Stochastic Exponential in Lie Groups and its Applications

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Abstract

The aim of this article is to develop new proves for the basic formulas of stochastic analysis in Lie groups, in particular the stochastic exponential and logarithm. These formulas will lead to simple proves of (multiplicative) Doob-Meyer decomposition and Girsanov theorem for semimartingales in Lie groups.

Key words: Lie groups, semimartingales, Doob-Meyer decomposition, Girsanov theorem.

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1 Introduction

Let G be a Lie group with the corresponding Lie algebra $\mathcal{G}$. We denote by ω the Maurer-Cartan form in G, i.e. if $v \in T_gG$, then $\omega_g(v) = L_{g^{-1}*}(v)$. It corresponds to the unique $\mathcal{G}$-valued left invariant 1–form in G. We recall that in the case of $G = (\mathbb{R}_{>0}, \cdot)$ the Maurer-Cartan form is $\omega_g = \frac{1}{g}dg$, and in the case of the general linear group $GL(n, \mathbb{R})$ the Maurer-Cartan form ω is $g^{-1}dg = (x_{ij})^{-1}(dx_{ij})$ where (x_{ij}) are the coordinate functions on $GL(n, \mathbb{R})$.

The aim of this articles is to develop a set of formulas which are basic in the construction of stochastic analysis in Lie groups, in particular we start

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with basic properties of the stochastic exponential and logarithm. These formulas will lead naturally to a Doob-Meyer decomposition and an extension of the Girsanov theorem for semimartingales in Lie groups.

We recall that given a 1–form θ in a differentiable manifold M, and a differentiable curve $\gamma : [a, b] \to M$, the integral of θ along γ is the well known line integral:

$$\int_\gamma \theta = \int \theta \ d\gamma = \int_a^b \theta(\dot{\gamma}(s)) \ ds.$$

(See e.g. the classical Spivak [12]). The generalization of this formula along γ to an integral of an adapted stochastic 1-form $\theta_{X_t} \in T^*_{X_t}M$ along an M-valued semimartingale X_t was proposed by Ikeda and Manabe [5] (see also [3], [9]). LOcally this integral can be described like that: let $(U, x^1, \dots, x^n)$ be a local system of coordinates in M. Then, with respect to this chart the 1-form θ can be written as $\theta_x = \theta^1(x) \ dx^1 + \dots \theta^n(x) \ dx^n$, where $\theta^1(x)$ are (C^∞, say) functions in M. Then, the Stratonovich integral of θ along X_t is defined by:

$$\int \theta \circ dX_t = \sum_{j=1}^{j=n} \int \theta^i(X_t) \ \circ dX_t^i.$$

Let M_t be a semimartingale in the Lie algebra $\mathcal{G}$. We recall that the (left) stochastic exponential $\epsilon(M)$ of M_t is the stochastic process X_t which is solution of the left invariant equation on G:

$$\begin{cases} dX_t = L_{X_t *} \ \circ dM_t, \\ X_0 = e. \end{cases}$$

An interesting geometric characterization of the exponential $\epsilon(M)$ is the fact that it corresponds to the stochastic development of $M_t \in T_eG$ to the group G with respect to the left invariant connection ∇^L, i.e. $\nabla^L_X Y = 0$ for all $X, Y \in \mathcal{G}$.

Let the logarithm of a process X_t on G is the following semimartingale in the Lie algebra:

$$(\log X)_t = \int_0^t \omega \circ dX_s.$$

where ω is the Maurer-Cartan form in G. One easily checks that the logarithm as defined above is the inverse of the stochastic exponential ϵ.

This article is organized as follows: in the next section we present new proves of the stochastic Campbell-Hausdorff formula in a simpler and more direct way compared to Hakim-Dowek and Lépingle [4] or Arnaudon [1]. In the last section we apply these formulas to obtain simple proves of the (multiplicative) Doob-Meyer and Girsanov theorems in Lie groups.

2 Main results

We recall the following result, which characterizes ∇^L–martingales in G.

Theorem 2.1 *A process X_t on G is a ∇^L–martingale if and only if $X_t = X_0\epsilon(M)$ for some local martingale M in $\mathcal{G}$.*

Proof:

See Hakim-Dowek and Lépingle [4].

□

Next lemma concerns pull-back of Maurer-Cartan forms by homomorphisms of Lie groups, the formula will be useful along the article.

Lemma 2.1 *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the pull-back $\varphi^*\omega_H = \varphi_*\omega_G$.*

Proof:

Let $v \in T_gG$, then a direct calculation leads to

$$\begin{aligned} \varphi^*\omega_H(v) &= L_{\varphi(g)^{-1}*}(\varphi_*(v)) \\ &= \varphi_*(L_{g^{-1}*}(v)) \\ &= \varphi_*(\omega_G(v)) \end{aligned}$$

□

We shall denote by $I_g : G \to G$ the adjoint in the group G given by $h \mapsto ghg^{-1}$. The map I_g is a automorphism of G and its derivative corresponds to the isomorphism of the Lie algebra called adjoint in $\mathcal{G}$ denoted by $Ad(g) = I_{g*} : \mathcal{G} \to \mathcal{G}$. We have the following well known relation of the adjoint of the Maurer-Cartan form and the pull-back by the right action:

Proposition 2.2 *The pull-back by the right action satisfies*

$$R_g^*\omega = Ad(g^{-1})\omega.$$

Proof:

The proof is a straightforward calculation from the definitions, see e.g. Kobayashi and Nomizu [8].

□

Proposition 2.3 *Let $m : G \times G \to G$ be the multiplication and $i : G \to G$ be the inverse in the group. Then the pull-backs satisfy:*

1. $m^*\omega = (\pi_2^* Ad^{-1})(\pi_1^*\omega) + \pi_2^*\omega$;

2. $i^*\omega = -Ad\ \omega$.

Proof:

Let $w = (u, v) \in T_{(g,h)}G \times G \simeq T_gG \times T_hG$. Then

$$\begin{aligned} m^*\omega(w) &= \omega(m_* w) = \omega(R_{h*}u + L_{g*}v) \\ &= L_{(gh)^{-1}*}(R_{h*}u + L_{g*}v) \\ &= L_{h^{-1}*}R_{h*}L_{g^{-1}*}u + L_{h^{-1}*}L_{g^{-1}*}L_{g*}v \\ &= Ad(h^{-1})\omega(u) + \omega(v). \end{aligned}$$

For the inverse function, consider the diagonal map $\Delta : G \to G \times G$ given by $\Delta(g) = (g, g)$. We have that $m \circ (Id \times i) \circ \Delta = e$ then the pull-back $(m \circ (Id \times i) \circ \Delta)^*\omega$ vanishes, hence:

$$\begin{aligned} 0 &= ((Id \times i) \circ \Delta)^* m^*\omega \\ &= ((Id \times i) \circ \Delta)^*((\pi_2^* Ad^{-1})\pi_1^*\omega) + \pi_2^*\omega) \\ &= (\pi_2 \circ (Id \times i) \circ \Delta)^* Ad^{-1}(\pi_1 \circ (Id \times i) \circ \Delta)^*\omega + (\pi_2 \circ (Id \times i) \circ \Delta)^*\omega \\ &= Ad\omega + i^*\omega \end{aligned}$$

□

Next lemma presents the main formulas which are useful in calculations with the logarithm.

Lemma 2.2 *Given semimartingales X and Y in G, we have the following formulas:*

1. *If $\varphi : G \to H$ is a homomorphism then*

$$\varphi_*(\log X) = \log(\varphi(X));$$

2. $\log(XY) = \int Ad(Y^{-1}) \circ d(\log X) + \log Y$;

3. $\log(X^{-1}) = \int Ad(X) \circ d(\log X)$.

Proof:

For the first formula, note that

$$\begin{aligned} \log(\varphi X) &= \int \varphi^*\omega_H \circ dX \\ &= \int \varphi_*\omega_G \circ dX \\ &= \varphi_* \log X. \end{aligned}$$

The second identity follows from the calculation:

$$\begin{aligned}
\log(XY) &= \int \omega \circ dm(X,Y) \\
&= \int m^*\omega \circ d(X,Y) \\
&= \int ((\pi_2^* Ad^{-1})(\pi_1^*\omega) + \pi_2^*\omega) \circ d(X,Y) \\
&= \int Ad(Y^{-1}) \circ d(\int \omega \circ dX) + \int \omega \circ dY \\
&= \int Ad(Y^{-1}) \circ d\log X + \log Y.
\end{aligned}$$

Finally, for the last formula we have that

$$\begin{aligned}
\log(X^{-1}) &= \int i^*\omega \circ dX \\
&= \int -Ad\ \omega \ \circ dX \\
&= -\int Ad(X) \circ d(\int \omega \ \circ dX) \\
&= -\int Ad(X) \circ d(\log X).
\end{aligned}$$

□

Corollary 2.4 *We have the following stochastic Campbell-Hausdorff formula:*

1. $\epsilon(M+N) = \epsilon\left(\int Ad(\epsilon(N)) \circ dM\right)\epsilon(N)$;
2. $\epsilon(M)^{-1} = \epsilon\left(-\int Ad(\epsilon(M)) \circ dM\right)$.

Proof:

For the first formula we just have to check that:

$$\begin{aligned}
&\log\left(\epsilon\left(\int Ad(\epsilon(N) \ \circ dM\right)\epsilon(N)\right) \\
= \ & \int Ad(\epsilon(N)^{-1}) \ \circ d\log\left(\epsilon\left(\int Ad(\epsilon(N)) \ \circ dM\right)\right) + \log(\epsilon(N)) \\
= \ & M+N.
\end{aligned}$$

And for the second formula:

$$
\begin{aligned}
& \log\left(\epsilon\left(-\int Ad(\epsilon(M) \ \circ dM\right)\right) \\
= & -\int Ad(\epsilon(M)) \ \circ dM \\
= & -\int Ad(\epsilon(M)) \ \circ d\log(\epsilon(M)) \\
= & \log(\epsilon(M)^{-1}).
\end{aligned}
$$

□

3 Applications

Our first application of these formulas is a multiplicative version of the Doob-Meyer decomposition. It was originally established by R. L. Karandikar in the case of group of matrices [7] and by M. Hakim-Dowek, D. Lepingle [4] (See also [1], [2]) in the general case.

Theorem 3.1 (Doob-Meyer decomposition in Lie groups) *Let $X = X_0\epsilon(M)$ be a semimartingale in G with $M = N + A$, where N is a local martingale and A is a process of finite variation in $\mathcal{G}$. Then we have that*

$$X = X_0 Y Z = X_0 Z' Y'$$

where Y, Y' are local martingales and Z, Z' are processes of finite variation. The relation between them are given by $Y = \int Ad(\epsilon(A)) \circ dN$, $Y' = \epsilon(N)$, $Z = \epsilon(A)$ and $Z' = \epsilon(\int Ad(\epsilon(N) \ dA)$.

Proof:

Apply the stochastic Campbell-Hausdorff formula to the classical Doob-Meyer decomposition $M = N + A$ and use Theorem 2.1.

□

We call the decomposition of the above theorem $X = X_0 Y Z$ ($X = X_0 Z' Y'$) the left (right) multiplicative Doob-Meyer decomposition of X. Now, we show a multiplicative version of the Girsanov theorem.

Theorem 3.2 (Girsanov-Meyer theorem in Lie groups) *Let P and Q be equivalent probability laws on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0})$ with Radon-Nikodyn derivative $A_t = \mathbf{E}_P(\frac{dQ}{dP} \mid \mathcal{F}_t)$. Let X be a semimartingale in G with* *left multiplicative Doob-Meyer decomposition X_0YZ with respect to P. Then X has left multiplicative Doob-Meyer decomposition X_0VW with respect to Q where*

$$V = \epsilon(\int A d\epsilon(\log Z + \int \frac{1}{A} d[A,B]) d(B - \int \frac{1}{A} d[A,B]))$$

and

$$W = \epsilon(\log Z + \int \frac{1}{A} d[A,B])$$

where B is the semimartingale $B_t = \log YZ - \log Z$.

Proof:

Apply the classical Girsanov-Meyer theorem (see e.g. [10, Thm. 20, p. 109]) to $\log(YZ)$ and the stochastic exponential.

$\square$

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