteq \mathbb{L}$.

$$\bar{\varepsilon}\left(\bigwedge Y\right) = \bigvee_{y \in Y} \bar{\varepsilon}(y) \quad \text{and} \quad \bar{\delta}\left(\bigvee Y\right) = \bigwedge_{y \in Y} \bar{\delta}(y)\,. \tag{32.8}$$

Erosions, dilations, anti-erosions, and anti-dilations exemplify the concept of morphological operator. Unfortunately, a rigorous mathematical definition of a morphological operator does not exist. According to Heijmans, any attempt to find a formal definition of a morphological operator would either be too restrictive or too general [28]. For the purposes of our article, it is sufficient to know that the four elementary operators erosion, dilation, anti-erosion, and anti-dilation are generally considered to be morphological ones [5].

If one of the four operators ε, δ, $\bar{\varepsilon}$, or $\bar{\delta}$ that we defined above is a mapping $[0,1]^{\mathbf{X}} \rightarrow [0,1]^{\mathbf{Y}}$ for some sets $\mathbf{X}$ and $\mathbf{Y}$ then we speak of a *fuzzy erosion*, a *fuzzy dilation*, a *fuzzy anti-erosion*, or a *fuzzy anti-dilation* [43, 17, 66]. The operators of erosion and dilation are often linked in terms of either one of the following relationships of duality: adjunction or negation.

Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices. Consider two arbitrary operators $\delta : \mathbb{L} \rightarrow \mathbb{M}$ and $\varepsilon : \mathbb{M} \rightarrow \mathbb{L}$. We say that (ε, δ) is an *adjunction* from $\mathbb{L}$ to $\mathbb{M}$ if we have

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y) \quad \forall\, x \in \mathbb{L}\,, y \in \mathbb{M}\,. \tag{32.9}$$

Adjunction constitutes a duality between erosions and dilations since they form a bijection which reverses the order relation in the complete lattice [28]. Moreover, if (ε, δ) is an adjunction, then δ is a dilation and ε is an erosion.

A second type of duality is based on *negation*. We define a *negation* on a complete lattice $\mathbb{L}$ as an involutive bijection $\nu_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$ which reverses the partial ordering. In the special case where $\mathbb{L} = [0, 1]$, we speak of a *fuzzy negation*. Examples of fuzzy negations include the following unary operators.

$$N_S(x) = 1 - x \quad \text{and} \quad N_D(x) = \frac{1-x}{1+px} \;\text{ for }\; p > -1\,. \tag{32.10}$$

Suppose that N is an arbitrary fuzzy negation and that $\mathbf{x} \in [0,1]^n$ and $W \in [0,1]^{m \times n}$. For simplicity, $N(\mathbf{x})$ denotes the component-wise fuzzy negation of the vector $\mathbf{x}$ and $N(W)$ denotes the entry-wise fuzzy negation of the matrix W.

Let Ψ be an operator mapping a complete lattice $\mathbb{L}$ into a complete lattice $\mathbb{M}$ and let $\nu_{\mathbb{L}}$ and $\nu_{\mathbb{M}}$ be negations on $\mathbb{L}$ and $\mathbb{M}$, respectively. The operator Ψ^ν given by

$$\Psi^\nu(x) = \nu_{\mathbb{M}}\left(\Psi\left(\nu_{\mathbb{L}}(x)\right)\right) \quad \forall\, x \in \mathbb{L}, \tag{32.11}$$

is called the *negation* or the *dual* of Ψ with respect to $\nu_{\mathbb{L}}$ and $\nu_{\mathbb{M}}$. The negation of an erosion is a dilation, and vice versa [28]. The preceding observations clarify that there is a unique erosion that can be associated with a certain dilation and vice versa in terms of either negation or adjunction.

An erosion, a dilation respectively, is usually associated with a *structuring element* (SE) which is used to probe a given image [57, 61]. In the fuzzy setting, the image $\mathbf{a}$ and the SE $\mathbf{s}$ are given by fuzzy sets [43, 17, 66]. For a fixed SE $\mathbf{s}$, a fuzzy dilation $\mathcal{D}(\cdot, \mathbf{s})$ is usually defined in terms of a supremum of fuzzy conjunctions C, where C commutes with the supremum operator in the second argument [17, 41, 66]. Similarly, a fuzzy erosion $\mathcal{E}(\cdot, \mathbf{s})$ can be defined in terms of an infimum of fuzzy disjunctions D or an infimum of fuzzy implications I, where D or I commutes with the infimum operator in the second argument.

If an ANN performs a (fuzzy) morphological operation at each node, we speak of a *(fuzzy)* *morphological neural network*. The neurons of an ANN of this type are called *(fuzzy) morphological neurons*. In particular, (fuzzy) neurons that perform dilations, erosions, anti-dilations, or anti-erosions are (fuzzy) morphological neurons. An AM that belongs to the class of fuzzy morphological neural networks is called a *fuzzy morphological associative memory* (FMAM).

32.3.2 Some Basic Operators of Fuzzy Logic

This article will show that - in their most general form - the neurons of FAMs are given in terms of a fuzzy conjunction, a fuzzy disjunction, or a fuzzy implication.

We define a *fuzzy conjunction* as an increasing mapping $C : [0,1] \times [0,1] \to [0,1]$ that satisfies $C(0,0) = C(0,1) = C(1,0) = 0$ and $C(1,1) = 1$. The minimum operator and the product obviously yield simple examples. In particular, a commutative and associative fuzzy conjunction $T : [0,1] \times [0,1] \to [0,1]$ that satisfies $T(x,1) = x$ for every $x \in [0,1]$ is called *triangular norm* or simply *t-norm* [47]. The fuzzy conjunctions C_M, C_P, and C_L below are examples of t-norms.

$$C_M(x,y) = x \wedge y\,, \tag{32.12}$$

$$C_P(x,y) = x \cdot y\,, \tag{32.13}$$

$$C_L(x,y) = 0 \vee (x+y-1)\,. \tag{32.14}$$

A *fuzzy disjunction* is an increasing mapping $D : [0,1] \times [0,1] \to [0,1]$ that satisfies $D(0,0) = 0$ and $D(0,1) = D(1,0) = D(1,1) = 1$. In particular, a commutative and associative fuzzy disjunction $S : [0,1] \times [0,1] \to [0,1]$ that satisfies $S(1,x) = x$ for every $x \in [0,1]$ is called *triangular co-norm*, for short *s-norm*. The following operators represent s-norms:

$$D_M(x,y) = x \vee y\,, \tag{32.15}$$

$$D_P(x,y) = x + y - x \cdot y\,, \tag{32.16}$$

$$D_L(x,y) = 1 \wedge (x+y)\,. \tag{32.17}$$

We would like to point out that in the literature of fuzzy logic, one often does not work with the overall class of fuzzy conjunctions and fuzzy disjunction but rather with the restricted class of t-norms and s-norms [47]. In particular, the FAM models presented in the next sections are based on t-norms and s-norms except for the FLBAM and the general FMAMs.

An operator $I : [0,1] \times [0,1] \to [0,1]$ that is decreasing in the first argument and that is increasing in the second argument is called a *fuzzy implication* if I extends the usual crisp implication on $\{0,1\} \times \{0,1\}$, i.e. $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. Some particular fuzzy implications, that were introduced by Gödel, Goguen, and Lukasiewicz can be found below [47, 17].

$$I_M(x,y) = \begin{cases} 1, & x \leq y \\ y, & x > y \end{cases}, \tag{32.18}$$

$$I_P(x,y) = \begin{cases} 1, & x \leq y \\ y/x, & x > y \end{cases}, \tag{32.19}$$

$$I_L(x,y) = 1 \wedge (y - x + 1)\,. \tag{32.20}$$

A fuzzy conjunction C can be associated with a fuzzy disjunction D or with a fuzzy implication I by means of a relationship of duality which can be either negation or adjunction. Specifically, we say that a fuzzy conjunction C and a fuzzy disjunction D are *dual operators with respect to a fuzzy negation* N if and only if the following equation holds for every $x, y \in [0,1]$:

$$C(x,y) = N\left(D(N(x), N(y)\right)\,. \tag{32.21}$$

In other words, we have that $C(x,\cdot) = D^N(N(x),\cdot)$ for all $x \in [0,1]$ or, equivalently, $C(\cdot,y) = D^N(\cdot,N(y))$ for all $y \in [0,1]$.

The following implication holds for fuzzy operators C and D that are dual with respect to N: If C is a dilation for every $x \in [0,1]$ then D is an erosion for every $x \in [0,1]$ and vice versa [28]. For example, note that the pairs (C_M, D_M), (C_P, D_P), and (C_L, D_L) are dual operators with respect to the standard fuzzy negation N_S. The dual operator of a (continuous) t-norm with respect to N_S is a (continuous) s-norm [47].

In this article, we will also consider the duality relationship of adjunction between a fuzzy conjunction C and a fuzzy implication I. We simply say that C and I form an adjunction if and only if $C(x, \cdot)$ and $I(x, \cdot)$ form an adjunction for every $x \in [0, 1]$. In this case, we also call C and I *adjoint* operators and we have that $C(z, \cdot)$ is a dilation and $I(z, \cdot)$ is an erosion for every $z \in [0, 1]$ [17]. Examples of adjunctions are given by the pairs (C_M, I_M), (C_P, I_P), and (C_L, I_L).

The fuzzy operations C, D, and I can be combined with the maximum or the minimum operation to yield the following matrix products. For $A \in [0, 1]^{m \times p}$ and $B \in [0, 1]^{p \times n}$, we define the max-$C$ product $C = A \circ B$ as follows:

$$c_{ij} = \bigvee_{k=1}^{p} C(a_{ik}, b_{kj}) \quad \forall\, i = 1, \ldots, m,\ j = 1, \ldots, n\,. \tag{32.22}$$

Similarly, the min-D product $D = A \bullet B$ and the min-I product $E = A \circledast B$ are given by the following equations:

$$d_{ij} = \bigwedge_{k=1}^{p} D(a_{ik}, b_{kj}) \quad \forall\, i = 1, \ldots, m,\ j = 1, \ldots, n\,, \tag{32.23}$$

$$e_{ij} = \bigwedge_{k=1}^{p} I(b_{kj}, a_{ik}) \quad \forall\, i = 1, \ldots, m,\ j = 1, \ldots, n\,. \tag{32.24}$$

Subscripts of the product symbols $\circ$, $\bullet$, or $\circledast$ indicate the type of fuzzy operators used in Equations 32.22, 32.23, or 32.24. For example, the symbol $\circ_M$ stands for the max-C product where the fuzzy conjunction C in Equation 32.22 is given by C_M.

32.4 Types of Neurons Used in Fuzzy Associative Memory Models

This section describes the most important types of fuzzy neurons that occur in FAM models. These models of artificial neurons can be formulated in terms of the max-C, min-D, and min-I matrix products that we introduced in Section 32.3.2.

Let us consider an arbitrary model of an artificial neuron. The symbol $\mathbf{x} = [x_1, \ldots, x_n]^T$ denotes the fuzzy input vector and y denotes the fuzzy output. The weights $w_i \in [0,1]$ of the neuron form a vector $\mathbf{w} = [w_1, \ldots, w_n]^T$. We use θ to denote the bias. A model without bias is obtained by setting $\theta = 0$ in Equations 32.26 and 32.27 or by setting $\theta = 1$ in Equations 32.28 and 32.29.

32.4.1 The Max-C and the Min-I Neuron

One of the most general classes of fuzzy neurons was introduced by Pedrycz [45] in the early 90's. The neurons of this class are called *aggregative logic neurons* since they realize an aggregation of the inputs and synaptic weights. We are particularly interested in the *OR-neuron* described by the following equation, where S is a s-norm and T is a t-norm.

$$y = \mathop{S}_{j=1}^{n} T\left(w_j, x_j\right) . \tag{32.25}$$

Let us adapt Pedrycz's original definition by introducing a bias term and by substituting the t-norm with a more general operation of fuzzy conjunction. We obtain a generalized OR-neuron or *S-C neuron*.

$$y = \left[\mathop{S}_{j=1}^{n} C\left(w_j, x_j\right)\right] \mathrm{s}\, \theta \, . \tag{32.26}$$

We refrained from replacing the s-norm by a fuzzy disjunction since associativity and commutativity are required in a neural model. If S equals the maximum operation, we obtain the *max-C*

neuron that is given by the following equation where $\mathbf{w}^T$ represents the transpose of $\mathbf{w}$.

$$y = \left[\bigvee_{j=1}^{n} C\left(w_j, x_j\right) \right] \vee \theta = \left(\mathbf{w}^T \circ \mathbf{x}\right) \vee \theta \,. \tag{32.27}$$

Particular choices of fuzzy conjunctions yield particular max-C neurons. Given a particular max-C neuron, we will indicate the underlying type of fuzzy conjunction by means of a subscript. For example, max-C_M will denote the neuron that is based on the minimum fuzzy conjunction. A similar notation will be applied to describe the min-I neuron and the min-D that will be introduced in Section 32.4.2. We define the *min-I neuron* by means of the following equation:

$$y = \left[\bigwedge_{j=1}^{n} I\left(x_j, w_j\right) \right] \wedge \theta = \left(\mathbf{w}^T \circledast \mathbf{x}\right) \wedge \theta \,. \tag{32.28}$$

We are particularly interested in min-I_T neurons where I_T denotes a fuzzy implication that forms an adjunction together with a t-norm T. This type of neuron occurs in the FLBAM model [7].

To our knowledge, the max-C neuron represents the most widely used model of fuzzy neuron in FAMs. The FLBAM of Bělohlávek, which consists of min-I neurons, represents an exception to this rule. For example, Kosko's FAM employs max-C_M or max-C_P neurons. Junbo's FAM and the FAM model of Liu are also equipped with max-C_M neurons. The generalized FAM of Chung and Lee as well as the IFAM models employ max-T neurons, where T is a t-norm.

Note that we may speak of a *max-C morphological neuron* if and only if $C(x, \cdot)$ is a dilation for every $x \in [0, 1]$ [17]. Examples of max-C morphological neurons include max-C_M, max-C_P, and max-C_L neurons.

32.4.2 The Min-D Neuron: A Dual Model of the Max-C Neuron

Consider the neural model that is described in terms of the equation below.

$$y = \left[\bigwedge_{j=1}^{n} D\left(w_j, x_j\right) \right] \wedge \theta = (\mathbf{w}^T \bullet \mathbf{x}) \wedge \theta \,. \tag{32.29}$$

We refer to neurons of this type as *min-D neurons*. For example, dual IFAMs are equipped with min-D neurons [71].

Suppose that C and D are dual operators with respect to N. In this case, the max-C neuron and the min-D neuron are *dual with respect to* N in the following sense. Let $\mathcal{W}$ denote the function computed by the max-C neuron, i.e. $\mathcal{W}(\mathbf{x}) = \left(\mathbf{w}^T \circ \mathbf{x}\right) \vee \theta$ for all $\mathbf{x} \in [0,1]^n$. If m_j denotes $N(w_j)$ and ϑ denotes $N(\theta)$ then we obtain the negation of $\mathcal{W}$ with respect to N as follows.

$$N(\mathcal{W}(N(\mathbf{x}))) = N\left(\left[\bigvee_{j=1}^{n} C(w_j, N(x_j)) \right] \vee \theta \right) \tag{32.30}$$

$$= \left[\bigwedge_{j=1}^{n} N(C(w_j, N(x_j)) \right] \wedge N(\theta) = \left[\bigwedge_{j=1}^{n} D(m_j, x_j) \right] \wedge \vartheta \,. \tag{32.31}$$

Note that the dual of a max-C morphological neuron with respect to a fuzzy negation N is a min-D morphological neuron that performs an erosion.

32.5 Kosko's Fuzzy Associative Memory and Generalizations

Kosko's FAMs constitute one of the earliest attempts to develop neural AM models based on fuzzy set theory. These models were introduced in the early 1990's and are usually referred as *max-min FAM* and *max-product FAM* [37]. Later, Chung and Lee introduced generalizations of Kosko's models that are known as *generalized fuzzy associative memories* (GFAMs) [14]. The models of Kosko and Chung $\&$ Lee share the same network topology and Hebbian learning rules with the Linear Associative Memory [2, 26, 34]. Thus, these FAM models exhibit a large amount of crosstalk if the inputs do not satisfy a certain orthonormality condition.

32.5.1 The Max-Min and the Max-Product Fuzzy Associative Memories

The max-min and the max-product FAM are both single layer feedforward ANNs. The max-min FAM is equipped with max-C_M fuzzy neurons while the max-product FAM is equipped with max-C_P fuzzy neurons. Thus, both models belong to the class of fuzzy morphological associative memories. Kosko's original definitions do not include bias terms. Consequently, if $W \in [0,1]^{m \times n}$ is the synaptic weight matrix of a max-min FAM and if $\mathbf{x} \in [0,1]^n$ is the input pattern, then the output pattern $\mathbf{y} \in [0,1]^m$ is computed as follows (cf. Equation 32.22).

$$\mathbf{y} = W \circ_M \mathbf{x} \,. \tag{32.32}$$

Similarly, the max-product FAM produces the output $\mathbf{y} = W \circ_P \mathbf{x}$. Note that both versions of Kosko's FAM perform dilations at each node (and overall). Thus, Kosko's models belong to the class FMAMs.

Consider a set of fundamental memories $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$. The learning rule used to store the fundamental memory set in a max-min FAM is called *correlation-minimum encoding*. In this learning rule, the synaptic weight matrix is given by the following equation

$$W = Y \circ_M X^T \,, \tag{32.33}$$

where $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n \times k}$ and $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{m \times k}$. In a similar fashion, the weight matrix of the max-product FAM is synthesized by setting $W = Y \circ_P X^T$. We speak of *correlation-product encoding* in this case.

Both the correlation-minimum and the correlation-product encoding are based on the Hebb's postulate which states that the synaptic weight change depends on the input as well as the output activation [27]. Unfortunately, Hebbian learning entails an extremely low storage capacity of one input-output pair per FAM matrix in Kosko's models. More precisely, Kosko only succeeded in showing the following proposition concerning the recall of patterns by a max-min and a max-product FAM [37].

Proposition 1. *Suppose that a single fundamental memory pair* $(\mathbf{x}^1, \mathbf{y}^1)$ *was stored in a max-min FAM by means of the correlation-minimum encoding scheme, then* $W \circ_M \mathbf{x}^1 = \mathbf{y}^1$ *if and only if* $\bigvee_{j=1}^n x_j^1 \leq \bigvee_{i=1}^m y_i^1$. *Moreover, we have* $W \circ_M \mathbf{x} \leq \mathbf{y}^1$ *for every* $\mathbf{x} \in [0,1]^n$.

Similarly, if a single fundamental memory pair $(\mathbf{x}^1, \mathbf{y}^1)$ *was stored in a max-product FAM by means of the correlation-product encoding scheme then* $W \circ_P \mathbf{x}^1 = \mathbf{y}^1$ *if and only if* $\bigvee_{j=1}^n x_j^1 = 1$. *Furthermore, we have* $W \circ_P \mathbf{x} \leq \mathbf{y}^1$ *for every* $\mathbf{x} \in [0,1]^n$.

In Section 32.5.2, we will provide conditions for perfect recall using a max-min or max-product FAM that stores several input-output pairs (cf. Proposition 2). Kosko himself proposed to utilize a *FAM system* in order to overcome the storage limitation of the max-min FAM and max-product FAM. Generally speaking, a FAM system consists of a bank of k FAM matrices W^ξ such that each FAM matrix stores a single fundamental memory $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, where $\xi = 1, \ldots, k$. Given an input pattern $\mathbf{x}$, a combination of the outputs of each FAM matrix in terms of a weighted sum yields the output of the system. Kosko argues that the separate storage of FAM associations consumes memory space but provides an "audit trail" of the FAM inference procedure and avoids crosstalk [37]. According to Chung and Lee, the implementation of a FAM system is limited to applications with a small amount of associations [14]. As to computational effort, the FAM system requires at least the synthesis of k FAM matrices.

32.5.2 Generalized Fuzzy Associative Memories of Chung and Lee

Chung and Lee generalized Kosko's FAMs by substituting the max-min or the max-product by a more general max-t product in Equations 32.32 and 32.33 [14]. The resulting model, called *generalized FAM* (GFAM), can be described in terms of the following relationship between an input pattern $\mathbf{x} \in [0,1]^n$ and the corresponding output pattern $\mathbf{y} \in [0,1]^m$. Here, the symbol $\circ_T$ denotes the max-C product (cf. Equation 32.22) where C is a t-norm.

$$\mathbf{y} = W \circ_T \mathbf{x} \quad \text{where} \quad W = Y \circ_T X^T \,. \tag{32.34}$$

We refer to the learning rule that is used to generate $W = Y \circ_T X^T$ as *correlation-t encoding*. Note that a GFAM performs a dilation at each node (and overall) if and only if the t-norm represents a dilation in $[0,1]$.

We could generalize the GFAM even further by substituting the t-norm with a more general fuzzy conjunction. However, the resulting model does not satisfy Proposition 2 below since it requires the associativity and the boundary condition $T(x,1) = x$ of a t-norm.

In the theory of linear associative memories trained by means of a learning rule based on Hebb's postulate, perfect recall of the stored patterns is possible if the patterns $\mathbf{x}^1, \ldots, \mathbf{x}^k$ constitute an orthonormal set [25, 26]. Chung and Lee noted that a similar statement, which can be found below, is true for GFAM models.

A straightforward fuzzification of the orthogonality and orthonormality concepts leads to the following definitions. Fuzzy patterns $\mathbf{x}, \mathbf{y} \in [0,1]^n$ are said *max-t orthogonal* if and only if $\mathbf{x}^T \circ_T \mathbf{y} = 0$, i.e. $T(x_j, y_j) = 0$ for all $j = 1, \ldots, n$. Consequently, we speak of a *max-t orthonormal* set $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ if and only if the patterns $\mathbf{x}^\xi$ and $\mathbf{x}^\eta$ are max-t orthogonal for every $\xi \neq \eta$ and $\mathbf{x}^\xi$ is a normal fuzzy set for every $\xi = 1, \ldots, k$. Recall that a fuzzy set $\mathbf{x} \in [0,1]^n$ is normal if and only if $\bigvee_{j=1}^n x_j = 1$, i.e. $\mathbf{x}^T \circ_T \mathbf{x} = 1$.

Based on the max-t orthonormality definition, Chung and Lee succeeded in showing the following proposition concerning the recall of patterns by a GFAM [14].

Proposition 2. *Suppose that the fundamental memories* $(\mathbf{x}^\xi, \mathbf{y}^\xi)$*, for* $\xi = 1, \ldots, k$*, are stored in a GFAM by means of the correlation-t encoding scheme. If the set* $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$ *is max-t orthonormal* *then* $W \circ_T \mathbf{x}^\xi = \mathbf{y}^\xi$ *for every* $\xi = 1, \ldots, k$.

In particular, Chung and Lee noted that the Lukasiewicz GFAM, i.e. the GFAM based on the Lukasiewicz fuzzy conjunction, will perfectly recall the stored patterns if the $\mathbf{x}^\xi$'s are such that $0 \vee (\mathbf{x}_j^\xi + \mathbf{x}_j^\eta - 1) = 0$ for every $\xi \neq \eta$ and $j = 1, \ldots, n$. In other words, we have $\mathbf{y}^\xi = W \circ_L \mathbf{x}^\xi$ for every $\xi = 1, \ldots, k$ if $\mathbf{x}_j^\xi + \mathbf{x}_j^\eta \leq 1$ for every $\xi \neq \eta$ and $j = 1, \ldots, n$. These inequalities hold true in particular for patterns $\mathbf{x}^\xi$'s that satisfy the usual condition $\sum_{\xi=1}^k x_j^\xi = 1$ for every $j = 1, \ldots, n$.

32.6 Variations of Max-Min Fuzzy Associative Memory

In this section, we will discuss two variations of Kosko's max-min FAM: the models of Junbo and Liu. Junbo *et al.* generate the weight matrix of their model according to the Gödel implicative learning scheme that we will introduce in Equation 32.35. Liu modified Junbo's model by incorporating a threshold at the input and output layer.

32.6.1 Junbo's Fuzzy Associative Memory Model

Junbo's FAM and Kosko's max-min FAM share the same topology and the same type of morphological neurons, namely max-C_M neurons [30]. Consequently, Junbo's FAM computes the output pattern $\mathbf{y} = W \circ_M \mathbf{x}$ upon presentation of an input pattern $\mathbf{x} \in [0, 1]^n$.

The difference between the max-min FAM and the Junbo's FAM lies in the learning rule. Junbo *et al.* chose to introduce a new learning rule for FAM which allows for the storage of multiple fuzzy fundamental memories. The synaptic weight matrix is computed as follows:

$$W = Y \circledast_M X^T \,. \tag{32.35}$$

Here, the symbol $\circledast_M$ denotes the min-I_M product of Equation 32.24. We will refer to this learning rule as *Gödel implicative learning* since it employs Gödel's fuzzy implication I_M [73, 71].

The following proposition shows the optimality (in terms of the perfect recall of the original patterns) of the Gödel implicative learning scheme for max-min FAMs [18, 30]. In particular, Proposition 3 reveals that the Junbo's FAM can store at least as many patterns as the max-min FAM of Kosko.

Proposition 3. *Let* $X = [\mathbf{x}^1, \ldots, \mathbf{x}^k] \in [0,1]^{n \times k}$ *and* $Y = [\mathbf{y}^1, \ldots, \mathbf{y}^k] \in [0,1]^{m \times k}$ *be the matrices whose columns are the fundamental memories. If there exist* $A \in [0,1]^{m \times n}$ *such that* $A \circ_M \mathbf{x}^\xi = \mathbf{y}^\xi$ *for all* $\xi = 1, \ldots, k$*, then* $W = Y \circledast_M X^T$ *is such that* $W \circ_M \mathbf{x}^\xi = \mathbf{y}^\xi$ *for all* $\xi = 1, \ldots, k$*.*

32.6.2 The Max-Min Fuzzy Associative Memory with Threshold of Liu

Proposition 3 shows that Gödel implicative learning guarantees the best possible storage capacity for a max-min FAM. Therefore, improvements in storage capacity can only be achieved by considering neural associative memories with a different architecture and/or different types of neurons. Since adding hidden layers to the max-min FAM also fails to increase the storage capacity, Liu proposes the following model whose recall phase is described by the following equation [38]:

$$\mathbf{y} = (W \circ_M (\mathbf{x} \vee \mathbf{c})) \vee \mathbf{d}\,. \tag{32.36}$$

The weight matrix $W \in [0,1]^{m \times n}$ is given in terms of Gödel implicative learning and the thresholds $\mathbf{d} \in [0,1]^m$ and $\mathbf{c} = [c_1, \ldots, c_n]^T \in [0,1]^n$ are of the following form:

$$\mathbf{d} = \bigwedge_{\xi=1}^{k} \mathbf{y}^\xi \quad \text{and} \quad c_j = \begin{cases} \bigwedge_{i \in D_j} \bigwedge_{\xi \in LE_{ij}} y_i^\xi & \text{if } D_i \neq \emptyset\,, \\ 0 & \text{if } D_j = \emptyset\,, \end{cases} \tag{32.37}$$

where $LE_{ij} = \{\xi : x_j^\xi \leq y_i^\xi\}$ and $D_j = \{i : LE_{ij} \neq \emptyset\}$.

Liu's model is also known as the *max-min FAM with threshold*. Note that Equation 32.36 boils down to adding bias terms to the single-layer max-min FAM. The following proposition concerns the recall of patterns using the max-min FAM with threshold.

Proposition 4. *Suppose the symbols W, $\mathbf{c}$, and $\mathbf{d}$ denote the weight matrix and the thresholds of a max-min FAM with threshold that stores the fundamental memories $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, where $\xi = 1, \ldots, k$. If there exists $A \in [0,1]^{m \times n}$ such that $A \circ_M \mathbf{x}^\xi = \mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$, then $\mathbf{y}^\xi = (W \circ (\mathbf{x}^\xi \vee \mathbf{c})) \vee \mathbf{d}$ for all $\xi = 1, \ldots, k$.*

Thus, the max-min FAM with threshold can store at least as many patterns as the FAM of Junbo and the max-min FAM of Kosko. In the next section, we will introduce a FAM model whose storage capacity is at least as high as that of Liu's model and which does not require the cumbersome computation of the threshold $\mathbf{c} \in [0,1]^n$.

32.7 Other Subclasses of Fuzzy Morphological Associative Memories

In this section, we discuss the IFAM , the dual IFAM, and FLBAM models. The IFAMs and the dual IFAMs can be viewed as extensions of morphological associative memories (MAMs) to the fuzzy domain [73, 71, 70]. Thus, these models maintain the features of the MAM models. In particular, an IFAM model computes a dilation whereas a dual IFAM performs an erosion whereas the FLBAM model computes an anti-dilation at every node [7].

32.7.1 Implicative Fuzzy Associative Memories

Implicative fuzzy associative memories (IFAMs) bear some resemblance with the GFAM model of Chung and Lee. Specifically, an IFAM model is given by a single layer feedforward ANN endowed with max-T neurons where T is a continuous t-norm. In contrast to the GFAM, the IFAM model includes a bias term $\boldsymbol{\theta} = [0,1]^n$ and employs a learning rule that we call *R-implicative fuzzy* *learning*. Note that a continuous t-norm represents a dilation in $[0,1]$. Therefore, the neurons of an IFAM are dilative and thus IFAMs belong to the class of FMAMs.

Consider a fundamental memory set $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$ and an IFAM model that is equipped with max-T neurons. Let I_T be the fuzzy implication such that I_T and the given continuous t-norm T are adjoint and let the symbol $\circledast_T$ denote the min-I_T product (cf. Equation 32.24). Given an input pattern $\mathbf{x} \in [0,1]^n$, the IFAM model produces the following output pattern $\mathbf{y} \in [0,1]^m$:

$$\mathbf{y} = (W \circ_T \mathbf{x}) \vee \boldsymbol{\theta}\,, \quad \text{where} \quad W = Y \circledast_T X^T \quad \text{and} \quad \boldsymbol{\theta} = \bigwedge_{\xi=1}^{k} \mathbf{y}^\xi\,. \tag{32.38}$$

The fuzzy implication I_T is uniquely determined by the following equation.

$$I_T(x,y) = \bigvee \{z \in [0,1] : T(x,z) \leq y\} \; \forall\, x,y \in [0,1]\,. \tag{32.39}$$

We refer to I_T as the *R-implication associated with the t-norm* T, hence the name R-implicative fuzzy learning. Particular choices of T, I_T respectively, lead to particular IFAM models. The name of a particular IFAM model indicates the choice of T and I_T. For example, the Gödel IFAM corresponds to the IFAM model given by the equation $\mathbf{y} = (W \circ_M \mathbf{x}) \vee \boldsymbol{\theta}$ where $W = Y \circledast_M X^T$ and $\boldsymbol{\theta} = \bigwedge_{\xi=1}^{k} \mathbf{y}^\xi$.

Note that the learning rule used in the Gödel IFAM model coincides with the Gödel implicative learning rule that is used in the FAM models of Junbo and Liu. Recall that Liu's max-min FAM with threshold can be viewed as an improved version of Junbo's FAM. Although the Gödel IFAM disposes of only one threshold term $\boldsymbol{\theta}$, its storage capacity is at least as high as the one of Liu's FAM [71]. In fact, the IFAM model can be considered a generalization of Liu's max-min FAM with threshold. The following proposition concerns the recall of patterns using an arbitrary IFAM model [71].

Proposition 5. *Consider the fundamental memory set* $\{(\mathbf{x}^\xi, \mathbf{y}^\xi) : \xi = 1, \ldots, k\}$*. If there exist a synaptic weight matrix* $A \in [0,1]^{m \times n}$ *and a bias vector* $\boldsymbol{\beta} \in [0,1]^m$ *such that* $\mathbf{y}^\xi = (A \circ_T \mathbf{x}^\xi) \vee \boldsymbol{\beta}$*, for every* $\xi = 1, \ldots, k$*, then* $A \leq W = Y \circledast_T X^T$*,* $\boldsymbol{\beta} \leq \boldsymbol{\theta} = \bigwedge_{\xi=1}^k \mathbf{y}^\xi$*, and* $\mathbf{y}^\xi = (W \circ_T \mathbf{x}^\xi) \vee \boldsymbol{\theta}$ *for all* $\xi = 1, \ldots, k$*.*

In the autoassociative case, we speak of the *autoassociative fuzzy implicative memory* (AFIM). The synaptic weight matrix and bias vector of an AFIM model are given by $W = X \circledast_T X^T$ and $\boldsymbol{\theta} = \bigwedge_{\xi=1}^k \mathbf{x}^\xi$, respectively. We can convert the AFIM into a dynamic model by feeding the output $(W \circ_T \mathbf{x}) \vee \boldsymbol{\theta}$ back into the memory. We refer to the patterns $\mathbf{x} \in [0,1]^n$ that remain fixed under an application of $W = X \circledast_T X^T$ as the fixed points of W. In sharp contrast to the GFAM models, one can store as many patterns as desired in an AFIM [73]. In particular, the storage capacity of the AFIM is at least as high as the storage capacity of the quantum associative memory if the stored patterns are binary [74]. Furthermore, we succeeded in showing the following proposition concerning the recall of patterns by an AFIM model and the fixed points of $W = X \circledast_T X^T$. The following proposition characterizes the fixed points of an AFIM as well as the output patterns in terms of the fixed points [71].

Proposition 6. *Consider a fundamental memory set* $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$*. If* $W = X \circledast_T X^T$ *and* $\boldsymbol{\theta} = \bigwedge_{\xi=1}^k \mathbf{x}^\xi$ *then for every input pattern* $\mathbf{x} \in [0,1]^n$*, the output* $(W \circ_T \mathbf{x}) \vee \boldsymbol{\theta}$ *of the AFIM is the* supremum *of* $\mathbf{x}$ *in the set of fixed points of* W *greater than* $\boldsymbol{\theta}$*, i.e* $(W \circ_T \mathbf{x}) \vee \boldsymbol{\theta}$ *is the smallest fixed point* $\mathbf{y}$ *of* W *such that* $\mathbf{y} \geq \mathbf{x}$ *and* $\mathbf{y} \geq \boldsymbol{\theta}$*.*

Moreover, a pattern $\mathbf{y} \in [0,1]^n$ *is a fixed point of* W *if* $\mathbf{y} = \mathbf{c}$ *for some constant vector* $\mathbf{c} = [c, c, \ldots, c]^T \in [0,1]^n$ *or if* $\mathbf{y}$ *is of the following form for some* $L_l \subseteq \{1, \ldots, k\}$ *and some* $k \in \mathbb{N}$*.*

$$\mathbf{y} = \bigvee_{l=1}^{k} \bigwedge_{\xi \in L_l} \mathbf{x}^\xi \,. \tag{32.40}$$

This proposition reveals that AFIM models exhibit a very large number of fixed points which include the original patterns $\mathbf{x}^\xi$, where $\xi = 1, \ldots, k$, and many spurious states. Moreover, the basin of attraction of an original pattern $\mathbf{x}^\xi$ only consists of patterns $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{x}^\xi$. In the near future, we intend to generalize Proposition 6 to include the heteroassociative case.

IFAM models have been successfully applied to several problems in prediction where they have outperformed other models such as statistical models and the FAM models that we mentioned above [71, 67]. The Lukasiewicz IFAM which exhibited the best performance in these simulations is closely related to the gray-scale morphological associative memory (MAM) [51, 70]. Both the MAM model as well as the general IFAM model are equipped with a dual model.

32.7.2 Dual Implicative Fuzzy Associative Memories

Recall that the IFAM model has dilative max-T neurons where T is a continuous t-norm. A dual IFAM model can be constructed by taking the dual neurons with respect to a certain fuzzy negation. We chose to consider only the standard fuzzy negation N_S.

Let us derive the dual model of a given IFAM. Suppose that we want to store the associations $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, where $\xi = 1, \ldots, k$, in a dual IFAM. Let us synthesize the weight matrix $\bar{W}$ and the bias vector $\bar{\boldsymbol{\theta}}$ of the IFAM using the fundamental memories $(N_S(\mathbf{x}^\xi), N_S(\mathbf{y}^\xi))$, where $\xi = 1, \ldots, k$. If M denotes $N_S(\bar{W})$ and if $\boldsymbol{\vartheta}$ denotes $N_S(\bar{\boldsymbol{\theta}})$ then an application of Equations 32.30 and 32.31 to Equation 32.38 yields the recall phase of the dual IFAM model [71, 73]:

$$\mathbf{y} = N_S\left[\left(\bar{W} \circ_T N_S(\mathbf{x})\right) \vee \bar{\boldsymbol{\theta}}\right] = (M \bullet_S \mathbf{x}) \wedge \boldsymbol{\vartheta}\,, \tag{32.41}$$

Here, the symbol $\bullet_S$ stands for the min-S product of Equation 32.23 based on the continuous s-norm S that is the dual operator of T with respect to N_S. We conclude that the dual IFAM model performs an erosion at each node.

In view of Equations 32.11 and 32.41, every statement concerning the IFAM model yields a corresponding dual statement concerning the dual IFAM model. Specifically, we obtain the corresponding dual statement from the statement about the IFAM model by replacing minimum with maximum, t-norm with s-norm, the product $\circ_T$ with $\bullet_S$, and vice versa [71].

32.7.3 Fuzzy Logical Bidirectional Associative Memory

The *fuzzy logical bidirectional associative memory* (FLBAM) [7] constitutes a recurrent model whose network topology coincides with the one of Kosko's bidirectional associative memory (BAM) [36]. In contrast to Kosko's BAM, the neurons of the FLBAM calculate min-I_T products where the fuzzy implication I_T is adjoint to some continuous t-norm T. Using a Hebbian style correlation-t encoding scheme, Bělohlávek constructs the weight matrix W for the forward direction of the FLBAM as follows:

$$W = Y \circ_T X^T . \tag{32.42}$$

The weight matrix for the backward direction simply corresponds to W^T. Thus, given an input pattern $\mathbf{x}_0 \in [0,1]^n$, the FLBAM generates the following sequence $(\mathbf{x}_0, \mathbf{y}_0), (\mathbf{x}_1, \mathbf{y}_0), (\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_1), \ldots$:

$$\mathbf{y}_k = W \circledast_T \mathbf{x}_k \quad \text{and} \quad \mathbf{x}_{k+1} = W^T \circledast_T \mathbf{y}_k \quad \text{for} \quad k = 0, 1, 2, \ldots . \tag{32.43}$$

The following proposition shows that a FLBAM reaches a stable state after one step in the forward direction and one step in the backward direction [7].

Proposition 7. *For an arbitrary input pattern $\mathbf{x}_0 \in [0,1]^n$, the pair $(\mathbf{x}_1, \mathbf{y}_0)$ is a stable state of the FLBAM.*

The following observations demonstrate that the FLBAM models belong to the FMAM class [72]. Specifically, we will show that the neurons of a FLBAM compute anti-dilations. Recall that the FLBAM has min-I_T neurons, where I_T is adjoint to some continuous t-norm T. The fact that I_T and T form an adjunction implies that I_T can be expressed in the form given by Equation 32.39.

Therefore, the following equations hold true for every $X \subseteq [0,1]$ and $y \in [0,1]$:

$$I_T\left(\bigvee X, y\right) = \bigvee\left\{z \in [0,1] : T\left(\bigvee X, z\right) \leq y\right\} \tag{32.44}$$

$$= \bigvee\left\{z \in [0,1] : \bigvee_{x \in X} T(x,z) \leq y\right\} \tag{32.45}$$

$$= \bigwedge_{x \in X}\left[\bigvee\{z \in [0,1] : T(x,z) \leq y\}\right] = \bigwedge_{x \in X} I_T(x,y)\,. \tag{32.46}$$

Consequently, $I_T(\cdot, y)$ represents an anti-dilation for every $y \in [0,1]$ and therefore the nodes of an FLBAM also calculate anti-dilations.

32.8 Experimental Results

32.8.1 Storage Capacity and Noise Tolerance Example

Consider the 12 patterns shown in Figure 32.1. These are gray-scale images $\mathbf{x}^\xi \in [0,1]^{56\times 46}$, $\xi = 1,\ldots,12$, from the faces database of AT&T Laboratories Cambridge [1]. This database contains files in PGM format. The size of each image is 92x112 pixels, with 256 gray levels per pixel. We downsized the original images using neighbor interpolation. Then, we obtained fuzzy patterns (vectors) $\mathbf{x}^1,\ldots,\mathbf{x}^{12} \in [0,1]^{2576}$ using the standard row-scan method.

We stored the patterns $\mathbf{x}^1,\ldots,\mathbf{x}^k$ in the Lukasiewicz, Gödel, and Goguen AFIMs and we verified that they represent fixed points of these models, i.e. the AFIMs succeeded in storing the fundamental memory set. In order to verify the tolerance of the AFIM model with respect to corrupted or incomplete patterns, we presented as input the patterns $\mathbf{r}^1,\ldots,\mathbf{r}^6$ displayed in Figure 32.2. The first three patterns, $\mathbf{r}^1, \mathbf{r}^2$, and $\mathbf{r}^3$, of Figure 32.2 were generated introducing pepper noise in $\mathbf{x}^1$ with probabilities 25%, 50%, and 75%, respectively. The other three patterns, $\mathbf{r}^4, \mathbf{r}^5$, and $\mathbf{r}^6$, were obtained excluding respectively 25%, 50%, and 75% of the original image. The respective recalled patterns are shown in Figure 32.3. Note that the Lukasiewicz AFIM succeeded in recalling

Figure 32.1: Fundamental memories set $\{\mathbf{x}^1, \ldots, \mathbf{x}^{12}\}$ used in Example 32.8.1.

the original pattern almost perfectly.

We also conducted the same experiment using the FAM models presented previously. We observed that the max-min and max-product FAMs, as well as the Lukasiewicz GFAM and the FLBAMs based on the implication of Lukasiewicz and Gödel, failed to demonstrate an adequate performance on this task due to a high amount of crosstalk between the stored patterns. Moreover, we noted that the FAM of Junbo and the max-min FAM with threshold produced the same outputs of the Gödel AFIM. Thus, although the threshold $\boldsymbol{\theta}$ of the Gödel AFIM captures the effects of the thresholds $\mathbf{c}$ and $\mathbf{d}$ of the max-min FAM with threshold of Liu, in this example, it did not improve the tolerance with respect to noise of the Gödel AFIM with respect to Junbo's FAM. The dual AFIMs succeeded in storing the fundamental memory set but failed to recall $\mathbf{x}^1$ when the patterns of Figure 32.2 where present as input. In fact, concerning a dual AFIM model, we can show that the basin of attraction of an original pattern $\mathbf{x}^\xi$ consists of patterns $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{x}^\xi$, i.e. the dual AFIM exhibit tolerance with respect to corrupted patterns $\tilde{\mathbf{x}}^\xi$ only if $\tilde{\mathbf{x}}^\xi \geq \mathbf{x}^\xi$ [71]. Table 32.1 presents the normalized error produced by the FAM models when the incomplete or corrupted patterns of Figure 32.2 are presented as input. For instance, the normalized error $E(\mathbf{r}^\eta)$ of the AFIM

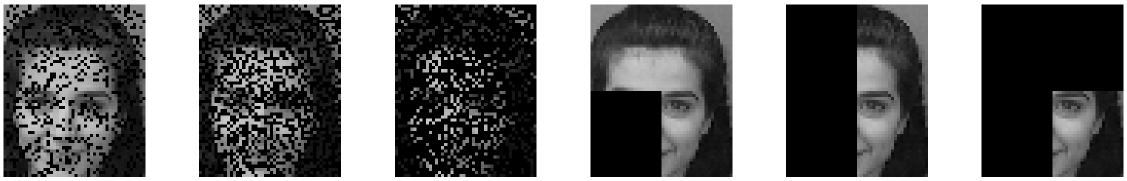

Figure 32.2: Patterns $\mathbf{r}^1, \ldots, \mathbf{r}^6$ representing corrupted or incomplete versions of pattern $\mathbf{x}^1$ used as input of the FMAM models.

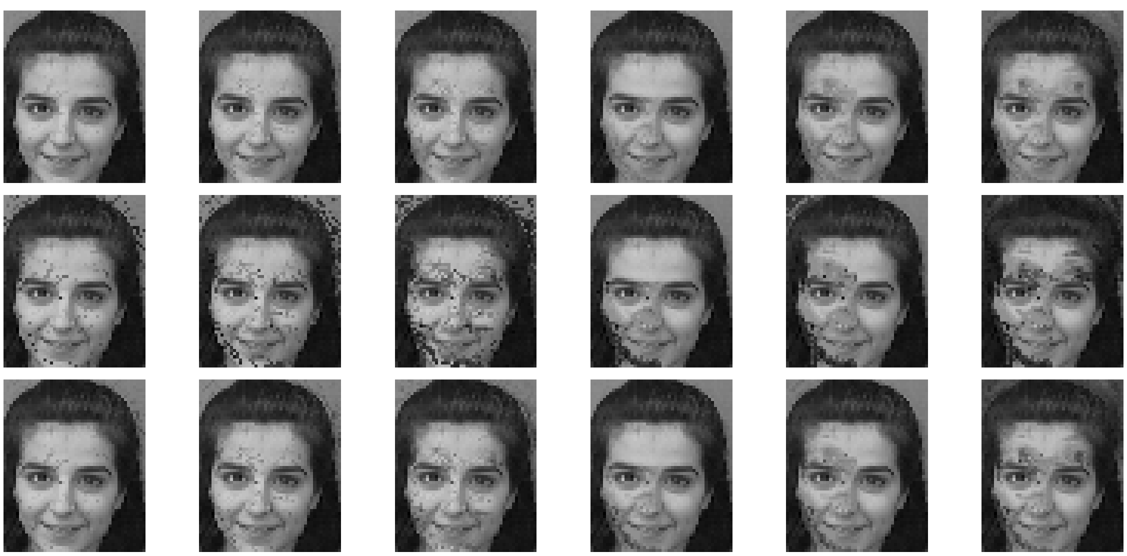

Figure 32.3: Patterns recalled by Lukasiewicz (first row), Gödel (second row), and Goguen (third row) when the patterns $\mathbf{r}^1, \ldots, \mathbf{r}^6$ of Figure 32.2 are presented as input.

Associative Memory	$E(\mathbf{r}^1)$	$E(\mathbf{r}^2)$	$E(\mathbf{r}^3)$	$E(\mathbf{r}^4)$	$E(\mathbf{r}^5)$	$E(\mathbf{r}^6)$
Lukasiewicz IFAM	0.0248	0.0388	0.0587	0.0842	0.1051	0.1499
Gödel IFAM	0.1132	0.1690	0.2440	0.1878	0.2419	0.3355
Goguen IFAM	0.0459	0.0682	0.1079	0.1241	0.1498	0.2356
max-min FAM of Kosko	0.8331	0.8309	0.8332	0.8257	0.8066	0.7555
max-prod FAM of Kosko	0.4730	0.4723	0.4706	0.4729	0.4736	0.4565
Lukasiewicz GFAM	0.5298	0.4944	0.4932	0.5061	0.6200	0.7013
FAM of Junbo	0.1132	0.1690	0.2440	0.1878	0.2419	0.3355
max-min FAM with threshold	0.1132	0.1690	0.2440	0.1878	0.2419	0.3355
Lukasiewicz FLBAM	0.3512	0.3667	0.3827	0.3932	0.4254	0.4574
Gödel FLBAM	0.2954	0.3156	0.3277	0.2994	0.3111	0.4982

Table 32.1: Normalized error produced by the FAM models when the patterns $\mathbf{r}^1, \ldots, \mathbf{r}^6$ of Figure 32.2 are presented as input.

models are computed as follows for $\eta = 1, \ldots, 6$:

$$E(\mathbf{r}^\eta) = \frac{\|\mathbf{x}^1 - [(W \circ_T \mathbf{r}^\eta) \vee \boldsymbol{\theta}]\,\|}{\|\mathbf{x}^1\|}\,. \tag{32.47}$$

32.8.2 Application of the Lukasiewicz IFAM in Prediction

Fuzzy associative memories can be used to implement mappings of fuzzy rules. In this case, a set of rules in the form of human-like IF-THEN conditional statements are stored. In this subsection, we present an application of a certain FMAM model to a problem of forecasting time-series. Specifically, we applied the Lukasiewicz IFAM to the problem of forecasting the average monthly streamflow of a large hydroelectric plant called Furnas, that is located in southeastern Brazil. This problem was previously discussed in [39, 40, 72].

First, the seasonality of the monthly streamflow suggests the use of 12 different predictor models, one for each month of the year. Let s_ξ, for $\xi = 1, \ldots, q$, be samples of a seasonal streamflow time series. The goal is to estimate the value of s_γ from a subsequence of $(s_1, s_2, \ldots, s_{\gamma-1})$. Here, we employ subsequences that correspond to a vector of the form

$$\mathbf{p}^{\gamma} = (s_{\gamma-h}, \ldots, s_{\gamma-1})^T , \tag{32.48}$$

where $h \in \{1, 2, \ldots, \gamma - 1\}$. In this experiment, our IFAM based model only uses a fixed number of three antecedents. For example, the values of January, February, and March were taken into account to predict the streamflow of April.

The uncertainty that is inherent in hydrological data suggests the use of fuzzy sets to model the streamflow samples. For $\xi < \gamma$, a fuzzification of $\mathbf{p}^{\xi}$ and s^{ξ} using Gaussian membership functions yields fuzzy sets $\mathbf{x}^{\xi} : \mathcal{U} \to [0, 1]$ and $\mathbf{y}^{\xi} : \mathcal{V} \to [0, 1]$, respectively, where $\mathcal{U}$ and $\mathcal{V}$ represent finite universes of discourse. A subset S of the resulting input-output pairs $\{(\mathbf{x}^{\xi}, \mathbf{y}^{\xi}), \xi < q\}$ is implicitly stored in the Lukasiewicz IFAM (we only construct the parts of the weight matrix that are actually used in the recall phase) [72]. We employed the *subtractive clustering method* to determine the set S [13]. Feeding the pattern $\mathbf{x}^{\gamma}$ into the IFAM model, we retrieve the corresponding output pattern $\mathbf{y}^{\gamma}$. For computational reasons, $\mathbf{x}^{\gamma}$ is modeled as a singleton on $\mathcal{U}$. A defuzzification of $\mathbf{y}^{\gamma}$ using the mean of maximum yields s_{γ} [72].

Figure 32.4 shows the forecasted streamflows estimated by the prediction model based on the Lukasiewicz IFAM for the Furnas reservoir from 1991 to 1998. Table 32.2 compares the errors that were generated by the IFAM model and several other models [40, 39]. In contrast to the IFAM-based model, the MLP, NFN, and FPM-PRP models were initialized by optimizing the number of the parameters for each monthly prediction. For example, the MLP considers 4 antecedents to predict the streamflow of January and 3 antecedents to predict the streamflow for February. Moreover, the FPM-PRP model also takes into account slope information which requires some additional “fine tuning”. We experimentally determined a variable number of parameters (including slopes) for the IFAM model such that MSE = 0.88×10^5, MAE = 157, and MPE = 15.

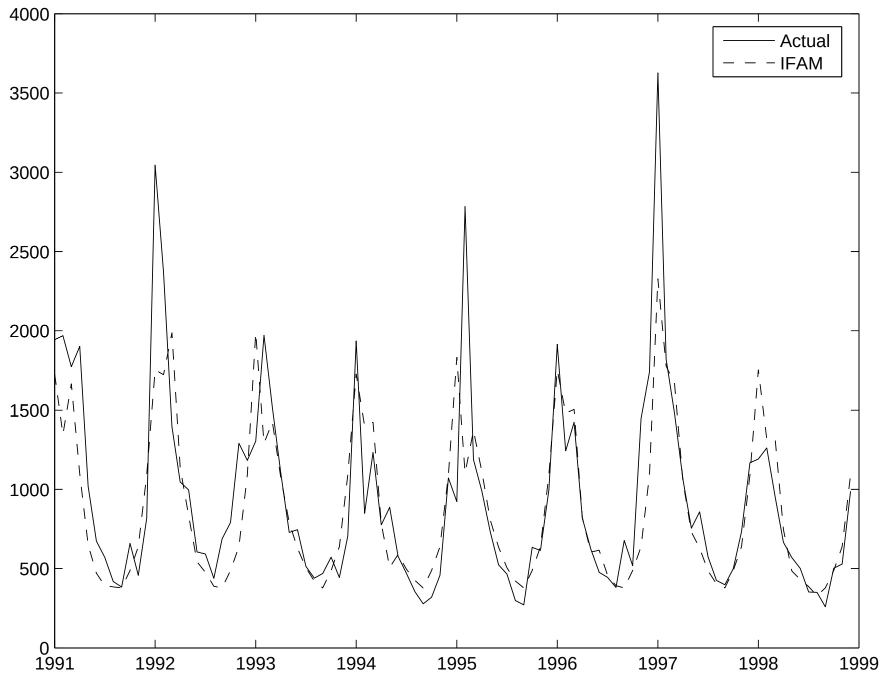

Figure 32.4: The streamflow prediction for the Furnas reservoir from 1991 to 1998. The continuous line corresponds to the actual values and the dashed line corresponds to the predicted values.

32.9 Conclusion and Suggestions for Further Research: Fuzzy and Granular Morphological Associative Memories

This article describes the most widely known models of fuzzy associative memory from the perspective of mathematical morphology. We showed that most FAM models compute an elementary operation of mathematical morphology at each node. Therefore, these models belong to the class of fuzzy morphological associative memories. Although a general theory of FMAMs has yet to be developed, a number of useful theoretical results have already been proven for a large subclass of FMAMs called implicative fuzzy associative memories [71]. In addition, certain FMAM models such as the Lukasiewicz IFAM have outperformed other FAM models in applications as fuzzy rule-based systems [67].

The mathematical basis for fuzzy (morphological) associative memories can be found in fuzzy

Methods	MSE ($\times 10^5$)	MAE (m^3/s)	MPE (%)
Lukasiewicz IFAM	1.42	226	22
PARMA	1.85	280	28
MLP	1.82	271	30
NFN	1.73	234	20
FPM-PRP	1.20	200	18

Table 32.2: Mean square, mean absolute, and mean relative percentage errors produced by the prediction models.

mathematical morphology which relies on the fact that the set $[0,1]^{\mathbf{X}}$ represents a complete lattice for any universe $\mathbf{X}$ [55, 43]. Recall that a fuzzy set represents a special case of an information granule, a concept that also encompasses intervals, rough sets, probability distributions, and fuzzy interval numbers [78, 31]. Information granules have been used in a variety of applications [6] but - to our knowledge - granular associative memories have yet to be formulated and investigated. We believe that the complete lattice framework of mathematical morphology may prove to be useful for developing a general theory and applications of FMAMs and granular (morphological) associative memories.

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