The Grobman-Hartman theorem

Now that we have studied the structure of solutions to linear differential equations in general, we wish to use that theory to study the local structure of the solutions to non-linear systems. If X is a C^r vector field, $r \geq 1$, defined in an open set $U \subset \mathbf{R}^n$, and $x_0 \in U$ is a non-singular point (i.e., $X(x_0) \neq 0$), then we have seen that there is a C^r change of coordinates which takes solutions near x_0 to straight lines. Thus, it remains to describe the solutions near a critical point. If the derivative $A = DX_{x_0}$ of X at x_0 has eigenvalues with real parts different from zero, we will see, that after a continuous change of coordinates, the structure of solutions of X near x_0 is the same as that of the linear system $\dot{y} = Ay$ near 0.

We now make the relevant definitions.

Let X be a C^r vector field as above with $r \geq 1$ with a critical point at x_0 (i.e., $X(x_0) = 0$). Let A be the derivative of X at x_0. Thus, $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map whose matrix in the standard coordinates on $\mathbf{R}^n$ is the Jacobian matrix of X at x_0.

Definition. The critical point x_0 of X is called *hyperbolic* if the eigenvalues of A all have non-zero real parts.

If X is a C^r vector field, recall that the *local flow* of X near x_0 is the function $\eta(t, x)$ defined in a neighborhood V of $(0, x_0)$ in $\mathbf{R}^{n+1}$ such that

1. $\eta(0, x) = x$ for $(0, x) \in V$
2. $t \to \eta(t, x)$ is a solution to the differential equation $\dot{x} = X(x)$ defined in a neighborhood of $t = 0$.

We also use the notation η_t for the local flow $\eta(t, x)$. We sometimes call η_t the local flow of the differential equation $\dot{x} = X(x)$ as well. We will also use the term *integral curve* of the vector field X for a solution curve.

Definition. A linear map $L : \mathbf{R}^N \to \mathbf{R}^N$ is called *hyperbolic* if its (possibly complex) eigenvalues have norm different from one.

Examples:

1. L is the map induced by the 2×2 matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
2. $L = e^A$ where A is a linear map whose eigenvalues have non-zero real parts.

If L is a hyperbolic linear map of $\mathbf{R}^N$, then there is a direct sum decomposition $\mathbf{R^N} = E^s \oplus E^u$ such that

1. $L(E^s) = E^s$ and $L(E^u) = E^u$
2. the eigenvalues of $L \mid E^s$ have norm < 1 and those of $L \mid E^u$ have norm > 1

Definition Let X be a C^r vector field defined in a neighborhood of x_0 in $\mathbf{R}^N$ having x_0 as a critical point. Let DX_{x_0} be the derivative of X at x_0. A C^0 linearization of X near x_0 is a homeomorphism h from a neighborhood U of x_0 in $\mathbf{R}^N$ to a neighborhood of 0 such that if η_t is the local flow of X near x_0, then $h\eta_t h^{-1}$ is the local flow of the linear differential equation $\dot{y} = DX_{x_0} \cdot y$ near 0.

One may similarly define C^k linearizations of a C^r vector field X for $1 \leq k \leq r$ by requiring that h be a C^k diffeomorphism from a neighborhood of x_0 to a neighborhood of 0.

Theorem 1. (Grobman-Hartman). *Suppose x_0 is a hyperbolic critical point of the C^1 vector field X. Then X has a C^0 linearization near x_0.*

Remark. For smooth linearizations, one has the following result.

Theorem. *Suppose that L is linear map on R^n. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L. For each positive integer k, there is a positive integer $N(k)$ with the following property. Suppose that for each $1 \leq i \leq n$ and each $n-$tuple $(m_1, m_2, \ldots, m_n)$ of non-negative integers satisfying $2 \leq \sum_{1 \leq j \leq n} m_j \leq N(k)$, we have $\lambda_i \neq \sum_{1 \leq j \leq n} m_j \lambda_j$.*

Then, any $C^{N(k)}$ vector field X with $X(x_0) = 0$ and $DX_{x_0} = L$ has a local C^k linearization near x_0.

Note that as a corollary of the Grobman-Hartman theorem, we have

Corollary. *Let x_0 be a hyperbolic critical point of a C^1 vector field X in $\mathbf{R}^n$. If all the eigenvalues of the derivative $L = DX_{x_0}$ have negative real parts, then x_0 is asymptotically stable. If L has at least one eigenvalue with positive real part, then x_0 is unstable.*

We will proceed toward the proof of Theorem 1. Note that we may assume that both X and L have local flows defined for $\mid t \mid \leq 1$.

In the course of the proof, it will be necessary to first linearize the time-one map η_1 of X near x_0. So, we first study the relevant linearization theorem for local diffeomorphisms.

Definition. Let f be a C^1 diffeomorphism from a neighborhood U of x_0 in $\mathbf{R}^n$ into $\mathbf{R}^n$ with $f(x_0) = x_0$. The fixed point x_0 is called *hyperbolic* if all the eigenvalues of Df_{x_0} have absolute values with norm different from one; i.e, Df_0 is a hyperbolic linear map.

Theorem 2.(Grobman-Hartman theorem for local diffeomorphisms). *Suppose x_0 is a hyperbolic fixed point of the local C^1 diffeomorphism f defined on a neighborhood U of x_0 in $\mathbf{R}^n$. Let $L = Df_{x_0}$. There is a neighborhood $U_1 \subseteq U$ of x_0 and a homeomorphism h from U_1 into $\mathbf{R}^n$ such that $h(x_0) = 0$ and $hf(x) = Lh(x)$ for $x \in U_1 \bigcap f^{-1}U_1$.*

Note that an equivalent formulation of

$$hf(x) = Lh(x) \text{ for } x \in U_1 \bigcap f^{-1}U_1$$

is

$$hfh^{-1}(y) = L(y) \text{ for } h^{-1}(y) \in U_1 \bigcap f^{-1}U_1$$

so the formulas in both theorems are analogous.
Remark.

1. The proofs we will give of the above theorems are valid if $\mathbf{R}^n$ is replaced by a Banach space.

2. A map h as in Theorem 2 is called a C^0 linearization of f. One may define C^k linearizations analogously for $k \geq 1$.

Definition. Let U be an open subset of $\mathbf{R}^n$ and let $g : U \to R^n$ be a mapping. We say g is *Lipschitz* (or *Lipschitz continuous*) if there is a constant $K > 0$ such that

$$\sup_{x \neq y} \frac{| g(x) - g(y) |}{| x - y |} \leq K < \infty.$$

When g is Lipschitz, we let $Lip(g) = \sup_{x \neq y} \dfrac{| g(x) - g(y) |}{| x - y |}$ and we call it the *Lipschitz constant* of g.

Note that if g is C^1 and $M = \sup_x | D_x g |$, then g is Lipschitz and $Lip(g) = M$. That is, the maximum of the norms of the derivatives of a C^1 map g equals the Lipschitz constant of g.

We will develop some machinery to prove Theorem 2. Then we will prove Theorem 1.

Let us first note that, replacing f by $x \to f(x + x_0) - x_0$, we may assume $x_0 = 0$.

Lemma 3. There is a C^∞ function $\alpha : \mathbf{R} \to [0, 1]$ such that

1. $\alpha(u) = 1$ for $u \leq \frac{1}{2}$.
2. $\alpha(u) = 0$ for $u \geq 1$.

Proof.
Let

$$\phi(u) = \begin{cases} exp(-\frac{1}{(\frac{1}{2}-u)(u-1)}) & \text{for } \frac{1}{2} < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, ϕ is C^∞.
Let

$$\psi(u) = \frac{\int_{-\infty}^{u} \phi(s)\, ds}{\int_{-\infty}^{1} \phi(s)\, ds}$$

Then, ψ is C^∞ and

$$\psi(u) = \begin{cases} 0 & \text{for } u \leq \frac{1}{2} \\ 1 & \text{for } u \geq 1 \end{cases}$$

and $\psi(u) \in [0, 1]$ for all u.

Let $\alpha(u) = 1 - \psi(u)$. Then, α has the required properties. (Details left as an exercise.)

Let f be as in the statement of Theorem 2. Our next lemma will show that we may assume there is a $\delta > 0$ such that f is defined on all of $\mathbf{R}^n$, $f(x) = L(x)$ for $| x | \geq \delta$ and $Lip(f - L)$ (on all of $\mathbf{R}^n$) is small.

Lemma 4. Let $\varepsilon > 0$. There are a $\delta > 0$ and a C^1 diffeomorphism $f_1 : \mathbf{R}^n \to \mathbf{R}^n$ such that

1. $f_1(x) = L(x)$ for $\mid x \mid \geq \delta$
2. $f_1(x) = f(x)$ for $\mid x \mid \leq \frac{\delta}{2}$
3. $Lip(f_1 - L) < \varepsilon$ and $\|\, f_1 - L \,\|_0 < \varepsilon$.

(Here, $\|\, f_1 - L \,\|_0 = \sup_{x \in \mathbf{R}^n} \mid f_1(x) - L(x) \mid$.)

Proof. Let $\varepsilon_1 \in (0,1)$, and let $\delta_1 \in (0,1)$ be small enough so that

(a) f is defined for $\mid x \mid \leq \delta_1$

(b) $\|\, D_x(f - L) \,\| < \varepsilon_1$
and

(c) $\mid f(x) - L(x) \mid < \varepsilon_1 \delta_1$ for $\mid x \mid \leq \delta_1$

Let α be as in Lemma 3, and let $K = \sup_{u \in \mathbf{R}} \mid \alpha'(u) \mid$.
Let $\gamma(x) = \alpha(\frac{\mid x \mid}{\delta_1})$. Note that $\mid D_x \gamma \mid \leq \frac{K}{\delta_1}$ for all x.

Now,

$$\gamma(x) = \begin{cases} 1 & \text{for } \mid x \mid \leq \frac{\delta_1}{2} \\ 0 & \text{for } \mid x \mid \geq \delta_1 \end{cases}$$

Let

$$\begin{aligned} f_1(x) &= \gamma(x) f(x) + (1 - \gamma(x)) L(x) \\ &= L(x) + \gamma(x)(f(x) - L(x)) \end{aligned}$$

Note that f_1 is the $\gamma-$average of f and L.
Now, $(f_1 - L)(x) = \gamma(x)(f(x) - L(x))$, so

$$\begin{aligned} \| f_1 - L \|_0 &= \sup_x | \gamma(x)(f(x) - L(x)) | \\ &\le \sup_{| x | \le \delta_1} | f(x) - L(x) | \le \quad \varepsilon_1 \end{aligned}$$

Also,

$$\begin{aligned} \| D_x(f_1 - L) \| &= | D_x\gamma \cdot (f(x) - L(x)) + \gamma(x)(D_x f - L) | \\ &\le | D_x\gamma \| f(x) - L(x) | + \| D_x f - L \|_0 \\ &\le \frac{K}{\delta_1}\varepsilon_1\delta_1 + \varepsilon_1 \end{aligned}$$

Note that we use the notation $D_x\gamma \cdot (f(x) - L(x))$ for the map $v \to D_x\gamma(v)(f(x) - L(x))$.

Now, given $\varepsilon \in (0,1)$, choose $\varepsilon_1 \in (0,1)$ small enough so that $\max(\varepsilon_1, K\varepsilon_1 + \varepsilon_1) < \varepsilon$. $\square$

Proofs of the Grobman-Hartman theorems

We will reduce the proofs of Theorems 1 and 2 to one main step, and then we will complete the proof of that step. We let *id* denote the identity map on $\mathbf{R}^n$, and we let $|| \phi ||_0$ denote the C^0 norm of a mapping.

(A) MAIN STEP. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a hyperbolic linear map. There is an $\varepsilon > 0$ depending on L such that the following holds.*

If $\phi_1 : \mathbf{R}^n \to \mathbf{R}^n, \phi_2 : \mathbf{R}^n \to \mathbf{R}^n$ are Lipschitz maps such that

$$|| \phi_i ||_0 \leq \varepsilon \tag{1}$$

and

$$Lip(\phi_i) < \varepsilon, \tag{2}$$

then, there is a unique continuous map $h_{\phi_1\phi_2} : \mathbf{R}^n \to \mathbf{R}^n$ such that $h_{\phi_1\phi_2} - id$ is a bounded continuous map and

$$(L + \phi_1) \circ h_{\phi_1\phi_2} = h_{\phi_1\phi_2} \circ (L + \phi_2). \tag{3}$$

Assuming (A), let us prove Theorems 1 and 2.

Proof of Theorem 2.

We may assume $f = L + \phi$ where $L = D_0 f,\ Lip(\phi) < \varepsilon$, and $\phi(x) = 0$ for $| x | \geq \varepsilon$ and ε is as small as we wish. Let ζ denote the zero map ($\zeta(x) = 0$ for all x).

By (A), there are unique maps $h_{\phi\zeta}$ and $h_{\zeta\phi}$ of bounded distance from the identity such that

$$f h_{\phi\zeta} = h_{\phi\zeta} L \tag{4}$$

and

$$L h_{\zeta\phi} = h_{\zeta\phi} f. \tag{5}$$

This gives us $fh_{\phi\zeta}h_{\zeta\phi} = h_{\phi\zeta}Lh_{\zeta\phi} = h_{\phi\zeta}h_{\zeta\phi}f$.

So, $h_{\phi\zeta}h_{\zeta\phi}$ is a continuous map and $h_{\phi\zeta}h_{\zeta\phi} - id$ is bounded. By uniqueness of the solution h to $fh = hf$, we have $h_{\phi\zeta}h_{\zeta\phi} = id$. Similarly, $h_{\zeta\phi}h_{\phi\zeta}L = Lh_{\zeta\phi}h_{\phi\zeta}$. By uniqueness of the solutions of $hL = Lh$, we have $h_{\zeta\phi}h_{\phi\zeta} = id$.

Thus, $h_{\phi\zeta}$ is a homeomorphism and Theorem 2 is proved. □

Proof of Theorem 1.

This proof illustrates a general principle of the theory of linearizations. To linearize a vector field near a critical point x_0, it is sufficient to linearize its time one map near x_0.

We will reduce the proof of Theorem 1, to the statement we proved in Theorem 2.

We may assume that $X = D_0X + \phi$ with $\phi(x) = 0$ for $\mid x \mid \geq \varepsilon$ and $Lip(\phi) < \varepsilon$ with ε small. Let ψ_t be the time t map of X, and let η_t be the time t map of D_0X. Then, $\psi_1 = \eta_1 + \gamma$ where $\gamma = 0$ off a small neighborhood of 0, and $Lip(\gamma)$ is small. By the above proof of Theorem 2, we know there is a unique continuous map $h : \mathbf{R}^n \to \mathbf{R}^n$ such that $h - id$ is bounded and $h^{-1}\eta_1 h = \psi_1$ which is equivalent to $\eta_1 h = h\psi_1$ and $\eta_1 h\psi_{-1} = h$.

Let

$$H(x) = \int_0^1 (\eta_t h\psi_{-t})(x)\, dt. \tag{6}$$

Then, for $0 \leq s \leq 1$,

$$\eta_s H\psi_{-s} = \int_0^1 \eta_{s+t}h\psi_{-(t+s)}\, dt \quad \text{(since each } \eta_s \text{ is linear).}$$

Making the change of variable $u = s + t - 1$, we get the last integral equal to

$$\int_{s-1}^{s} \eta_{u+1}h\psi_{-1-u}\, du = \int_{s-1}^{0} \eta_{u+1}h\psi_{-1-u}\, du + \int_0^s \eta_u\eta_1 h\psi_{-1}\psi_{-u}\, du$$

Now set $v = u + 1$ in the first integral of the preceding equation and use the fact that $\eta_1 h\psi_{-1} = h$ in the second one.

We get the sum of integrals to be

$$\int_s^1 \eta_v h\psi_{-v}\, dv + \int_0^s \eta_u h\psi_{-u}\, du = H$$

.

Thus we have obtained that for $0 \leq s \leq 1$,

$$\eta_s H \psi_{-s} = H. \tag{7}$$

Thus, H conjugates each η_s to ψ_s. We would like to show that H is a homeomorphism.

Let us first show that H is continuous and $H - id$ is bounded.

Continuity is easy since if y is near x, the functions $t \to \eta_t h \psi_{-t}(x)$ and $t \to \eta_t h \psi_{-t}(y)$ are uniformly close.

Let us now see that $H - id$ is bounded.

We have

$$H(x) - x = \int_0^1 \eta_t h \psi_{-t}(x) dt - \int_0^1 x dt$$

For each $t \in [0,1]$, we have

$$\begin{aligned} \mid \eta_t h \psi_{-t}(x) - x \mid &= \mid \eta_t h \psi_{-t}(x) - \psi_t \psi_{-t}(x) \mid \\ &= \mid \eta_t h(u) - \psi_t(u) \mid \text{ for } u = \psi_{-t}(x) \\ &= \mid \eta_t h(u) - \eta_t(u) + \eta_t(u) - \psi_t(u) \mid \end{aligned}$$

But, $\eta_t(u) = \psi_t(u)$ for all $\mid t \mid \leq 1$ if $\mid u \mid$ is large.

Hence, if we let $L_0 = D_0 X$, then $\eta_t = e^{tL_0}$. Thus,

$$\mid \eta_t h \psi_{-t}(x) - x \mid \leq \sup_{0 \leq s \leq 1} \mid e^{sL_0} \mid \cdot || \, h - id \, ||_0 + C_1$$

uniformly in t

where

$$C_1 = \sup_{\mid t \mid \leq 1, u \in \mathbf{R}^n} \mid \eta_t(u) - \psi_t(u) \mid .$$

This gives us that $H - id$ is bounded.

Now,

$$H - id \text{ is continuous and bounded }, \tag{8}$$

and, for $s = 1$,

$$\eta_1 H = H\psi_1 \tag{9}$$

But, h also satisfies (8) and (9), and we showed that there was a *unique* map satisfying these conditions. Thus, $H = h$, and the map h constructed for the time-one maps actually satisfies (7) for $0 \leq s \leq 1$.

We also want $\eta_s h \psi_{-s} = h$ for $-1 \leq s \leq 0$.

But, for $s \in [-1, 0]$, we have $-s \in [0, 1]$, so $\eta_{-s} h \psi_{-(-s)} = h$. Now compose on the left with η_s and on the right with ψ_{-s} and we get $h = \eta_s h \psi_{-s}$.

This proves Theorem 1. □

Proofs of the Grobman-Hartman theorems - Continued

We now develop the necessary results to prove the Main Step (A) above.

Lemma 1. Suppose $H : V \to V$ is a bounded linear self-map of the Banach space V with $\mid H \mid < 1$. Let I denote the identity map, $Ix = x$. Then, $I - H$ is an isomorphism and

$$\mid (I-H)^{-1} \mid \leq \frac{1}{1 - \mid H \mid} \tag{1}$$

Proof.

Let $T = \sum_{i=0}^{\infty} H^i$. Then, T is a bounded linear operator, and

$$(I-H)T = T(I-H) = I.$$

Therefore, $I - H$ is an isomorphism with inverse T.
Moreover,

$$\mid (I-H)^{-1} \mid = \mid T \mid \leq \sum_{i=0}^{\infty} \mid H \mid^i = \frac{1}{1 - \mid H \mid} \quad QED.$$

Lemma 2. *If $V = V_1 \oplus V_2$ is a direct sum decomposition of the Banach space V, and $H : V \to V$ is an isomorphism such that $H(V_i) = V_i$ for $i = 1, 2$, $\mid H \mid V_1 \mid < 1$, and $\mid H^{-1} \mid V_2 \mid < 1$, then $I - H$ is an isomorphism. If V is given the maximum norm, then*

$$\mid (I-H)^{-1} \mid \leq \max \left(\frac{1}{1 - \mid H \mid V_1 \mid}, \frac{\mid H^{-1} \mid V_2 \mid}{1 - \mid H^{-1} \mid V_2 \mid} \right). \tag{2}$$

Proof.

For $u = u_1 + u_2$ with $u_i \in V_i$, define

$$T(u) = T(u_1 + u_2) = \sum_{i=0}^{\infty} H^i(u_1) + (-\sum_{i=1}^{\infty} H^{-i}(u_2)).$$

Then, $(I - H)T = T(I - H) = I$. QED

Lemma 3. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map all of whose eigenvalues have norm less than one. Let $\tau_1 = \sup\{| \lambda | : \lambda$ is an eigenvalue of $L\}$. Let $\tau \in (\tau_1, 1)$. Then, there is a new norm $|| \cdot ||$ on $\mathbf{R}^n$ such that $|| L(v) || \leq \tau|| v ||$ for all $v \in \mathbf{R}^n$. That is, with respect to the norm $||| \cdot |||$ on L induced by the norm $|| \cdot ||$, we have $||| L |||< \tau$.*

Proof. Using the fact that $L = S + N$ where S is semi-simple (complex diagonalizable) and N is nilpotent, one sees that there is a constant $C > 0$ such that $m \geq 0$ implies that $| L^m v | \leq C(\tau^m)| v |$ for all $v \in \mathbf{R}^n$. Thus, for each v, the quantity $\alpha(v) = \sup(\{| L^m v |\tau^{-m} : m \geq 0\})$ is finite. Set $|| v || = \alpha(v)$. Then, it is easy to see that $|| \cdot ||$ is a norm on $\mathbf{R}^n$.

On the other hand,

$$\begin{aligned}
|| Lv || &= \sup(\{| L^m Lv |\tau^{-m} : m \geq 0\}) \\
&= \tau\tau^{-1} \sup(\{| L^m Lv |\tau^{-m} : m \geq 0\}) \\
&= \tau \sup(\{| L^m Lv |\tau^{-m-1} : m \geq 0\}) \\
&= \tau \sup(\{| L^{m+1} v |\tau^{-m-1} : m \geq 0\}) \\
&= \tau \sup(\{| L^m v |\tau^{-m} : m \geq 1\}) \\
&\leq \tau|| v ||. \quad QED
\end{aligned}$$

Remark. If we were dealing with a Banach space E instead of $\mathbf{R}^n$, we would just let τ_1 be the spectral radius of the operator L above.

Proposition 4. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear hyperbolic isomorphism. That is, no eigenvalues of L have norm 1. Let $\tau \in (0, 1)$ be such that the eigenvalues of L inside the unit circle have norm < 1, and those outside the unit circle have norm $> \tau^{-1}$. Then, there is a direct sum decomposition $\mathbf{R}^n = V_1 \oplus V_2$ and a new norm $|| \cdot ||$ on $\mathbf{R}^n$ such that*

$$L(V_1) = V_1, \ L(V_2) = V_2 \tag{3}$$

and

$$|| \ L \mid V_1 \ || < \tau, \ || \ L^{-1} \mid V_2 \ || < \tau \tag{4}$$

.

Proof. Let $\mathbf{R}^n = V_1 \oplus V_2$ be the direct sum decomposition such that $L \mid V_1$ has eigenvalues less than τ in norm, and $L \mid V_2$ has eigenvalues greater than τ^{-1} in norm. Note that $L^{-1} \mid V_2$ has eigenvalues of norm less than τ. By Lemma 3, there are norms $|| \cdot ||_1$ and $|| \cdot ||_2$ on V_1 and V_2, respectively, such that (4) holds. For $v = (v_1, v_2)$ with $v_i \in V_i$, let $|| \ v \ || = \max(|| \ v_1 \ ||, || \ v_2 \ ||)$.
QED

Proof of Main Step (A)**.**

We show that the equation

$$(L + \phi_1) \circ (id + u_1) = (id + u_1) \circ (L + \phi_2) \text{ with } Lip(\phi_i) < \varepsilon \tag{5}$$

has a unique solution $u_1 \in C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ for ε small.

Equation (5) is equivalent to

$$L \circ id + L \circ u_1 + \phi_1 \circ (id + u_1) = L + \phi_2 + u_1 \circ (L + \phi_2)$$

or,

$$u_1 - L^{-1} u_1 \circ (L + \phi_2) = L^{-1} \phi_2 - L^{-1} \phi_1 \circ (id + u_1). \tag{6}$$

Let $H : C_b^0(\mathbf{R}^n, \mathbf{R}^n) \to C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ be defined by

$$H(u) = L^{-1} \circ u \circ (L + \phi_2),$$

and let $H_1 = I - H$ with I the identity transformation of $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$. Then, both H and H_1 are bounded linear maps, and equation (6) becomes

$$H_1(u_1) = L^{-1} \phi_2 - L^{-1} \phi_1 \circ (id + u_1). \tag{7}$$

$$\text{Claim } (B): \qquad H_1 \text{ is an isomorphism and } \mid H_1^{-1} \mid \leq \frac{1}{(1-\tau)}.$$

Exercise. (Lipschitz Inverse Function Theorem). *Let $(V, |\cdot|)$ be a Banach space, and suppose $f : V \to V$ is 1-1, onto, and Lipschitz with Lipschitz inverse. There is an $\varepsilon > 0$ such that if $g = f + \phi$ where ϕ is Lipschitz with $\| \phi \|_0 < \varepsilon$ and $Lip(\phi) < \varepsilon$, then g is 1-1, onto, and Lipschitz with Lipschitz inverse.*

Proof of Claim (B)**.** Note that by the exercise, for ε small, $(L+\phi_2)^{-1}$ exists and is Lipschitz. This gives that H is an isomorphism with inverse $u \to L \circ u \circ (L+\phi_2)^{-1}$.

Let $\bar{V}_i = C_b^0(\mathbf{R}^n, V_i)$ for $i = 1, 2$. Then, $C_b^0(\mathbf{R}^n, \mathbf{R}^n) = \bar{V}_1 \oplus \bar{V}_2$, $H(\bar{V}_i) = \bar{V}_i$, $\mid H \mid \bar{V}_2 \mid < \tau$, and$\mid H^{-1} \mid \bar{V}_1 \mid < \tau$. Thus, H is hyperbolic on $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$. By Lemma 2, we have that H_1 is an isomorphism and $\mid H_1^{-1} \mid \leq \frac{1}{1-\tau}$ which is Claim (B).

Now, (6) becomes

$$H_1(u_1) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1)$$

or

$$u_1 = H_1^{-1}(L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id+u_1)) = H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id+u_1))$$

which means we want a fixed point in $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ of the map

$$T : u \to H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u))$$

We show that T is a contraction if ε is small.

We have,

$$\begin{aligned}
\| Tu - Tv \|_0 &= \| H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u)) \\
&\quad -(H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + v))) \|_0 \\
&= \| H_1^{-1}(L^{-1}\phi_1 \circ (id + v)) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u)) \|_0 \\
&\leq \mid H_1^{-1} \| L^{-1} \| \| \phi_1 \circ (id + u) - \phi_1 \circ (id + v) \|_0 \\
&\leq \mid H_1^{-1} \| L^{-1} \mid (Lip(\phi_1)) \| u - v \|_0.
\end{aligned}$$

So, if

$$Lip(\phi_1)|\ L^{-1}\ |\frac{1}{1-\tau} < 1,$$

then T is a contraction.

This completes the proofs of Theorems 1 and 2.