1. Find the residue at $z = 0$ of the function

(a) $\dfrac{1}{z+z^2}$; *(b)* $z\cos\left(\dfrac{1}{z}\right)$; *(c)* $\dfrac{z-\sin z}{z}$; *(d)* $\dfrac{\cot z}{z^4}$; *(e)* $\dfrac{\sinh z}{z^4(1-z^2)}$.

Ans. *(a)* 1; *(b)* $-1/2$; *(c)* 0; *(d)* $-1/45$; *(e)* $7/6$.

2. Use Cauchy's residue theorem (Sec. 70) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$; *(b)* $\dfrac{\exp(-z)}{(z-1)^2}$; *(c)* $z^2\exp\left(\dfrac{1}{z}\right)$; *(d)* $\dfrac{z+1}{z^2-2z}$.

Ans. *(a)* $-2\pi i$; *(b)* $-2\pi i/e$; *(c)* $\pi i/3$; *(d)* $2\pi i$.

3. Use the theorem in Sec. 71, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\dfrac{z^5}{1-z^3}$; *(b)* $\dfrac{1}{1+z^2}$; *(c)* $\dfrac{1}{z}$.

Ans. *(a)* $-2\pi i$; *(b)* 0; *(c)* $2\pi i$.

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

(a) $z\exp\left(\dfrac{1}{z}\right)$; *(b)* $\dfrac{z^2}{1+z}$; *(c)* $\dfrac{\sin z}{z}$; *(d)* $\dfrac{\cos z}{z}$; *(e)* $\dfrac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; *(b)* $\dfrac{1-\exp(2z)}{z^4}$; *(c)* $\dfrac{\exp(2z)}{(z-1)^2}$.

Ans. *(a)* $m = 1$, $B = -1/2$; *(b)* $m = 3$, $B = -4/3$; *(c)* $m = 2$, $B = 2e^2$.

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$; *(b)* $\left(\dfrac{z}{2z+1}\right)^3$; *(c)* $\dfrac{\exp z}{z^2+\pi^2}$.

Ans. *(a)* $m = 1$, $B = 3$; *(b)* $m = 3$, $B = -3/16$; *(c)* $m = 1$, $B = \pm i/2\pi$.

2. Show that

(a) $\displaystyle \operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi)$;

(b) $\displaystyle \operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;

(c) $\displaystyle \operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \quad (|z| > 0, 0 < \arg z < 2\pi)$.

3. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle *(a)* $|z-2| = 2$; *(b)* $|z| = 4$.

Ans. *(a)* πi; *(b)* $6\pi i$.

Show that

(a) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$;

(b) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}.$

Ans. $\pi/2$.

2. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}.$

Ans. $\pi/4$.

3. $\displaystyle\int_0^\infty \frac{dx}{x^4+1}.$

Ans. $\pi/(2\sqrt{2})$.

4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}.$

Ans. $\pi/6$.

5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}.$

Ans. $\pi/200$.

Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$

7. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)}.$

Ans. $-\pi/5$.