A Multilevel Wavelet Scheme for Evolution Equations with Time Adaptivity

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Abstract

The study in this paper is devoted to the stability and consistency analyses of an adaptive multilevel time discretization proposed by Bacry, Mallat and Papanicolau [1]. The main idea is to evolve the components in a multirresolution representation of the numerical solution by means of an explicit algorithm, adapting the time step according to each scale level. For a model problem, and in the context of biorthogonal wavelets, it is proved that the stability condition and consistency order are the same as in the original non-adapted scheme.

Resumo

Este trabalho é dedicado ao estudo de estabilidade e consistência para um esquema em multinível, com adaptabilidade temporal, para equações evolutivas. Em tal esquema, a solução numérica é representada como a soma de várias componentes, em diferentes níveis de escala. Para fazer a evolução temporal, adota-se um algoritmo explícito de referência que seja estável. Mas o passo de tempo não é o mesmo para todas as componentes. Em cada nível, escolhe-se um passo de tempo que satisfaça os princípios que regem a estabilidade do algoritmo de referência em tal nível de escala. Escolhendo um problema modelo, e um contexto de multiescala definido por wavelets biortogonais, prova-se que as condições de estabilidade e consistência do algortimo explícito de referência são mantidas no esquema adaptativo associado.

1 Introduction

For stability or accuracy reasons, the computation of approximate solutions to partial differential equations for time-dependent problems requires the adjustment of the time discretization to the spacial resolution. For instance, for parabolic problems, as for the heat equation, a stable explicit scheme typically requires the division of the time step by four if the spacial resolution is increased by a factor of two. For flow problems, the compromise between time and space steps is usually done by introducing a CFL number. The time step is then obtained by multiplying the space step by the CFL number divided by an estimate of the maximum local speed.

In many problems, the solutions may present all sort of features at different scale levels that pose considerable additional computational challenge. In such cases, a multilevel framework may be helpful. Splitting the solutions into several components of different scale levels, each component requires different time steps to be stably evolved. Therefore, a time-adaptivity strategy may improve the efficiency of such methods. For instance, in the solution of dissipative problems with multilevel scheme of nonlinear Galerkin type [8], the main idea is to compute differently the low and high modes, since their physical significances are different.

Nowadays, there is ample numerical evidence that significant improvements in accuracy and computational efficiency may be obtained by economically adapting the mesh points according to the occurrence of localized singular features, such as boundary layers, shocks or rarefaction waves. Adaptivity may simplify the numerical simulation, with no waste of fine grid cells where the solution is smooth, and refinement only close to irregularities, where it is actually needed. In such adaptive contexts, it is also tempting to consider a higher degree of efficiency by using local time steps that depends on the level of refinement, i.e. in some parts a large time step may be satisfactory, but in other parts a small time step is necessary [12].

In wavelet analysis, space adaptivity appears naturally since the wavelet coefficients can be used as local regularity indicators. In such context, time adaptivity, combined with space discretization by means of adapted orthonormal wavelet expansion, was firstly considered in [1]. The algorithm modifies the time discretization at each wavelet scale level in a way that allows each component to be evolved with the time step satisfying the corresponding stability constraint. Numerical results for the one-dimensional Burgers equation are presented in [1] concerning the efficiency of the method on accuracy, stability and complexity.

In the present paper, we consider the algorithm proposed in [1] in the extended context of biorthogonal wavelets. Focalizing only the time-adaptation aspect, we develop a classical consistency and stability analysis. For a simpler and more clear development of the general concepts and for the analysis of the method, we consider the one-dimensional heat equation. However, the analysis can be straightforward extended for general linear equations with constant coefficients and periodic boundary conditions in higher dimensions. We show that, for the model problem, the same stability and consistency conditions hold for the reference scheme as for its time-adaptive version.

The next section contains a brief overview of the main aspects of biorthogonal multiresolution analysis which are required in the subsequent parts of this paper. In Section 3, the reference algorithm is described and analysed. Section 4 is the main part of this paper. It is dedicated to the definition of the adaptive scheme and its consistency and stability analyses. Some concluding remarks are presented in Section 5.

2 Biorthogonal Multiresolution Analysis

In a multiresolution analysis, a sequence of embeded approximating spaces $V_j \subset L^2(\mathbb{R})$ are considered with corresponding Riesz bases $\{\phi_{j,k}(x), k \in \Gamma_j\}$. The index j is associated with the scale level and k indicates the local position in space. A fundamental aspect is the possibility of multilevel representations in terms of direct sums

$$V_j = V_J \oplus W_J \oplus \cdots \oplus W_{j-1},$$

where J is a chosen coarsest level and W_l contains the details between two consecutive levels l and $l+1$. Riesz bases $\{\psi_{l,k}(x), k \in \Lambda_l\}$ for the complementary spaces W_l are usually called wavelets. For the applications of this paper, we consider shift invariant spaces V_j with basis of the form

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),\ \ k \in \mathbb{Z}.$$

The basic function $\phi(x)$ is called scaling function and satisfies a scale relation

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h(k)\phi(2x-k). \tag{1}$$

In the Fourier domain, the scale relation reads

$$\widehat{\phi}(\xi) = H(\xi/2)\widehat{\phi}(\xi/2), \tag{2}$$

where

$$H(\xi) = \sum_{k\in\mathbb{Z}} h(k)e^{-ik\xi}. \tag{3}$$

For the construction of the wavelets, a dual multiresolution analysis V_j^* may be considered. It is determined by a scaling function $\phi^*(x)$, with corresponding scale relation

$$\phi^*(x) = 2\sum_{k\in\mathbb{Z}} h^*(k)\phi^*(2x-k), \tag{4}$$

satisfying the biorthogonal relation

$$\int_{\mathbb{R}} \phi^*(x)\phi(x-k)dx = \delta_k.$$

Complementary spaces W_l and W_l^* are defined, such that

$$V_{l+1} = V_l + W_l \qquad V_{l+1}^* = V_l^* + W_l^*$$

by choosing the bases $\psi_{l,k}(x) = 2^{l/2}\psi(2^l x - k)$ (respectively $\psi_{l,k}^*(x) = 2^{l/2}\psi^*(2^l x - k)$) associated to the mother wavelets

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g(k)\phi(2x-k) \quad \text{e} \quad \psi^*(x) = 2\sum_{k\in\mathbb{Z}} g^*(k)\phi^*(2x-k),$$

where $g^*(k) = (-1)^{k+1}h(1-k)$ e $g(k) = (-1)^{k+1}h^*(1-k)$. Thus, the following biorthogonal relations hold

$$\int_{\mathbb{R}} \psi^*(x)\psi(x-k)dx = \delta_k, \tag{5}$$

$$\int_{\mathbb{R}} \phi^*(x)\psi(x-k)dx = \int_{\mathbb{R}} \psi^*(x)\phi(x-k)dx = 0. \tag{6}$$

In such framework, approximations $\mathcal{P}^j f$ of functions f are found in V_j by means of the biorthogonal projection operator

$$\mathcal{P}^j f(x) := \sum_{k\in\Gamma_j} c^j(k)\phi_{j,k}(x), \tag{7}$$

where

$$c^j(k) := \mathcal{D}^j f(k) = \int_{\mathbb{R}} f(x)\phi_{j,k}^*(x)dx. \tag{8}$$

It can also be represented in a multilevel setting

$$\begin{aligned} \mathcal{P}^j f(x) &= \mathcal{P}^J f(x) + \sum_{l=J}^{j-1} \mathcal{Q}^l f(x) \\ &= \sum_{k\in\Gamma_J} c^J(k)\phi_{J,k}(x) + \sum_{l=J}^{j-1}\sum_{k\in\Lambda_l} d^l(k)\psi_{l,k}(x), \end{aligned} \tag{9}$$

where $\mathcal{Q}^l f(x)$ are projections on W_l, and d_k^l are the wavelet coefficients

$$d_k^l := \mathcal{G}^l f(k) = \int_{\mathbb{R}} \psi_{j,k}^*(x) f(x)dx. \tag{10}$$

The transformations relating the information at the finest level $\{c^j(k)\}$ and its multilevel representation $\{c^J(k)\} \cup \{d^J(k)\} \cup \cdots \cup \{d^{j-1}(k)\}$ are known as Mallat algorithms and are defined by the iterative application of the formulas

$$c^{j-1}(k) = 2\sum_{s\in\mathbb{Z}} h^*(s-2k)c^j(s), \tag{11}$$

$$d^{j-1}(k) = 2\sum_{s\in\mathbb{Z}} g^*(s-2k)c^j(s). \tag{12}$$

Conversely,

$$c^j(k) = \sum_{s\in\mathbb{Z}} h(h-2s)c^{j-1}(s) + \sum_{s\in\mathbb{Z}} g(k-2s)d^{j-1}(s). \tag{13}$$

In the applications of this paper, the functions are 1-periodic. All the concepts of biorthogonal multiresolution analysis hold for 1-periodic functions by simply considering 2^j-periodic sequences c^j in the expansions

$$f(x) = \sum_{k\in\mathbb{Z}} c^j(k)\phi_{j,k}(x)$$

defining of the spaces V^j. In such case, the Riesz basis property implies that there exist constants $0 < A < B$ such that, for all 2^j-periodic sequences c^j,

$$A\|c^j\|_j \leq \|\sum_{k\in\mathbb{Z}} c^j(k)\phi_{j,k}(x)\|_{\mathbf{L}^2} \leq B\|c^j\|_j,$$

where $\|.\|_{\mathbf{L}^2}$ stands for the norm in $\mathbf{L}^2([0,1])$ and

$$\|c^j\|_j = 2^{-j}\sum_{k=0}^{2^j-1} |c^j(k)|^2.$$

2.1 Accuracy

For shift-invariant approximating spaces, the approximation power is determined by the Strang-Fix condition. A function $\phi(x)$ is said to satisfy the Strang-Fix condition of order p if $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(\xi)$ has zeros of order $p+1$ at $\xi = 2k\pi$, $k \in \mathbb{Z}$. In such case, the polynomials of degree less or equal to p can be locally reproduced by linear combinations of the scaling functions $\phi_{j,k}(x)$. If ϕ and ϕ^* are integrable scaling functions of compact support, and ϕ satisfies a Strang-Fix condition of order p, then for functions in the Sobolev space $\mathbf{H}^{p+1}(\mathbb{R})$ the biorthogonal projection $\mathcal{P}^j f$ on V_j verifies the accuracy property [5, 10]

$$\|f - \mathcal{P}^j f\|_{\mathbf{H}^s} \lesssim 2^{-j(p+1-s)}||f||_{\mathbf{H}^{p+1}}, \tag{14}$$

for $0 \leq s \leq \min\{r, p+1\}$, where r is degree of regularity of ϕ, so that $\phi \in \mathbf{H}^r(\mathbb{R})$. Consequently, the following estimate also holds

$$\|\mathcal{Q}^j f\|_{\mathbf{H}^s} \lesssim 2^{-j(p+1-s)}||f||_{\mathbf{H}^{p+1}}, \tag{15}$$

Similar results are valid in the periodic case [9].

2.2 Cases of Interest

We have particular interest in the family of biorthogonal multiresolution analyses introduced by Cohen, Daubechies and Feauveau [6]. Let N^* and N be two positive integers of same parity such that $N^*+N=M$ is an even integer. $\phi^*=\phi_{N^*}$ is chosen as the B-spline de order N^*. For even $N^*=2l^*$ the corresponding scaling filter is

$$H(\xi)=\left(\cos\frac{\xi}{2}\right)^{N^*}.$$

If $N=2l$, then scaling functions $\phi(x)=\phi_{N^*,N}(x)$ may be found with scaling filters

$$H(\xi)=\left(\cos\frac{\xi}{2}\right)^{N}\sum_{k=0}^{l+l^*-1}\begin{pmatrix} l+l^*-1+k \\ k \end{pmatrix}\left(\sin\frac{\xi}{2}\right)^{2k}.$$

Similarly, for odd $N^*=2l^*+1$, and $N=2l+1$, the corresponding filters are

$$H^*(\xi)=e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^{N^*}$$

and

$$H(\xi)=e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^{N}\sum_{k=0}^{l+l^*}\begin{pmatrix} l+l^*+k \\ k \end{pmatrix}\left(\sin\frac{\xi}{2}\right)^{2k}.$$

For these families, all the basic functions have compact support. ϕ^* is a C^{N^*-2} piecewise polynomials of degree N^*-1, and ϕ has increasing regularity with increasing N. ϕ^* and ϕ are symmetric functions centered at $x=0$, for even N^* and N, and centered at $x=\frac{1}{2}$, for odd N^* and N. They satisfy Strang-fix conditions or order N^*-1 and $N-1$, respectively.

In the extreme case $N^*=0$, $\phi^*(x)=\delta(x)$ is the Dirac distribution and $\theta_M(x)=\phi_{0,M}$ correspond to the interpolating scaling functions defined by Delauries and Dubuc [7]. It can be shown that

$$\theta_M(x)=\int_{\mathbb{R}}\phi_{N^*}(y)\phi_{N,N^*}(y+x)dy,$$

independently of the choices of N,N^* such that $M=N+N^*$[11].

3 The Reference Numerical Scheme

In this paper, we are concerned with the numerical solution of evolution equations

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t}=\mathcal{L}u(t,x), x\in\mathbb{R},\ t\geq 0, \\ \\ u(0,x)=u_0(x), \end{cases} \tag{16}$$

with periodic boundary condition $u(t, x+1) = u(t, x)$, where $\mathcal{L}$ is a differential operator acting in the x variable.

We consider pairs of shift-invariant spaces $\{V_j, V_j^*\}$ forming a biorthogonal multiresolution analysis, and having sufficient regularity. We define the discretization of problem (16) by following two basic steps.

⋆ <u>*Space discretization*</u>.

At each time step, an approximate solution $u_j(t, x)$ is sought in the approximating space V_j. It is defined by imposing a Petrov-Galerkin orthogonally property (the residual is orthogonal to V_j^*). The result is the semidiscrete ODE problem

$$\frac{\partial u_j(t, x)}{\partial t} = [\mathcal{L}^j u_j(t, \cdot)](x), \tag{17}$$

where $\mathcal{L}^j$ is the discrete version of $\mathcal{L}$ given by

$$\mathcal{L}^j u_j(t, \cdot) = \mathcal{P}^j \mathcal{L} u_j(t, \cdot). \tag{18}$$

⋆ <u>*Time discretization*</u>.

The ODE system (17) is discretized by an appropriate ODE solver. For instance, when adopting the forward Euler scheme, we get

$$u_j(t + \Delta_t, x) = [(I + \Delta_t \mathcal{L}^j) u_j(t, \cdot)](x) =: [\mathcal{K}^j u_j(t, \cdot)](x). \tag{19}$$

The analysis of this paper holds for constant coefficients differential operators

$$\mathcal{L}u = \sum_{\nu=0}^{m} A_\nu \frac{\partial^\nu u}{\partial x^\nu}.$$

Their discrete version may be expressed in terms of the 2^j-periodic coefficients $u^{n,j}(s)$ in the expansion

$$u_j(t_n, x) = \sum_{s \in \mathbb{Z}} u^{n,j}(s) \phi(2^j x - s)$$

by the formula

$$(\mathcal{L}^j \mathbf{u}^{n,j})(s) = \sum_{\nu=0}^{m} A_\nu 2^{j\nu} \sum_{k \in \mathbb{Z}} u^{n,j}(k) \beta^\nu(s-k),$$

with

$$\beta^\nu(q) = \int_{\mathbb{R}} \phi^*(x) \frac{d^\nu \phi}{dx^\nu}(x) dx.$$

Therefore, the discrete formulation (19) can also be expressed in matrix form by

$$\begin{aligned} u^{n+1,j}(k) &= [(I+\Delta_t \mathcal{L}^j)\mathbf{u}^{n,j}](k) \\ &= [\mathcal{K}^j \mathbf{u}^{n,j}](k), \end{aligned} \tag{20}$$

where $\mathcal{K}^j = \mathcal{K}(\Delta_t, 2^{-j}) = (I + \Delta_t \mathcal{L}^j)$. This reference scheme can be interpreted as a finite difference approximation.

In the applications of this paper, we shall also adopt the spline multiresolution framework. In such case, the coefficients $\beta^\nu(q)$ depend on the choice of the parameter M, but are independent of the particular choice of basic dual functions $\{\phi_{N^*,N}, \phi_{N^*}\}$ such that $M = N + N^*$ [11]. Precisely

$$\beta^\nu(q) = \frac{d^\nu \theta_M}{dx^\nu}(q),$$

which also corresponds to the collocation scheme based on the interpolating scaling functions $\theta_M(x)$. It can also be proved that, under the posed conditions on $\mathcal{L}$, apart from the choice of the initial data, the numerical scheme (20) is also equivalent to the Galerkin method based on the orthogonal Daubechies' scaling functions supported on $[0, M-1]$ [2]. However, if the discrete operator $\mathcal{L}^j$ is represented in the multilevel context, then the formulations differ according to the considered multiresolution analysis.

3.1 Stability

Given the periodic conditions and the constant coefficients in $\mathcal{L}$, $\mathcal{K}^j$ results to be a circulant matrix. Therefore, the system (20) can be diagonalized by the Fourier matrix. This means that

$$\begin{aligned} \widehat{u}^{n+1,j}(k) &= \left[1 + \Delta_t \sum_{\nu=0}^{m} 2^{\nu j} \widetilde{\beta}^{(\nu)}(\xi_k^j)\right] \widehat{u}^{n,j}(k) = \widehat{\mathcal{K}}(\xi_k^j)\widehat{u}^{n,j}(k) \\ &= \left[\widehat{\mathcal{K}}(\xi_k^j)\right]^n \widehat{u}^{0,j}(k), \end{aligned} \tag{21}$$

where $\widehat{u}^{n,j}$ stands for the discrete Fourier transform of order 2^j of the numerical solution coefficients $u^{n,j}$, and

$$\widetilde{\beta}^{(\nu)}(\xi) = \sum_{s\in\mathbb{Z}} \beta^{(\nu)}(s) e^{-is\xi}, \quad \xi_k^j = 2\pi k 2^{-j}.$$

The symbol $\widehat{\mathcal{K}}(\xi) = \widehat{\mathcal{K}}(\xi, \Delta_t, 2^{-j})$ is the amplification factor, since its magnitude indicates how the amplitude $\widehat{u}^{n,j}(k)$ of each frequency present in the numerical solution is amplified during one time step.

As indicated in [3],Theorem 5.2.1, the scheme is stable if, and only if, there are constants C_e, α_e such that

$$|\widehat{\mathcal{K}}^n(\xi)| \le C_e e^{\alpha_e t_n}, \qquad \xi \in [0, 2\pi].$$

As an example, with $\mathcal{L}u = u_{xx}$ for the heat equation, consider

$$\widehat{\mathcal{K}}(\xi) = 1 + \tau\widetilde{\beta}^{(2)}(\xi),$$

where $\tau = 2^{2j}\Delta_t$. If τ is kept constant, them stability is attained if $|\widehat{\mathcal{K}}(\xi)| \leq 1$, that is,

$$\left|1 + \tau\widetilde{\beta}^{(2)}(\xi)\right| \leq 1, \qquad \xi \in [0, 2\pi]. \tag{22}$$

As shown in Figure 1(a), $\beta^{(2)}(\xi)$ is a negative function, with minimum value at $\xi = \pi$. Therefore, stability occurs if

$$0 < \tau \leq \frac{2}{\max\limits_{\xi\in[0,2\pi]} |\widetilde{\beta}^{(2)}(\xi)|} = \frac{2}{|\widetilde{\beta}^{(2)}(\pi)|} = \tau_{\max}. \tag{23}$$

In Table 1, the numerical values for $|\widetilde{\beta}^{(2)}(\pi)|$ are displayed for $M = 6, 8$ e 10 and the corresponding $\tau_{\max}$. The curves in Figures 1b-1d illustrate the behaviour of the symbol $|\widehat{\mathcal{K}}(\xi)|$, for $M = 6$, $M = 8$ and $M = 10$, with τ within the stability region (23).

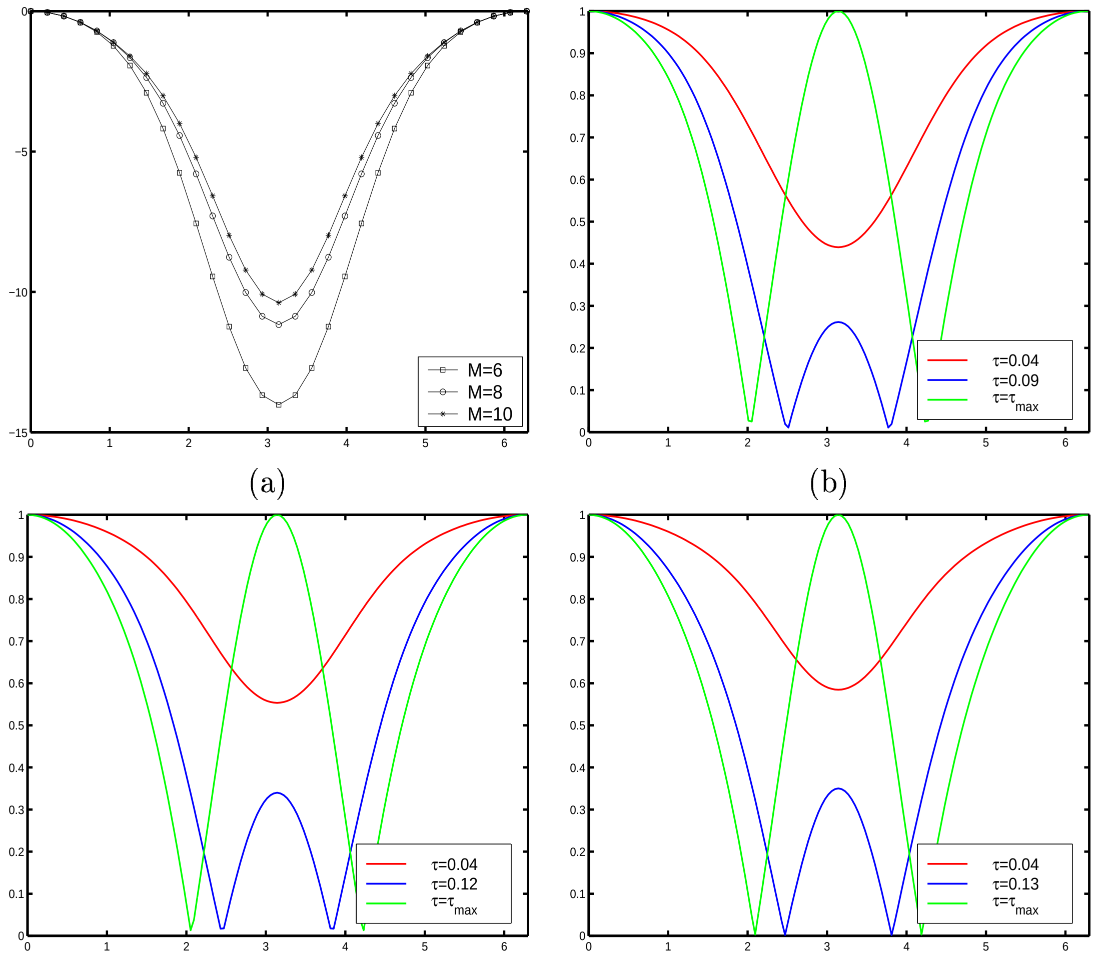
Table 1: $|\widetilde{\beta}^{(2)}(\pi)|$ and corresponding $\tau_{\max}$

M	$\lvert\widetilde{\beta}^{(2)}(\pi)\rvert$	$\tau_{\max}$
6	$\frac{1472}{105} \sim 14.019047619$	0.142663043
8	$\frac{1339264}{119945} \sim 11.165650923$	0.179120770
10	$\frac{21066447454208}{2028319032915} \sim 10.386160713$	0.192563937

3.2 Consistency

The truncation error is defined by

$$\begin{aligned}
(ET^{n,j})(k) &= \frac{1}{\Delta_t}\left[u(t_n + \Delta_t, x_k^j) - (\mathcal{K}^j u)(t_n, x_k^j)\right] \\
&= \frac{1}{\Delta_t}\left[u(t_n + \Delta_t, x_k^j) - u(t_n, x_k^j)\right] - (\mathcal{L}^j u)(t_n, x_k^j) \\
&= \left\{\frac{1}{\Delta_t}\left[u(t_n + \Delta_t, x_k^j) - u(t_n, x_k^j)\right] - \frac{\partial u}{\partial t}(t_n, x_k^j)\right\} \\
&+ \left[(\mathcal{L} - \mathcal{L}^j)u\right](t_n, x_k^j).
\end{aligned}$$



(a) (b) (c) (d)

Figure 1: (a) $\widetilde{\beta}^{(2)}(\xi), M = 6, 8, 10$; Symbol $|\widehat{\mathcal{K}}(\xi)|$ for: (b) $M = 6$; (c) $M = 8$; (d) $M = 10$.

It is the result of two kinds of dicretization errors. The first part being

$$\frac{1}{\Delta_t}\left[u(t_n + \Delta_t, x_k^j) - u(t_n, x_k^j)\right] - \frac{\partial u}{\partial t}(t_n, x_k^j)$$

gives the error in the time discretization by the Euler scheme, which is of first order. The second part

$$\left[(\mathcal{L} - \mathcal{L}^j)u\right](t_n, x_k^j)$$

is the truncation error in the discretization of $\mathcal{L}$ in the biorthogonal framework. As shown in [9, 4], its consistency is of order $M - m + \alpha$, where $\alpha = 0$ for even m, and $\alpha = 1$ otherwise.

Therefore,

$$|ET^{n,j}| = \mathcal{O}(\Delta_t) + \mathcal{O}(2^{-j(M-m+\alpha)}),$$

which means that the scheme is consistent of order $(1, M-m+\alpha)$. According to Lax-Richtmyer Equivalence Theorem [13], convergence holds in the stability region.

4 The Multilevel Scheme: Time-Adaptivity

In an biorthogonal multiresolution analysis framework, there is the possibility of representing the numerical solution $u_j(t,x)$ in a multilevel setting. Therefore, we may consider the idea of modifying the reference scheme in the same way as proposed in [1]. For a simplier and more clear development of the general concepts, and for the analysis of the method, we consider the one-dimensional heat equation. As described in the previous section, for stability of the scheme with spacial resolution 2^{-j}, we must choose the time step Δ_t^j in the stability region $\Delta_t^j \leq 4^{-j}\tau_{\max}$. At the next coarser level $j-1$, $\Delta_t^{j-1} = 4\Delta_t^j$. This means that, at level $j-1$, the solution can be updated at $t + \Delta_t^{j-1}$ by the expression

$$u_{j-1}(t+\Delta_t^{j-1}, x) = \left(I + \Delta_t^{j-1}\mathcal{L}^{j-1}\right) u_{j-1}(t,x). \tag{24}$$

At level j, the time step is four times smaller, an thus requires four iterations to update the solution at $t + \Delta_t^{j-1}$. That is,

$$u_j(t+\Delta_t^{j-1}, x) = u_j(t+4\Delta_t^j, x) = (I + \Delta_t^j\mathcal{L}^j)^4 u_j(t,x). \tag{25}$$

Using the two-level decomposition $V_j = V_{j-1} + W_{j-1}$, the component in W_{j-1} needs to be evolved with the time step Δ_t^j, but it is natural to consider the evolution of the component in V_{j-1} with time step Δ_t^{j-1}. Using the representation $\mathcal{P}^j = \mathcal{P}^{j-1} + \mathcal{Q}^{j-1}$, the dicretization $\mathcal{L}^j = \mathcal{P}^j\mathcal{L}\mathcal{P}^j$ may be decomposed as

$$\begin{aligned}
\mathcal{L}^j &= \mathcal{P}^{j-1}\mathcal{L}\mathcal{P}^{j-1} + \mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1} + \mathcal{Q}^{j-1}\mathcal{L}\mathcal{P}^{j-1} + \mathcal{Q}^{j-1}\mathcal{L}\mathcal{Q}^{j-1} \\
&= \mathcal{L}^{j-1} + \mathcal{T}^j,
\end{aligned} \tag{26}$$

where $\mathcal{L}^{j-1}$ is the discretization of $\mathcal{L}$ at level $j-1$ and $\mathcal{T}^j$ is the operator

$$\mathcal{T}^j = \mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1} + \mathcal{Q}^{j-1}\mathcal{L}\mathcal{P}^{j-1} + \mathcal{Q}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}$$

which acts on or returns detail components. Equations (24), (25) and (26) imply that

$$u_j(t+4\Delta_t^j, x) = [(I + \Delta_t^j\mathcal{T}^j) + \Delta_t^j\mathcal{L}^{j-1}]^4 u_j(t,x). \tag{27}$$

Following [1], we modify the scheme by neglecting high order terms involving powers greater than one of $\Delta_t^j\mathcal{L}^{j-1}$. The result is the modified scheme

$$
\begin{aligned}
u_j(t+4\Delta_t^j,x) &= [(I+\Delta_t^j\mathcal{T}^j)^4+4\Delta_t^j\mathcal{L}^{j-1}]u_j(t,x) \\
&=: \mathcal{K}_a^j u_j(t,x).
\end{aligned} \tag{28}
$$

For this modified scheme, the solution $u_j(t+4\Delta_t^j,x)$ at $t+4\Delta_t^j$ is obtained by the evolution of $\mathcal{L}^{j-1}u_j(t,x)$ with time step $\Delta_t^{j-1}=4\Delta_t^j$, while the components of higher scale level $\mathcal{T}^j u_j(t,x)$ uses the appropriate time step Δ_t^j.

4.1 Matrix Structure

The adaptive scheme (28) can be formulated in terms of the multiresolution coefficients of the numerical solution. To simplify the analysis, we shall consider the multiresolution analysis V_j defined by the interpolating scaling functions $\theta_M(x)$, with regularity $r\geq 2$. A function $v\in V_j$ may be expressed as

$$
\begin{aligned}
v(x) &= \sum_{k\in\mathbb{Z}} v^j(k)\theta(2^jx-k) \\
&= \sum_{k\in\mathbb{Z}} v^{j-1}(k)\theta(2^{j-1}x-k)+\sum_{k\in\mathbb{Z}} d^{j-1}(k)\psi(2^{j-1}x-k),
\end{aligned}
$$

where $v^j(k)=v(k2^{-j})$ and $d^{j-1}(k)=\mathcal{G}^{j-1}v(k)$. Developing each term in (26), we get

$$
(\mathcal{L}^{j-1}v)(x) = (\mathcal{P}^{j-1}\mathcal{L}\mathcal{P}^{j-1}v)(x) \tag{29}
$$

$$
= 2^{2j}\sum_{k\in\mathbb{Z}}\lambda_{0,0}(k)\theta(2^{j-1}x-k), \tag{30}
$$

where

$$
\lambda_{0,0}(k)=\frac{1}{4}\sum_{s\in\mathbb{Z}} v^{j-1}(s)\beta^{(2)}(s-k).
$$

Similarly, using the Mallat's formulas (11) and (12), we obtain

$$
\begin{aligned}
(\mathcal{T}^jv)(x) &= (\mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}v)(x)+(\mathcal{Q}^{j-1}\mathcal{L}\mathcal{P}^{j-1}v)(x)+(\mathcal{Q}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}v)(x) \\
&= 2^{2j}\left\{\sum_{k\in\mathbb{Z}}\lambda_{0,1}(k)\theta(2^{j-1}x-k)\right. \qquad (31) \\
&+ \left.\sum_{k\in\mathbb{Z}}[\lambda_{1,0}(k)+\lambda_{1,1}(k)]\psi(2^{j-1}x-k)\right\}, \qquad (32)
\end{aligned}
$$

where the coefficients are

$$\begin{aligned}
\lambda_{0,1}(k) &= \frac{1}{4}\sum_{s\in\mathbb{Z}} d^{j-1}(s)\vartheta^{(2)}(s-k),\\
\lambda_{1,0}(k) &= \frac{1}{4}\sum_{s\in\mathbb{Z}} v^{j-1}(s)\chi_1(s-k),\\
\lambda_{1,1}(k) &= \frac{1}{4}\sum_{s\in\mathbb{Z}} d^{j-1}(s)\chi_2(s-k),
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^{(m)}(k) &= \frac{d^m\psi}{dx^m}(k),\\
\chi_1(s) &= \sum_{n\in\mathbb{Z}} g^*(n)\beta^{(2)}(n/2+s),\\
\chi_2(s) &= \sum_{n\in\mathbb{Z}} g^*(n)\vartheta^{(2)}(n/2+s).
\end{aligned}$$

Therefore, $I+\Delta_t^j\mathcal{T}^j$ has the form

$$(I+\Delta_t^j\mathcal{T}^j v)(x) = \sum_{k\in\mathbb{Z}} a^{j-1}(k)\theta(2^{j-1}x-k) + \sum_{k\in\mathbb{Z}} b^{j-1}(k)\psi(2^{j-1}x-k),$$

with

$$a^{j-1}(k) = v^{j-1}(k) + \tau\lambda_{0,1}(k), \tag{33}$$

$$b^{j-1}(k) = d^{j-1}(k) + \tau(\lambda_{1,0}(k)+\lambda_{1,1}(k)). \tag{34}$$

To describe the action of the operator $I+\Delta_t^j\mathcal{T}^j$ in matrix form, we consider a vector containing the multilevel coefficients of $v(x)$, sorted in the following order

$$[v^{j-1}(0), d^{j-1}(0), \ldots, v^{j-1}(k), d^{j-1}(k), \ldots, v^{j-1}(2^{j-1}-1), d^{j-1}(2^{j-1}-1)]^T.$$

Let

$$[a^{j-1}(0), b^{j-1}(0), \ldots, a^{j-1}(k), b^{j-1}(k), \ldots, a^{j-1}(2^{j-1}-1), b^{j-1}(2^{j-1}-1)]^T,$$

be the corresponding vector for the multiresolution discrete values of $(I+\Delta_t^j\mathcal{T}^j)\mathbf{v}^j$. Bearing in mind the formulas (33)-(34), the convolution form of the expressions for $\lambda_{0,1}$, $\lambda_{1,0}$ and $\lambda_{1,1}$, while considering this kind of data sorting, then $(I+\Delta_t^j\mathcal{T}^j)$ has a block circulant structure, with 2×2 blocks. Namely, $I+\Delta_t^j\mathcal{T}^j = \mathrm{circ}(A_0, A_1, \ldots, A_{2^{j-1}-1})$, where

$$A_k = \begin{pmatrix} \delta_k & \vartheta(k) \\ \chi_1(k) & \delta_k + \chi_2(k) \end{pmatrix}.$$

Consequently, in the Fourier domain, $(I+\Delta_t^j\mathcal{T}^j)$ is transformed into a block diagonal matrix, with 2×2 blocks placed in the diagonal, as stated in the next lemma.

Lemma 4.1 Let $\widehat{a}^{j-1}$ and $\widehat{b}^{j-1}$ be the discrete Fourier transform of order 2^{j-1} of the coefficients $a^{j-1}(k)$ and $b^{j-1}(k)$ given in (33) and (34). Then

$$\begin{pmatrix} \widehat{a}^{j-1}(k) \\ \widehat{b}^{j-1}(k) \end{pmatrix} = \begin{pmatrix} 1 & \frac{\tau}{4}\widetilde{\vartheta}^{(2)}(2\xi_k^j) \\ \frac{\tau}{4}\widetilde{\chi}_1(2\xi_k^j) & 1+\frac{\tau}{4}\widetilde{\chi}_2(2\xi_k^j) \end{pmatrix} \begin{pmatrix} \widehat{v}^{j-1}(k) \\ \widehat{d}^{j-1}(k) \end{pmatrix},$$

where

$$\begin{aligned}
\widetilde{\chi}_1(\xi) &= \overline{G_P^*(\xi)}\widetilde{\beta}^{(2)}(\xi) + \overline{G_I^*(\xi)}\widetilde{\beta}^{(2)}_{1/2}(\xi), \\
\widetilde{\chi}_2(\xi) &= \overline{G_P^*(\xi)}\widetilde{\vartheta}^{(2)}(\xi) + \overline{G_I^*(\xi)}\widetilde{\vartheta}^{(2)}_{1/2}(\xi), \\
\widetilde{\beta}^{(2)}(\xi) &= \sum_{s\in\mathbb{Z}} \beta^{(2)}(s)e^{-i\xi s}, \\
\widetilde{\vartheta}^{(2)}(\xi) &= \sum_{s\in\mathbb{Z}} \vartheta^{(2)}(s)e^{-i\xi s}, \\
G_P^*(\xi) &= \sum_{s\in\mathbb{Z}} g^*(2s)e^{-i\xi s}, \\
G_I^*(\xi) &= \sum_{s\in\mathbb{Z}} g^*(2s+1)e^{-i\xi ks}.
\end{aligned}$$

As a consequence of Lemma 4.1, the following results hold for the adaptive operator $\mathcal{K}_a^j = (I+\Delta_t^j\mathcal{T}^j)^4 + 4\Delta_t^j\mathcal{L}^{j-1}$.

Corollary 4.2 Consider the vector $\widehat{\mathbf{u}}_{MR}^{n,j}$ formed by the components of the discrete Fourier transforms $\widehat{\mathbf{u}}^{n,j-1}$ and $\widehat{\mathbf{d}}^{n,j-1}$, sorted in the following order

$$[\widehat{u}^{n,j-1}(0), \widehat{d}^{n,j-1}(0), \widehat{u}^{n,j-1}(1), \widehat{d}^{n,j-1}(1)\ldots, \widehat{u}^{n,j-1}(2^{j-1}-1), \widehat{d}^{n,j-1}(2^{j-1}-1)]^T.$$

On this form, the action of the operator $\mathcal{K}_a^j = (I+\Delta_t^j\mathcal{T}^j)^4 + 4\Delta_t^j\mathcal{L}^{j-1}$, corresponding to the adaptive scheme (28), can be expressed by the formula

$$\widehat{\mathbf{u}}_{MR}^{n+1,j} = \widehat{\mathcal{K}}_a^j \widehat{\mathbf{u}}_{MR}^{n,j},$$

where $\widehat{\mathcal{K}}_a^j$ is a block diagonal matrix, with 2×2 blocks in the diagonal, defined by $\widehat{\mathcal{K}}_a(2\xi_k^j,\tau)$, $0 \le k \le 2^{j-1}-1$, such that

$$\begin{pmatrix} \widehat{u}^{n+1,j-1}(k) \\ \widehat{d}^{n+1,j-1}(k) \end{pmatrix} = \widehat{\mathcal{K}}_a(2\xi_k^j,\tau) \begin{pmatrix} \widehat{u}^{n,j-1}(k) \\ \widehat{d}^{n,j-1}(k) \end{pmatrix}.$$

The amplification matrices $\widehat{\mathcal{K}}_a(\xi;\tau)$ have the formula as follows

$$\widehat{\mathcal{K}}_a(\xi,\tau) \;=\; \left[\begin{pmatrix} 1 & \frac{\tau}{4}\widetilde{\vartheta}^{(2)}(\xi) \\ \frac{\tau}{4}\widetilde{\chi}_1(\xi) & 1+\frac{\tau}{4}\widetilde{\chi}_2(\xi) \end{pmatrix}^4 + \tau\begin{pmatrix} \widetilde{\beta}^{(2)}(\xi) & 0 \\ 0 & 0 \end{pmatrix}\right].$$

4.2 Stability Analysis

As described in the section 3.1, the stability region for the reference scheme is $\tau \leq \tau_{\max}$. The question here is to see whether the same characterization holds for the adaptive scheme.

As proved in Corollary 4.2, the adaptive scheme (28) can be formulated, in the Fourier domain, by the relation

$$\widehat{\mathbf{u}}^{n+1,j}_{MR} = \widehat{\mathcal{K}}^j_a \widehat{\mathbf{u}}^{n,j}_{MR}.$$

Therefore, the stability analysis can be stated in terms of spectral properties of the amplification matrix $\widehat{\mathcal{K}}_a$. As described in [3], Theorem 5.2.2, a necessary condition for stability is that the eigenvalues ν_k of $\widehat{\mathcal{K}}_a$ satisfy the von Neumann condition

$$|\nu_k| \leq e^{\alpha_e \Delta_t}. \tag{35}$$

Theorem 5.2.3 in [3] shows that such condition is sufficient in the case where $\widehat{\mathcal{K}}_a$ can be uniformly diagonalized in the sense that there is a matrix $T = T(2^{-j},\xi)$ such that

$$T^{-1}\widehat{\mathcal{K}}_a T = diag(\nu_1, \nu_2, \ldots, \nu_{2^j}), \tag{36}$$

with $\|T\|\,\|T^{-1}\| \leq C$, for C independent of ξ and the resolution level j.

For the adaptive scheme under study, $\widehat{\mathcal{K}}^j_a$ is a block diagonal matrix, with 2×2 blocks $\widehat{\mathcal{K}}_a(2\xi^j_k;\tau)$, $0 \leq k \leq 2^{j-1}-1$ placed in the diagonal. Therefore, we simply need to analyse the spectral properties of such blocks.

The eigenvalues $\nu_{k,\ell},\ \ell = 1,2$ of each block $\widehat{\mathcal{K}}_a(\xi^j_k,\tau)$ may be obtained directly from the block entries, and the spectral radius $\rho(\widehat{\mathcal{K}}_a(\xi,\tau))$ can be expressed as a function of ξ and τ. Numerical experiments show that, for fixed τ, the maximum of $\rho(\widehat{\mathcal{K}}_a(\xi;\tau))$ occurs at $\xi = \pi$. In Figure 2 the graphs of $\rho(\widehat{\mathcal{K}}_a(\pi;\tau))$ are plotted as functions of τ, for $M = 6, 8, 10$ e 12. In each case, it and can be noticed that, for $\tau > \tau_{\max}$, $\rho(\widehat{\mathcal{K}}_a(\pi,\tau)) > 1$. Therefore, for $\tau > \tau_{max}$ the adaptive scheme, as well as its reference scheme, is unstable.

Figure 3 displays level sets for the spectral radius $\rho(\widehat{\mathcal{K}}_a(\xi;\tau))$, for $M = 6, 8, 10, 12$. The horizontal red dotted line indicates $\tau = \tau_{\max}$, which is the upper level for stability. As expected, for $\tau \leq \tau_{\max}$, the necessary condition $\rho(\widehat{\mathcal{K}}_a(\xi,\tau)) \leq 1$ is verified. We can also note that $\xi = \pi$ is a critical point in the sense that, for each level curve, the minimum value for τ is reached at $\xi = \pi$.

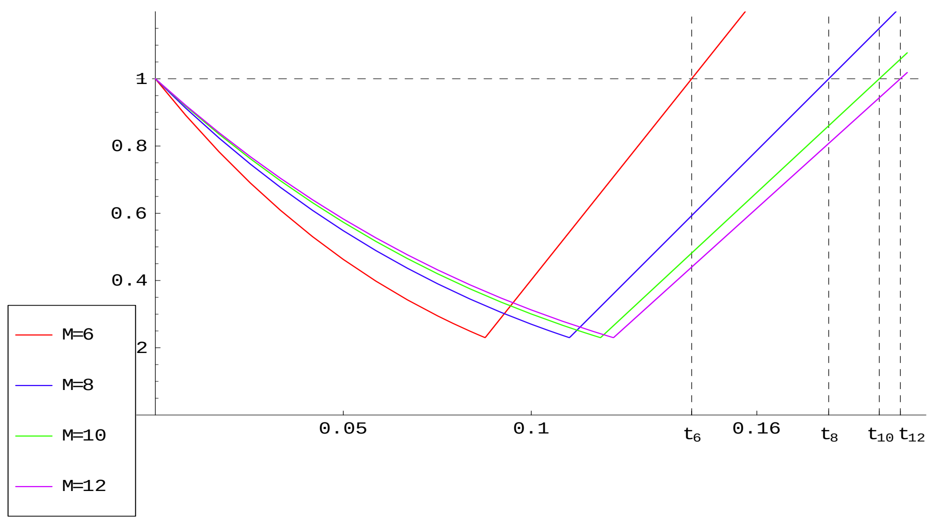


Figure 2: The spectral radius $\rho(\widehat{\mathcal{K}}_a(\pi;\tau))$.

Figure 4 is for fixed $\tau = \tau_{\max}$. It shows the behaviour of the eigenvalues of $\widehat{\mathcal{K}}_a(\xi, \tau_{\max})$, for different choices of the parameter M. As expected,

$$|\nu_{k,\ell}| \leq 1 \quad \text{for } \ell = 1, 2,$$

which shows that the necessary condition (35) for the stability is verified with $\alpha_e = 0$.

¿From Figure 4, we also conclude that the eigenvalues are distinct for $\xi \in (0, 2\pi)$. Similar behaviour holds for $\tau \leq \tau_{\max}$. This fact indicates that the matrix T mentioned in (36) may be taken as a block diagonal matrix, with 2×2 blocks formed by eigenvectors of $\hat{\mathcal{K}}_a(2\xi_k^j, \tau)$. For $\xi = 0$ and $\xi = 2\pi$, $\widehat{\mathcal{K}}_a(\xi)$ is simply the 2×2 identity matrix. The behaviour of the Euclidian norm $\|T(\xi, \tau)\|_2$ is illustrated in Figure 5, for $M = 6, 8, 10$ e 12. It suggests that, for $\xi \in [0, 2\pi]$ and $0 \leq \tau \leq \tau_{\max}$, $\|T(\xi, \tau)\|_2$ is a bounded function which is also bounded away from zero. Therefore, based on this numerical evidence, we argue that the condition on T that guarantees stability is verified by the adaptive scheme.

4.3 Consistency

Let us consider the truncation error for the adaptive scheme

$$4\Delta_t (ET_a^{n,j})(s) = u(t_n + 4\Delta_t, x_s^j) - (\mathcal{K}_a^j u)(t_n, x_s^j),$$

where $\mathcal{K}_a^j = (I + \Delta_t^j \mathcal{T}^j)^4 + 4\Delta_t^j \mathcal{L}^{j-1}$. It can be split into three terms

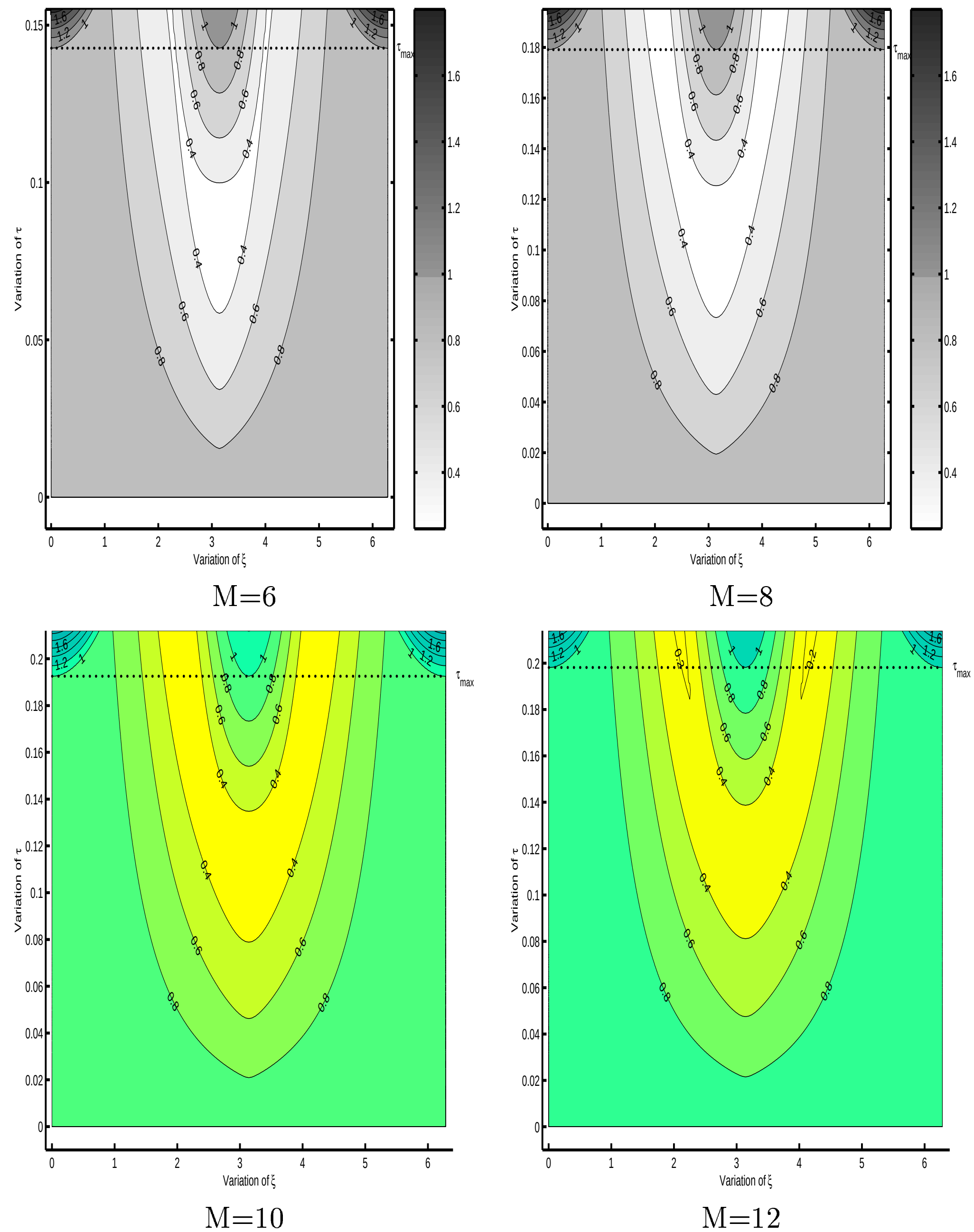


Figure 3: Spectral radius $\rho(\widehat{\mathcal{K}}_a(\xi;\tau))$, for $\xi \in [0, 2\pi]$

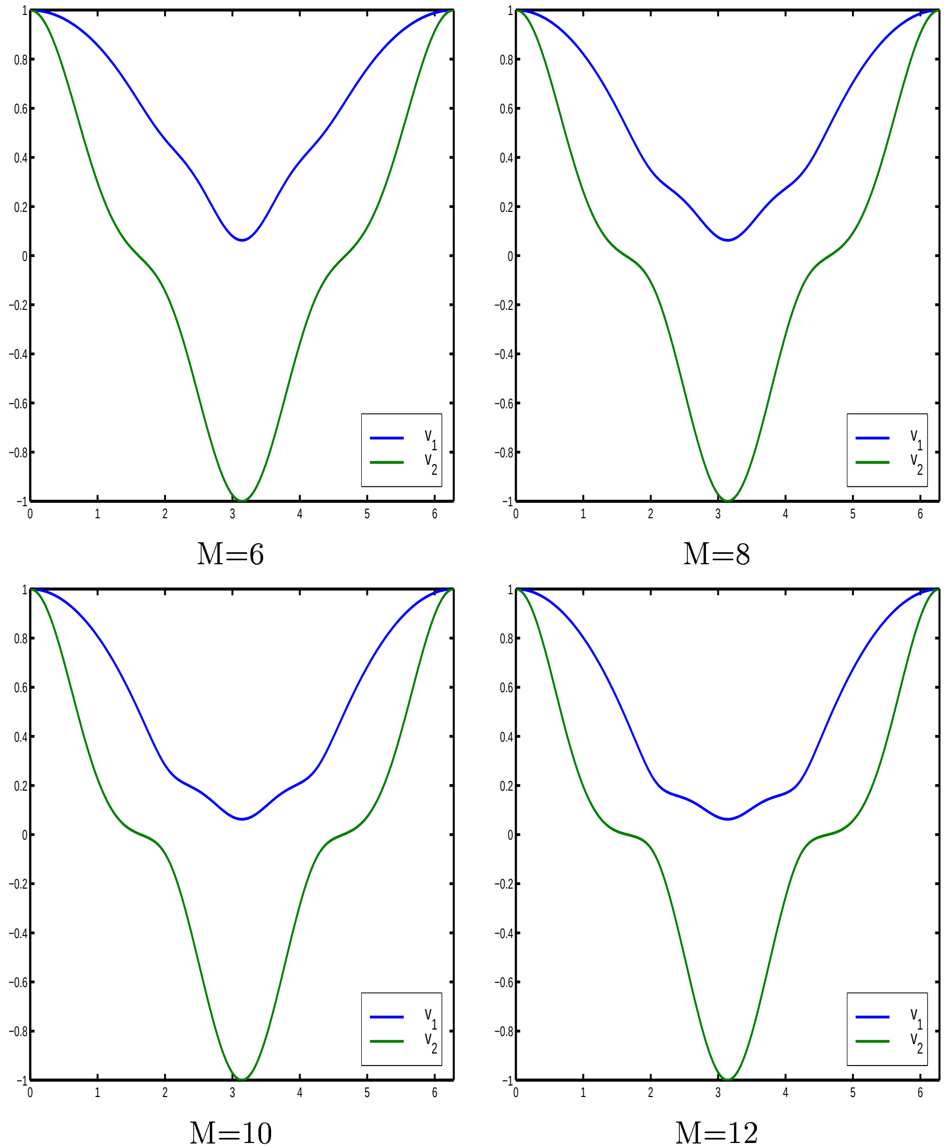
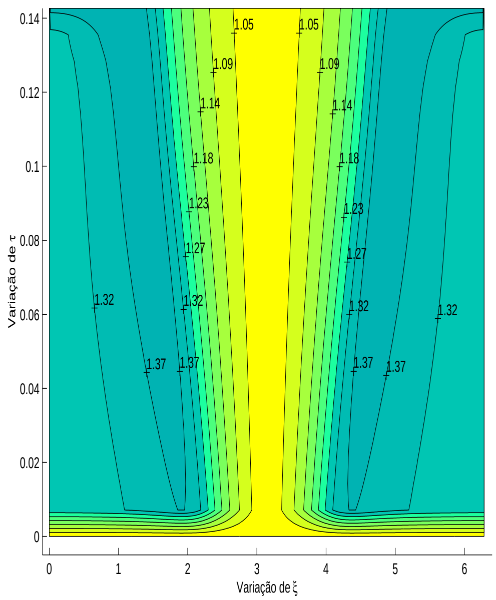
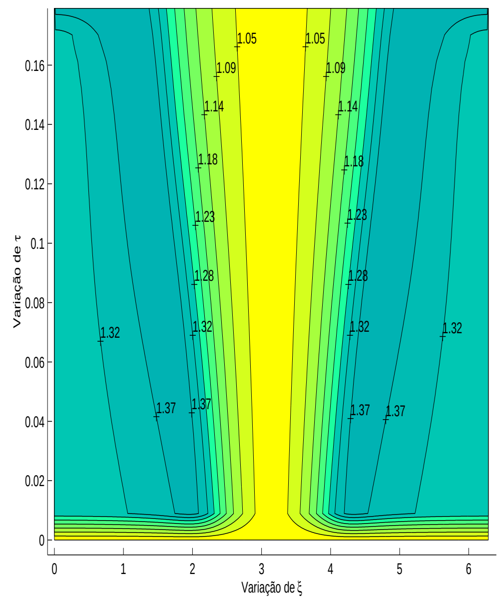


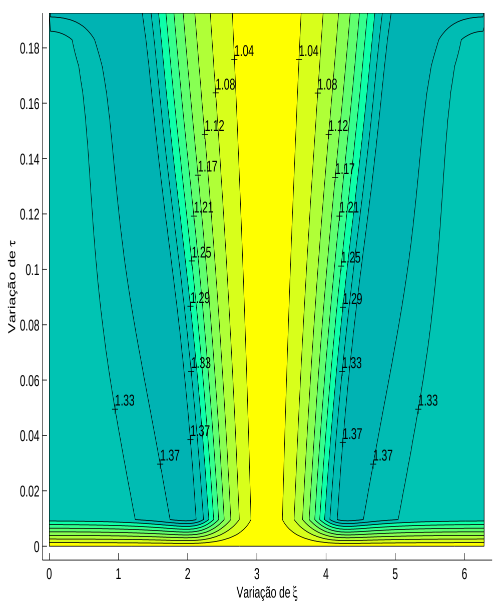
Figure 4: Eigenvalues for $\widehat{\mathcal{K}}_a(\xi, \tau_{\max})$



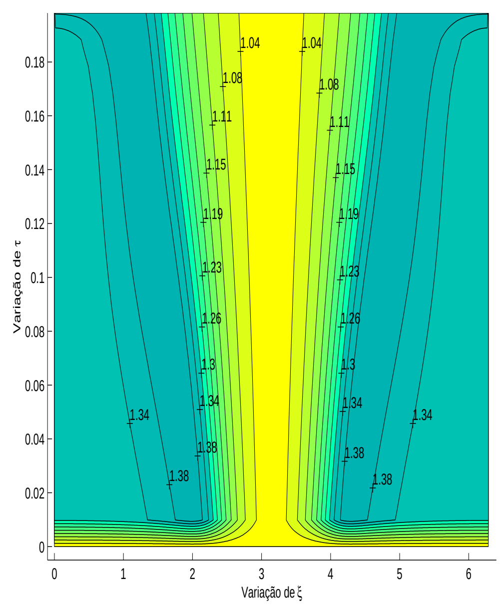
M=6



M=8



—M=10



M=12

Figure 5: $\|T(\xi,\tau)\|_2$ for $\xi \in [0,2\pi]$ and $\tau \in [0,\tau_{\max}]$

$$
\begin{aligned}
(ET_a^{n,j})(s) &= \left\{\frac{1}{4\Delta_t}\left[u(t_n+4\Delta_t, x_s^j) - u(t_n, x_s^j)\right] - \frac{\partial u}{\partial t}(t_n, x_k^j)\right\} \\
&+ \left\{(\mathcal{L}u - \mathcal{L}^j u)(t_n, x_k^j)\right\} \\
&- \frac{1}{4}\left[6\Delta_t(\mathcal{T}^j)^2 + 4\Delta_t^2(\mathcal{T}^j)^3 + \Delta_t^3(\mathcal{T}^j)^4\right] u(t_n, x_s^j). \qquad (37)
\end{aligned}
$$

The first and second terms form the truncation error for the reference scheme. As we shall prove next, the perturbation introduced by the adaptive strategy, corresponding to the third term, produces errors or higher orders.

We shall give the estimates in terms of the norm in $\mathbf{L}^2([0,1])$, since in V_j it is equivalent to norm $\|\cdot\|_j$ of the coefficients. Given the definition of $\mathcal{T}^j$, we get

$$
\|\mathcal{T}^j u\|_{\mathbf{L}^2} \leq \|\mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2} + \|\mathcal{Q}^{j-1}\mathcal{L}\mathcal{P}^{j-1}u\|_{\mathbf{L}^2} + \|\mathcal{Q}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2}.
$$

Having in mind the estimates (14) and (15) for the projections occurring in the interpolatory multiresolution context defined by $\theta_M(x)$ $(p = M-1)$, we obtain

$$
\begin{aligned}
\|\mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2} &\leq \|\mathcal{P}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}u - \mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2} + \|\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2} \\
&\lesssim 2^{-(j-1)s}\|\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{H}^s} + \|\mathcal{Q}^{j-1}u\|_{\mathbf{H}^2} \\
&\lesssim 2^{-(j-1)s}\|\mathcal{Q}^{j-1}u\|_{\mathbf{H}^{s+2}} + 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M} \\
&\lesssim 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M},
\end{aligned}
$$

where $0 \leq s \leq r-2$, and r is the regularity order of θ_M. Similarly, for the second term we have

$$
\begin{aligned}
\|\mathcal{Q}^{j-1}\mathcal{L}\mathcal{P}^{j-1}u\|_{\mathbf{L}^2} &\leq \|\mathcal{Q}^{j-1}\mathcal{L}(\mathcal{P}^{j-1}u - u)\|_{\mathbf{L}^2} + \|\mathcal{Q}^{j-1}\mathcal{L}u\|_{\mathbf{L}^2} \\
&\lesssim 2^{-(j-1)s}\|\mathcal{L}(\mathcal{P}^{j-1}u - u)\|_{\mathbf{H}^s} + 2^{-(j-1)(M-2)}\|\mathcal{L}u\|_{\mathbf{H}^{M-2}} \\
&\leq 2^{-(j-1)s}\|\mathcal{P}^{j-1}u - u\|_{\mathbf{H}^{s+2}} + 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M} \\
&\lesssim 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M}.
\end{aligned}
$$

Finally, for the last term

$$
\begin{aligned}
\|\mathcal{Q}^{j-1}\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{L}^2} &\lesssim 2^{-(j-1)s}\|\mathcal{L}\mathcal{Q}^{j-1}u\|_{\mathbf{H}^s} \\
&\leq 2^{-(j-1)s}\|\mathcal{Q}^{j-1}u\|_{\mathbf{H}^{s+2}} \\
&\lesssim 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M}.
\end{aligned}
$$

Combining all the three estimations, we get

$$
\|\mathcal{T}^j u\|_{\mathbf{L}^2} = 2^{-(j-1)(M-2)}\|u\|_{\mathbf{H}^M}.
$$

Therefore, we conclude that in the adaptive scheme the order of truncation error is determined by behaviour of the truncation error in the reference scheme. That is,

$$\|ET_a^{n,j}\| \leq \mathcal{O}(\Delta_t) + \mathcal{O}(2^{-j(M-2)}).$$

The results of Section 4.2 and Section 4.3 are summarized in the following theorem.

Theorem 4.3 For the examples analysed in this paper, the adaptive scheme (28) present the same stability and consistency properties of its reference scheme (20).

5 Conclusions

In this paper, the adaptive multilevel scheme proposed in [1] is considered in the biorthogonal wavelet context. The adaptive scheme is formulated in terms of the multiresolution coefficients of the numerical solution. Using two levels, and having periodic boundary conditions and constant coefficients, it turns out that the matrix making the connection between the solution at one time step to the next one has a 2×2 block circulant structure. So that, in the Fourier domain, it is transformed into a block-diagonal structure, with 2×2 blocks. Therefore, the stability of the scheme is determined by the spectral properties of each of these simple blocks, which can be derived from the behaviour of an easely computable function of two variables $(\tau, \xi), \tau \geq 0, 0 \leq \xi \leq 2\pi$. This is because the four components of each block are expressed as functions of the parameter $\tau = \tau(\Delta_t, 2^{-j})$ and the sample values of known 2π-periodic functions, which are defined in terms of the scaling filters and symbols associated to the finite difference coefficients. The dependence on the scale level appears, explicitely, on the sampling step.

As a model problem, we consider the one-dimensional heat equation. In the adaptive multilevel scheme, the advantage is that the time step used to update the components in a certain scale level is multiplied by four if the resolution is decrased by a factor of two. We show that this adaptive strategy does not affect the stability condition and consistency order, which are maintaned the same as in the original reference scheme.

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