Asymptotic angular stability in non-linear systems: rotation numbers and winding numbers

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Abstract

The asymptotic angular stability of a dynamical system may be quantified by its rotation number or its winding number. These two quantities are shown to result from different assumptions, made about the flow generating the Poincaré map which results from the sequence of homeomorphisms in S^1. An ergodic theorem of existence a.s. of the rotation number for non-linear systems is given. The advantages and disadvantages of both the rotation and winding numbers are discussed. Numerical calculations of the distribution of rotation number and winding number arising from different initial conditions are presented for three different chaotic maps.

1 Introduction

In Ruffino (2000) we introduced the concept of rotation number for a composition of stationary random homeomorphisms which preserve orientation in S^1, hence extending the definition of Ruelle (1985) for a product of random matrices. An ergodic theorem of existence almost surely (a.s.) of this rotation number is proven here for non-linear systems. We demonstrate the consistency of our concept (see Ruffino, 2000) by showing that this definition is adequate for application to continuous linear processes in $\mathbb{R}^2$ which have been discretely sampled in time. We stated a random Nyquist theorem (see e.g. Oppenheim and Schafer, 1989; Papoulis, 1984) for linear processes in $\mathbb{R}^2$: the rotation number (normalised by $1/T$) of the discrete sampled process with period T converges almost surely to the rotation number of the original continuous process when T goes to zero. In simpler words, the stroboscopic observation of the system can retrace its original frequency a.s. if the frequency of observation is high enough. In this paper we deal with the open question arising in Ruffino (2000) concerning the application of our concept to non-linear maps in the plane with an invariant probability measure. In particular we investigate a number of chaotic systems.

The following provides a mathematical description of the problem. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a local diffeomorphism with μ a Borel invariant probability measure on $\mathbb{R}^2$ for the map F. It is well known that in this case the asymptotic radial stability is determined by the Lyapunov exponents of the product of the derivatives $dF(x)$ of F along the trajectory of an initial point p:

$$dF^{(n)}(p) = dF(F^{(n-1)}(p)) \circ \dots dF(p). \tag{1}$$

The existence μ–almost surely of these characteristic exponents is the content of the celebrated multiplicative ergodic theorem, due originally to Oseledets (1968). This theorem corresponds to a stochastic linear algebra for the product of random matrices. For a detailed description of the applications of this result to random dynamical systems we refer the reader to Arnold (1998) and the references therein, in particular we mention the classical article by Ruelle (1979).

In this paper we study the asymptotic angular behaviour, which provides the counterpart of the multiplicative ergodic theorem. This asymptotic angular behaviour, which is induced by a sequence of linear operators (derivatives of F), may be quantified either by its *rotation number* or its *winding number*. In this paper, we develop a general theory of rotation numbers and winding numbers for a sequence of homeomorphisms in the circle S^1. While the winding number is more intuitive and likely to suffice for most applications in physics or applied mathematics, there are some cases where it may not be the correct quantity to measure. We clarify the differences between the rotation number and winding number using simple examples and discuss their advantages and disadvantages.

This paper is organised as follows: in sections 2 and 3 we define the rotation number and winding number for a sequence of homeomorphisms. In section 4 we state the main results from the theory of rotation numbers presented in Ruffino (2000). At the end of this section we compare our definition with that given in Ruelle (1985). In section 5 we present several examples, some of them with numerical simulations to illustrate the physical meaning of both the rotation and winding numbers.

2 The rotation number

We use Homeo(S^1) to denote the space of homeomorphisms in the circle S^1, Homeo$^+(S^1)$ for the group of orientation preserving maps and Homeo$^-(S^1)$ for those which reverse the orientation. We recall the classical definition of the rotation number of a fixed homeomorphism $f \in$ Homeo$^+(S^1)$. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f, i.e., if $\pi : \mathbb{R} \to S^1$ is the canonical projection $\pi(x) = \exp\{2\pi i x\}$ then $f \circ \pi = \pi \circ F$. We consider the argument of a complex number such that $\arg(\pi(x)) = x \pmod 1$. One easily sees the following facts: a) F is an increasing function; b) if F is a lift, then $F + k$, with $k \in \mathbb{Z}$, is another lift; c) if Id denotes the identity in $\mathbb{R}$ then $F - Id$ is periodic with period 1. The rotation number $\rho(f)$ of f is the asymptotic average velocity of the evolution of the dynamics of a lift F in the space state $\mathbb{R}$, i.e.:

$$\rho(f) = \lim_{n\to\infty} \frac{F^{(n)}(x) - x}{n} \qquad (\mathrm{mod}1), \tag{2}$$

where $F^{(n)}$ is the iteration of F n–times. It is well known that the limit always exists and is independent of the starting point $x \in \mathbb{R}$. For these and other properties see Nitecki (1971) for example. The rotation number for a sequence of elements in Homeo$^+$ is defined, analogously, as the asymptotic average velocity of the evolution of the dynamics of a lift F in the universal covering space.

3 The oriented winding number

Let $f_1, f_2, \ldots : S^1 \to S^1$ be a sequence of homeomorphisms in the circle (here we consider a deterministic sequence of maps, no randomness is assumed). Given a fixed initial point $p \in S^1$, one of the most natural ways of calculating the average rotation of the orbit of p around S^1 is to construct a monotone non-decreasing sequence of real numbers which provides the angle of the orbit at each step $n \in \mathbb{N}$. That is, let us define inductively the sequence $(\theta_n^p)_{n\geq 1}$ by $\theta_1^p = \arg(p)$ and $\theta_n^p \geq \theta_{n-1}^p$ such that $\pi(\theta_n^p) = f_n \circ \ldots f_1(p)$.

The anti-clockwise winding number is then defined by the average rate of increasing of this ordered, lifted to $\mathbb{R}$, orbit of p:

$$w_{f_1,f_2,\ldots}(p) = \lim_{t\to\infty} \frac{\theta_n^p - \theta_1^p}{n},$$

when the limit exists. It is known (see Ruffino, 2000, Prop. 2.1) that the winding number generalises the classical definition of rotation number in the case of a constant sequence of orientation preserving homeomorphism $f : S^1 \to S^1$, that is, if $f_n \equiv f$ for all $n \in \mathbb{N}$, then , the rotation number $\rho(f)$ of equation (2) equals the winding number $w_f(p)$ for all $p \in S^1$ (in this case we simplify the notation to w_f). Example 1 in section 5 shows that for a non-constant sequence these two concepts may differ.

The next theorem states some others properties of the winding number for the iteration of a single homeomorphism f as above.

Theorem 3.1 *Winding number satisfies the following properties:*

a) *If* $f \in \mathrm{Homeo}^-(S^1)$ *then there are 2 distinct points* $a, b \in S^1$ *such that* $w_f(a) = w_f(b) = 0$ *and for all* $p \in S^1 \setminus \{a, b\}$ *we have* $w_f(p) = 1/2$.

b) *If* $f \in \mathrm{Homeo}^-(S^1)$ *and* $\varphi \in \mathrm{Homeo}(S^1)$ *then* $w_{\varphi^{-1}\circ f\circ\varphi}(p) = w_f(\varphi^{-1}(p))$.

c) *If* $f \in \mathrm{Homeo}^+(S^1)$ *then* $w_{\varphi^{-1}\circ f\circ\varphi} = \pm w_f$ *according to* $\varphi \in \mathrm{Homeo}^{\pm}(S^1)$.

d) *If* $f, g \in \mathrm{Homeo}^+(S^1)$ *commute then* $w_{f\circ g} = w_f + w_g \pmod 1$.

Proof:

For (a), note that a lift of a orientation reversing homeomorphism is a continuous monotone decreasing function on $\mathbb{R}$, hence, in the interval $[0, 1]$ it must cross the Id (mod1) exactly twice (and at different points), say at the values corresponding to the angle of the points a and b. Once they are fixed points of f we have $w_f(a) = w_f(b) = 0$. These two points disconnect the circle S^1 into two open arcs, say U and V (see figure 1). The lift of f also shows that the dynamic of f is simply flipping points from the disconnected open arcs U to V and vice-versa. Hence, for a point $p \in S^1 \setminus \{a, b\}$, say in U, after $2n$ iteration of f we have $\theta^p_{2n} = n + \delta_n$ where δ_n is bounded by the length of U. Hence $w_f(p) = 1/2$. Item (b) is a direct corollary of (a).

Item (c) follows from the facts that if $f \in \mathrm{Homeo}^+(S^1)$ then $w_f(p) = \rho(f)$ for every point $p \in S^1$ and the invariance of the rotation number $\rho(f)$ by conjugacy (see Ruffino, 2000, Lemma 4.1 and remark afterwards). Item (d) follows by the same property of rotation numbers of a single homeomorphism.

□

The advantage of the winding number compared with the rotation number for sequences of homeomorphisms (see section 4) is, besides its simplicity and intuitive appeal, the fact that it makes sense for general homeomorphisms, even if they do not preserve orientation. On the other hand, the disadvantage is the fact that in general the winding number depends strongly on the initial point p. Moreover, though intuitive, in the case of a reversing-orientation homeomorphism, a flipping dynamics (which gives a winding number equal to $1/2$ except in 2 points) does not correspond properly to any physical rotation.

4 Theory of rotation numbers

In this section we recall the main theoretical results concerning rotation numbers as introduced in Ruffino (2000). Let $f_1, f_2, \ldots : S^1 \to S^1$ be a sequence in Homeo$^+(S^1)$. Let $F_1, F_2, \ldots : \mathbb{R} \to \mathbb{R}$ be the corresponding lifts such that $F_n(0) \in (-1/2, 1/2]$. The rotation number of this sequence of homeomorphisms is:

$$\rho(f_1, f_2, \ldots) = \lim_{n\to\infty} \frac{F_n \circ \ldots F_1(x) - x}{n} \pmod{1}, \tag{3}$$

when the limit exists. This obviously extends the classical definition of rotation number for a constant sequence $f_n \equiv f$ for all $n \in \mathbb{N}$. Moreover, the rotation number, in contrast to the winding number, does not depend on the initial point $x = \arg(p)$ for $p \in S^1$, (see Ruffino, 2000, Prop. 3.1).

[Figure 1 about here.]

We use a geometrical illustration (see figure 1) to demonstrate the dynamical difference between the rotation number and winding number for a sequence of homeomorphisms in $Homeo^+(S^1)$. Essentially, note that the sequence of homeomorphisms represent the Poincaré map of a (time dependent) flow which may be reconstructed according to a particular rule: the rotation number employs one rule while the winding number uses a different one.

Suppose, for instance, that the sequence of homeomorphisms $(f_n)_{n\geq 1}$ is such that f_n is a rotation by $\pi/6$ if n is odd and f_n is a rotation by $-\pi/6$ if n is even. The rotation number (see figure 1a) is obtained by employing the rule that the flow between consecutive integers carries e_1 to the interval $(-1/2, 1/2]$. Alternatively (see figure 1b) the winding number is obtained by using the rule that the trajectory of the starting point rotates anti-clockwise (independently of what happens to the other trajectories). One easily verifies that in this example, the rotation number is $\rho = 0$ and the winding number is $\omega = 1/2$.

The next two theorems summarise the main properties of rotation number. To establish a nomenclature, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta : \Omega \to \Omega$ be an ergodic transformation. Given a random homeomorphism $f : \Omega \to \mathrm{Homeo}^+(S^1)$ with distribution μ in $\mathrm{Homeo}^+(S^1)$, let ν be the invariant measure in S^1 for the action of a sequence of elements i.i.d. with the same distribution of f, that is, ν is invariant by convolution $\mu * \nu = \nu$. Let $F(\omega, x)$ be the lift of $f(\omega, s)$ and consider $\beta(\omega, s) \in (-1/2, 3/2]$ the function which measures the angular displacement induced by $f(\omega, s)$, that is, $\beta(\omega, s)$ is the unique function in this interval such that

$$F(\omega, x) = x + \beta(\omega, \pi(x))$$

Now, considering the sequence of random stationary homeomorphisms $(f \circ \theta^{n-1})_{n \geq 1}$, we have the following ergodic theorem of existence.

Theorem 4.1 (Ergodic theorem for rotation number) *The rotation number $\rho(f, \theta)$ of the sequence $(f \circ \theta^{n-1})_{n \geq 1}$ exists and is constant $\mathbb{P}$-a.s.. Precisely:*

$$\rho(f, \theta) = \mathbb{E}\left[\int_{S_1} \beta(\omega, s) \ d\nu(s) \ \right] \quad \mathbb{P} - a.s..$$

Note that ergodicity of ν is not relevant in this case since the rotation number, if it exists, is constant in S^1. The next theorem shows that the concept of rotation number proposed in (3) (in contrast to the winding number) makes sense from a physical perspective. The classical version of this theorem is known as Nyquist theorem which states that: given a real signal (function) $s(t), t \in \mathbb{R}$, if f_0 is the maximum frequency of its Fourier spectrum, then, the whole signal can be retraced if we sample in time this signal at a frequency greater than $2f_0$. This frequency is called the Nyquist rate (see e.g. Oppenheim and Schafer, 1989; Papoulis, 1984). We shall state the theorem for the stochastic case; for an alternative proof of the particular case of deterministic systems or for the general case of random dynamical systems (whose proofs use the same technique), see Ruffino (2000, Thm. 5.1 and 5.2).

Consider the following Stratonovich stochastic linear system in $\mathbb{R}^2$:

$$dx_t = Ax_t \ dt + \sum_{i=1}^{m} B^i x_t \ \circ dW_t^i \quad , \tag{4}$$

where $A, B^1, \ldots, B^m$ are 2×2–matrices, $(W_t^1, \ldots, W_t^m)_{t \geq 0}$ are m-independent linear Brownian motions with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let

$\varphi(\omega, t)$ be the stochastic linear solution flow of the equation above. It is well known, see e.g. Ruffino (1997), that the rotation number of φ is given by:

$$\rho(\varphi) = \int_{S^1} f(s)\ d\nu(s) \qquad a.s.,$$

where μ is the invariant probability measure of the induced process in S^1. In this situation, the stochastic Nyquist theorem states that the rotation number of the discrete sampled in time at intervals T of the trajectories of any point $0 \neq x \in \mathbb{R}^2$ converges to the rotation number of the continuous system up to a parameter $1/T$ which normalise the angular velocity. We recall the co-cycle property that $\varphi(\omega, nT) = \varphi(\theta_{(n-1)T}(\omega), T) \circ \ldots \varphi(\theta_T(\omega), T) \circ \varphi(\omega, T)$ where $\theta_s : \Omega \to \Omega$ is the ergodic transformation given by the shift in the noise.

Theorem 4.2 (Stochastic Nyquist theorem) *The rotation number for the sampled (in time) system $\rho(\varphi(T, \omega), \theta_T)$ of the sequence of random matrices $\{\varphi(T, \theta_{T(n-1)}(\omega)\ \}_{n \geq 1}$ satisfies*

$$\lim_{T \to 0} \frac{1}{T}\ \rho(\varphi(T, \omega), \theta_T) \ = \ \rho(\varphi) \qquad a.s..$$

4.1 Comments on Ruelle's rotation number

Ruelle (1985) constructed a definition of rotation number for the particular case of a product of a sequence of random matrices in the semi-simple Lie group $G = Sl^+(2, \mathbb{R})$. His method, in contrast to ours, uses the intrinsic property of the linear action of the matrices (polar decomposition), it involves lifting the matrices to the covering space $\widetilde{G}$ of $Sl^+(2, \mathbb{R})$ with projection $p : \widetilde{G} \to Sl^+(2, \mathbb{R})$. It is not our purpose here to compare the definition presented above with Ruelle's definition; but we emphasise that any physically interesting definition of rotation number should be consistent with the discretisation of continuous linear systems in the sense of Theorem 4.2 above. Another interesting point in our definition it that, when applied to maps in a parallelisable two-dimensional manifold (though it also depends on the trivialisation of the tangent bundle) it does not depend on the isotropy to the identity, in contrast to Ruelle's definition (see Ruelle, 1985). Moreover, in Ruelle (1985) the support of the invariant measure has to be connected. Note also that there is an ambiguity in Ruelle's definition in Theorem 1 (Ruelle, 1985, p.110): consider for instance (using his notation) the probability space $M = \{a, b\}$ with $\mathbb{P}\{a\} = \mathbb{P}\{b\} = 1/2$ and $f : M \to M$

the flipping on M. If $p(T(a)) = Id$ and $p(T(b)) = -Id$, one can assign $\Theta(T(a)) = 0$ and $\Theta(T(b)) = 3/2$, which leads to rotation number $3/4$, which obviously does not agree with the physical meaning of the rotation number, which in this case should be $1/2$.

5 Asymptotic angular stability in non-linear systems

In this section we consider a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which has an (ergodic) invariant probability measure μ in $\mathbb{R}^2$ (more generally, this could be a local diffeomorphism on a locally parallelisable two-dimensional manifold). The asymptotic angular behaviour of the product of the matrices given by the derivative of f along a trajectory is given by the rotation number of the action of these matrices in the circle S^1. That is, $(\mathbb{R}^2, \mu)$ plays the role of the probability space $(\Omega, \mathbb{P})$, f plays the role of the ergodic transformation θ, and the homeomorphisms are the induced action of the derivative df along the trajectory $f^{(n)}(x)$ of a certain element $x \in \mathbb{R}^2$. The composition of homeomorphisms are given by the product of the random matrices:

$$df^{(n)}(x) = df(f^{(n-1)}(x)) \circ \ldots \circ df(p).$$

In what follows we show some examples which illustrates this asymptotic analysis both in terms of rotation number and in terms of winding number.

Example 1:

Consider the map

$$f(x, y) = \left(x^3 - 2x + xy, \quad y\right).$$

The unique period-2 orbit is given by the points $(1, 0)$ and $(-1, 0)$. The linearisation of f at these points are given by

$$df_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$df_{(-1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

In both of these points the derivatives preserve the direction of e_1, hence the rotation number is $\rho(f) = 0$. For this initial direction, the winding number $\omega(f)(e_1)$ also vanishes. On the other hand, for any point (x, y) with $y \neq 0$ one easily sees that the winding number is $\omega(f)((x, y)) = 1/2$.

This example shows that, in general, not only does the winding number differ from rotation number, but also that the winding number depends on the initial direction one choose in the tangent space of the initial condition.

In the next examples we calculate numerically the rotation number and the winding number for well-known chaotic systems. While these systems may have many non-trivial ergodic invariant probability measures, the procedure for this calculation consists in taking an initial condition, $p = (0.1, 0.1)$ in all the cases, and computing 10^3 iterations of the map to guarantee that this point (if not already in the support of an ergodic invariant measure) will converge to some ergodic component. The calculation of the rotation number starts after these iterations. Similarly, for the winding number, we consider the same initial condition, $p = (0.1, 0.1)$, with the initial direction given by e_1.

[Figure 2 about here.]

Example 2:

Ikeda (1979) investigated the plane-wave model of a bistable ring cavity and noticed that it exhibited period doubling cascades to chaos. Hammel et al. (1985) extracted a complex difference equation relating the field amplitude at the $(i+1)^{th}$ cavity pass to that of a round trip earlier. The amplitude x and phase y, corresponding to the real and imaginary parts of the field are related by

$$f(x, y) = [1 + \mu(x\cos\theta - y\sin\theta), \quad \mu(x\sin\theta + y\cos\theta)] \tag{5}$$

where $\theta = a - b/(x^2 + y^2 + 1)$. This map is chaotic for $a = 0.4$, $b = 6$ and $\mu = 0.9$ and yields the attractor shown in figure 2a. The convergence of the estimates of $\rho(k)$ and $\omega(k)$ as a function of iteration k are illustrated in figures 3a and 3b and their distribution at the maximum iteration $k = 2^{15}$ is shown in figures 3c and 3d. Note that neither $\rho(k)$ nor $\omega(k)$ are normally distributed. We describe these distributions for $k = 2^{15}$ by giving the minimum, median and maximum values along with the mean and standard deviation in table 1 and 2. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = -0.29$ and $\langle\omega\rangle = 4.27$ respectively.

[Figure 3 about here.]

Example 3:

The Sinai map (see Sinai, 1972) is given by

$$f(x, y) = [(x + y + a\cos 2\pi y) \mod 1, \quad (x + 2y) \mod 1] \tag{6}$$

A value of $a = 0.1$ yields a chaotic system; the system state space is shown in figure 2b. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = 0.00$ and $\langle\omega\rangle = 3.02$ respectively (see tables 1 and 2).

Example 4:

The Tinkerbell map (see Alligood et al., 1996) is given by

$$f(x, y) = \left(x^2 - y^2 + ax + by, \quad 2xy + cx + dy\right) \tag{7}$$

where $a = -0.3$, $b = -0.6$, $c = 2$ and $d = -0.27$. The Tinkerbell attractor is shown in figure 2c. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = 1.94$ and $\langle\omega\rangle = 2.31$ respectively (see tables 1 and 2).

[Table 1 about here.]

[Table 2 about here.]

Acknowledgements

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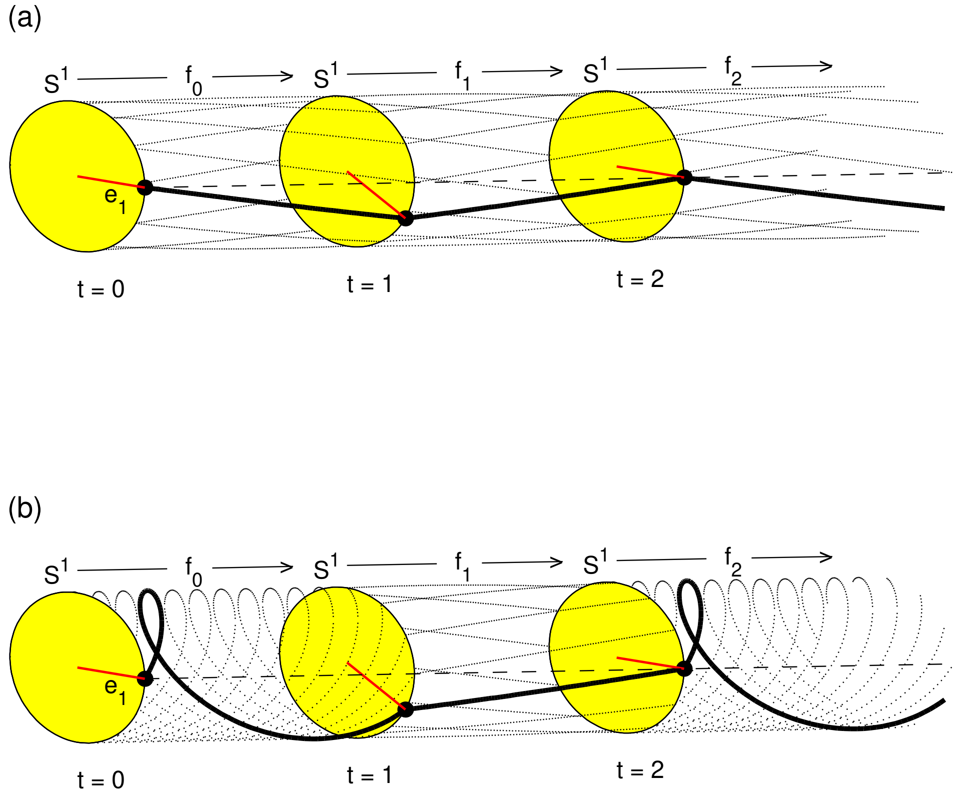

Figure 1: Stroboscopic sampling of flow on the cylinder (or torus) yields the Poincaré maps given by the sequence of homeomorphisms in S^1.

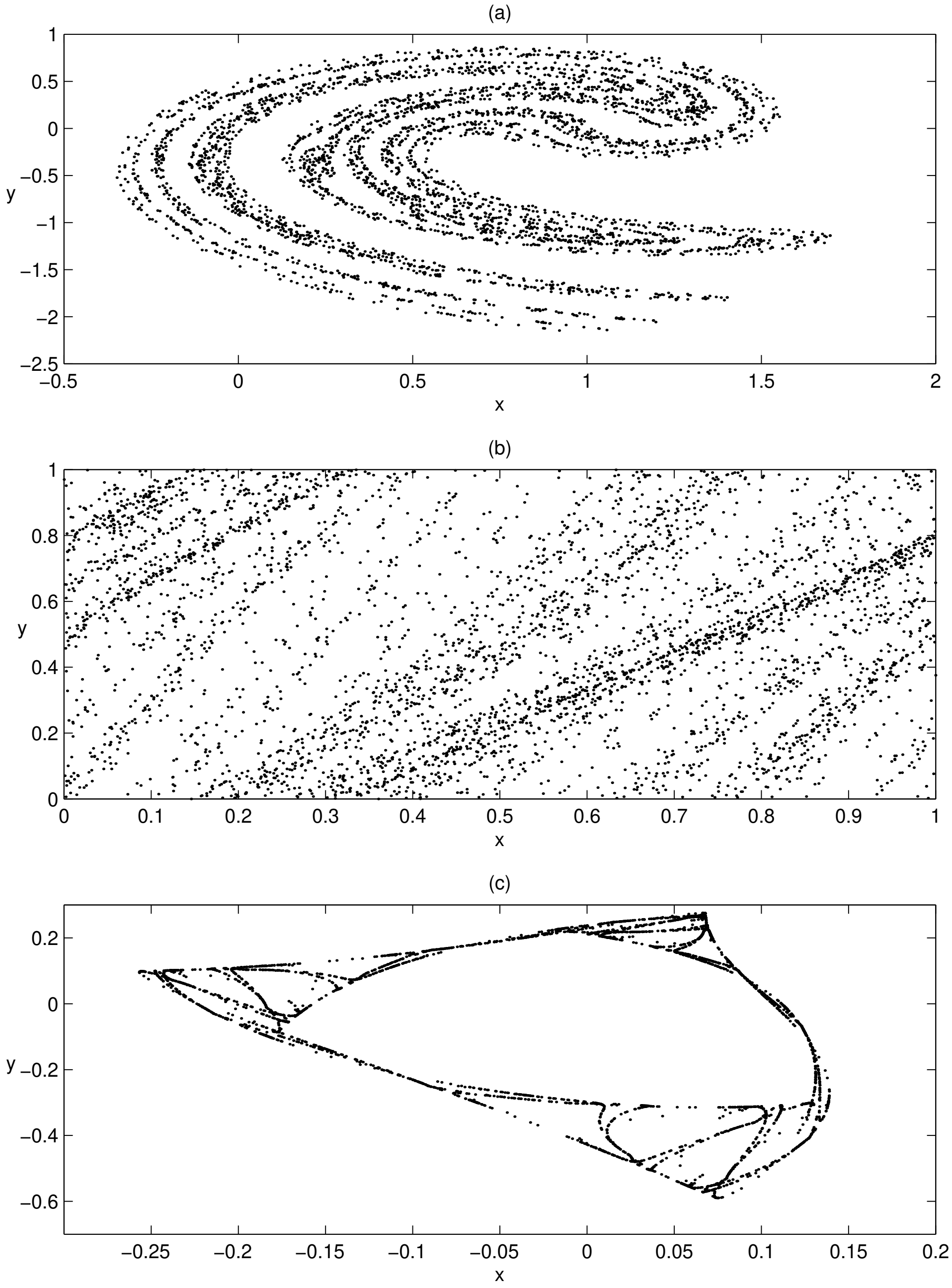

Figure 2: Attractors generated by two-dimensional chaotic maps: (a) Ikeda map, (b) Sinai map and (c) Tinkerbell map.

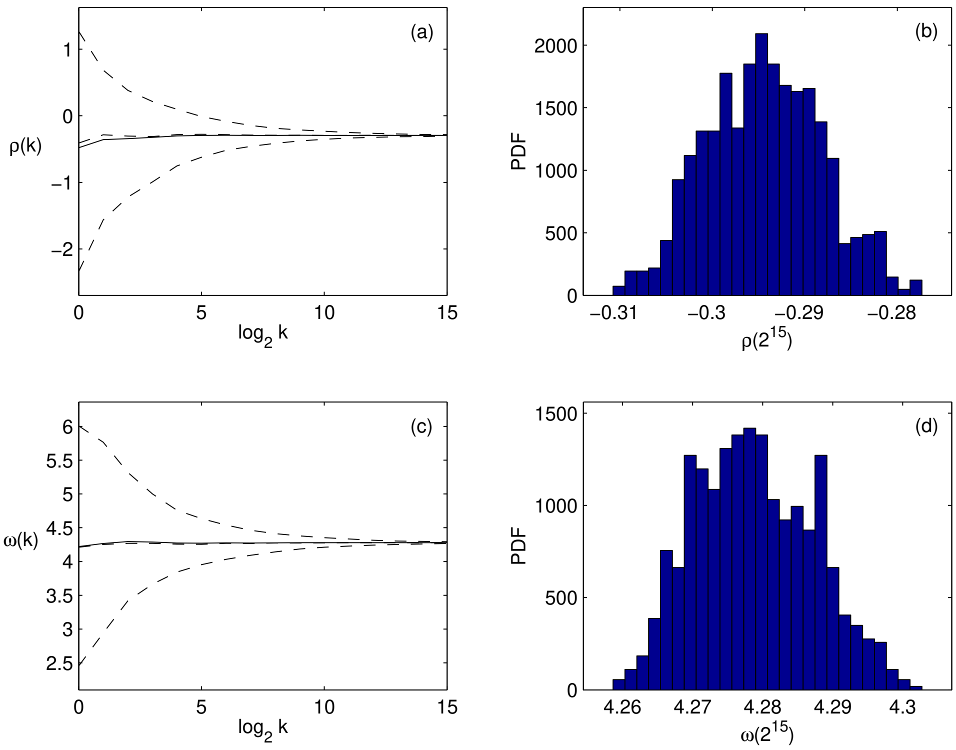

Figure 3: Rotation dynamics for the Ikeda map: (a) convergence of $\rho(k)$ with iteration k, (b) PDF of $\rho(k)$ for $k = 2^{15}$, (c) convergence of $\omega(k)$ with k and (d) PDF of $\omega(k)$ for $k = 2^{15}$.

List of Tables

System	mean	std	min	med	max
Ikeda	-0.2944	0.0062	-0.3106	-0.2944	-0.2774
Sinai	0.0000	0.0000	0.0000	0.0000	0.0000
Tinkerbell	1.9400	0.0049	1.9223	1.9400	1.9574

Table 1: Rotation numbers $\rho(k)$ for $k = 2^{15}$.

System	mean	std	min	med	max
Ikeda	4.2788	0.0084	4.2588	4.2782	4.3026
Sinai	3.0240	0.0109	2.9905	3.0237	3.0504
Tinkerbell	2.3091	0.0059	2.2905	2.3088	2.3259

Table 2: Winding numbers $\omega(k)$ for $k = 2^{15}$.