Moment Lyapunov Exponents for i.i.d. Random Products and Semigroups

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We consider an independent and identically distributed (i.i.d.) random sequence y_n in a semi-simple Lie group G with common law μ and form its random product $g_n = y_n \cdots y_1$. (For example if $G = \mathrm{Sl}(d, \mathbb{R})$ then g_n is a product of random matrices.) The purpose is to describe geometric properties of the semigroup S_μ generated by the support of μ via the asymptotics of the random product g_n.

The asymptotic properties of g_n are described by limits of cocycles $\rho_\lambda(g, x)$ over the flag manifolds. These cocycles are defined after the Iwasawa decomposition $G = KAN$ through the function $\rho : G \times K \to A$ given by $gu = v\rho(g, u)n$ with $v \in K$, $\rho(g, x) \in A$ and $n \in N$ and the parameter λ is a linear map on $\mathfrak{a} = \log A$.

The Lyapunov exponents of the random product g_n are defined by

$$\Lambda_\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \rho_\lambda(g_n, x).$$

In case $G = \mathrm{Sl}(d, \mathbb{R})$ then $\mathfrak{a}$ is the space of diagonal matrices and if $\lambda_1(D)$ is the first eigenvalue of the diagonal matrix D then $\rho_{\lambda_1}(g, x)$ is the cocycle $\|gv\| / \|v\|$ over the projective space whose Lyapunov exponents are given by the multiplicative ergodic theorem. The same way the cocycles $\rho_{\lambda_1 + \cdots + \lambda_k}(g, x)$ over the Grassmannians yield sums of Lyapunov exponents.

The moment Lyapunov exponent of the random product depending on $\lambda \in \mathfrak{a}^*$ and x is defined by

$$\begin{aligned}\gamma_\lambda(x) &= \limsup_{n\to\infty} \frac{1}{n} \log \int \rho_\lambda(g, x)\, \mu^{*n}(dg) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\rho_\lambda(g_n, x)\right]\end{aligned}$$

where μ^{*n} is the n-th convolution power of μ.

We assume that $\mathrm{int} S_\mu \neq \emptyset$ (that is, μ is an *étalée* measure). The so-called flag type of S_μ is a flag manifold associated to it that reveals several geometric and algebraic properties of the semigroup (for instance the Jordan form of its elements).

We relate the flag type of S_μ with the behavior as $p \to -\infty$ of the functions $p \mapsto \gamma_{p\lambda}(x)$. The point is that for the interesting values of λ, $\frac{1}{n} \log \rho_\lambda(g_n, x)$ converges a.s. to a constant ("top" Lyapunov exponent) which is positive. Hence the behavior of $\mathbb{E}[\rho_\lambda(g_n, x)^p]$ for large $p < 0$ tells if there are "late comers" to the limit of $\frac{1}{n} \log \rho_\lambda(g_n, x)$ showing the location of g_n in G.

The moment Lyapunov exponent $\gamma_\lambda(x)$ is related to the spectral radius of the operator

$$(U_\lambda(\mu) f)(x) = \int_G \rho_\lambda(g, x) f(gx) \mu(dg)$$

acting in spaces of continuous functions. The good properties of $\gamma_\lambda(x)$ are obtained via the pertubation theory of these operators.

An stochastic differential equation

$$dg = X(g)\, dt + \sum_{j=1}^{m} Y_j(g) \circ dW_j$$

on G (where X and Y_j are right invariant vector fields) yields a one-parameter semigroup μ_t of probability measures. When the results are applied to the measures μ_t the moment Lyapunov exponents are related to the principal eigenvalue of the differential operator

$$L_\lambda^\Theta = \widetilde{L} + \frac{1}{2} \sum_{j=1}^{m} \lambda\left(q_{Y_j}\right) \widetilde{Y}_j + \lambda(q_X) + \frac{1}{2} \sum_{j=1}^{m} \lambda\left(r_{Y_j}\right) + \frac{1}{2} \sum_{j=1}^{m} \left(\lambda\left(q_{Y_j}\right)\right)^2$$

where $\widetilde{L} + \frac{1}{2} \sum_{j=1}^{m} \lambda\left(q_{Y_j}\right) \widetilde{Y}_j$ is a second order hyperbolic differential operator and the other terms are functions defined from the vector fields X and Y_j.

References

[1] Arnold, L.: *A formula connecting sample and moment stability of linear stochastich systems.* SIAM J. Appl. Math. **44** (1984), 793-802.

[2] Arnold, L; W. Kliemann and E. Oeljeklaus: *Lyapunov Exponents of Linear Stochastic Systems.* In Lyapunov Exponents, Proceedings Bremen 1984, Lecture Notes in Mathematics - Springer Verlag **1186** (1985), 85-128.

[3] Arnold, L; E. Oeljeklaus and E. Pardoux: *Almost Sure and Moment Stability for Linear Ito Equations.* In Lyapunov Exponents, Proceedings Bremen 1984, Lecture Notes in Mathematics - Springer Verlag **1186** (1985), 129-159.

[4] Guivarc'h, Y. and A. Raugi: *Frontière de Furstenberg, proprietés de contraction et théorèmes de convergence*. Probability theory and related topics (ZW). **69** (1985), 187-242.

[5] San Martin, L.A.B.: Moment Lyapunov Exponents and Semigroups in Semi-simple Lie Groups. To appear.