BOOTSTRAP PREDICTION IN UNIVARIATE VOLATILITY MODELS WITH LEVERAGE EFFECT

CARLOS TRUCIOS MAZA AND LUIZ K. HOTTA

INTRODUCTION

The prediction of future values is a key objective in time series analysis, and it is of interest in many areas of knowledge. In general, prediction intervals (PI) is calculated under the assumption that the model is known and has normal distribution errors, because under these assumptions, PI may be easily obtained knowing only the mean and standard deviation. The field of financial time series is no exception, there are also few works on procedures to obtain PI for returns and volatilities. Because financial time series have some especial features like leverage effect and asymmetric distribution the innovations, the usual approach is not adequate. An alternative is to obtain PI using bootstrap procedures, which do not require the choice of a distribution to the innovations.

METHODOLOGY

We present an adaptation of the PRR algorithm proposed by Pascual, Romo, and Ruiz (2006) for the GARCH process for application in EGARCH (Nelson,1991) and GJR-GARCH models (Glosten et al., 1993). These modifications keep the original idea of PRR, i.e., incorporating the element of uncertainty in parameter estimation and making no assumptions about the distribution of the innovations.

ALGORITHM FOR GJR-GARCH MODELS

Consider R_T a sequence of T observations generated by the process GJR-GARCH(1,1). The algorithm is described for process GJR-GARCH (1,1), but it is easily generalized to a GJR-GARCH (p,q) process in a straightforward way.

- Step 1: Estimate the model with the observed data and calculate the centered residuals. Denote by $\hat{F}_T$ the empirical distribution of the centered residuals.

- Step 2: Generate a bootstrap series $r_t^*, t = 1, ..., T$ using the following recursion:

$$\begin{aligned}\hat{\sigma}_t^{*2} &= \hat{\omega} + \hat{\alpha} r_{t-1}^{*2} + \hat{\beta}\hat{\sigma}_{t-1}^{*2} + \hat{\gamma} r_{t-1}^{*2} I(r_{t-1}^* < 0)) \\ r_t^* &= \varepsilon_t^* \hat{\sigma}_t^*,\end{aligned} \tag{1}$$

  where $\varepsilon_t^* \sim$ i.i.d $\hat{F}_T$. Finally adjust the bootstrap sequence r_T^* to obtain the bootstrap estimates $(\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)$.

- Step 3: Calculate forecasts of returns and volatilities h steps ahead, $h = 1, 2, ...$ using the following recursion:

$$\begin{aligned}\hat{\sigma}_T^{*2}(h) &= \hat{\omega}^* + \hat{\alpha}^* r_T^{*2}(h-1) + \hat{\beta}^* \hat{\sigma}_T^{*2}(h-1) \\ &\quad + \hat{\gamma}^* r_T^{*2}(h-1) I(r_T^*(h-1) < 0) \\ r_T^*(h) &= \varepsilon_T^*(h)\hat{\sigma}_T^*(h),\end{aligned} \tag{2}$$

  where $\varepsilon_T^*(h) \sim$ i.i.d $\hat{F}_T$, $r_T^* = r_T$ and

$$\begin{aligned}\sigma_T^{*2}(0) &= \frac{\omega^*}{1-\alpha^*-\beta^*-\gamma^* P} + \alpha^* \sum_{i=0}^{T-2} \beta^{*i} r_{T-i-1}^2 \\ &\quad - (\alpha^* + \gamma^* P)\frac{\omega^*}{1-\alpha^*-\beta^*-\gamma^* P} \sum_{i=0}^{T-2} \beta^{*i} \\ &\quad + \gamma^* \sum_{i=0}^{T-2} \beta^{*i} r_{T-i-1}^2 I(r_{T-i-1} < 0),\end{aligned} \tag{3}$$

  where $P = Prob(r_t < 0)$.

- Step 4: Repeat steps 2 and 3, B times to obtain B bootstrap replicates $(r_T^{*(1)}(h), ..., r_T^{*(B)}(h))$ and $(\sigma_T^{*(1)}(h), ..., \sigma_T^{*(B)}(h))$. Then a $100(1-\gamma)\%$ PI for $r_T^*(h)$ is given by:

$$[Q_{r,B}^*(\tfrac{\gamma}{2}), Q_{r,B}^*(1-\tfrac{\gamma}{2})],$$

  where: $Q_{r,B}^* = G_{r,B}^{*-1}$, and $G_{r,B}^*(h) = \frac{\#(r_T^{*b}(h) \leq h)}{B}$. Similarly, obtain a PI for volatility.

ALGORITHM FOR EGARCH MODELS

Let R_T be a sequence of T observations generated by the EGARCH(1,1) process. Steps 1, 2 and 4 are similar to those described in the previous algorithm. Step 3 is modified as follows:

- Step 3: Calculate forecasts of returns and volatilities h steps ahead, $h = 1, 2, ...$ using the following recursion:

$$\begin{aligned}\log(\hat{\sigma}_T^{*2}(h)) &= \hat{\omega}^* + \hat{\alpha}^* \frac{r_T^*(h)}{\sigma_T^*(h)} + \hat{\gamma}^*[\left|\frac{r_T^*(h-1)}{\sigma_T^*(h-1)}\right| - M] \\ &\quad + \hat{\beta}^* log(\sigma_T^{*2}(h-1)) \\ r_T^*(h) &= \varepsilon_T^*(h)\hat{\sigma}_T^*(h),\end{aligned} \tag{4}$$

  where $\varepsilon_T^*(h) \sim$ i.i.d $\hat{F}_T$, $M = E(|\varepsilon_T|)$, $r_T^* = r_T$ and:

$$\begin{aligned}\log(\hat{\sigma}_T^{*2}(0)) &= \frac{\omega^*}{1-\beta^*} + \alpha^* \sum_{i=0}^{T-2} \beta^{*j} \frac{r_{T-1-j}}{\sigma_{T-1-j}^*} \\ &\quad + \gamma \sum_{i=0}^{T-2} \beta^{*j} \left|\frac{r_{T-1-j}}{\sigma_{T-1-j}^*}\right| - \gamma E(|\varepsilon_T|) \sum_{i=0}^{T-2} \beta^{*j}\end{aligned} \tag{5}$$

SIMULATION STUDY

Here we analyze the performance of the suggested algorithm through Monte Carlo simulation. We consider four models, EGARCH (1,1) and GJR-GARCH(1,1) models with skew-normal (SN) and skew-t (ST) distributions for the innovations. The models are:

- $EGARCH(1,1)$

$$\begin{aligned}log(\sigma_t^2) &= -0.3474 - 0.1420\varepsilon_t + 0.2195(|\varepsilon_{t-1}| \\ &\quad - E(|\varepsilon_{t-1}|)) + 0.9496 log(\sigma_{t-1}^2)\end{aligned} \tag{6}$$

- $GJR - GARCH(1,1)$

$$\begin{aligned}\sigma_t^2 &= 0.01 + 0.05\varepsilon_{t-1}^2 + 0.15\varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0) \\ &\quad + 0.83\sigma_{t-1}^2\end{aligned} \tag{7}$$

where ε_t are innovations. In the EGARCH(1,1) model we considered innovations with SN($\lambda = 0.7$) and ST($\lambda = 0.8$, *shape* $= 6$) distributions, while for the GJR-GARCH(1,1) model we considered innovations with SN($\lambda = 0.7$) and ST ($\lambda = 0.8$, *shape* $= 7$) distributions. The distributions were parameterized as in Fernandez and Steel (1998), such that, when $\lambda = 1$, the ST (λ,*shape*) distribution is symmetric. For each model we ran 1000 replications

SIMULATION ALGORITHM

For each replication we have three steps. For the k-th replication the steps are:
Step 1: Generate a series with size $T = 1000$. That is generate $\{r_t, t = 1, \cdots, 1000\}$ and $\{\sigma_t, t = 1, \cdots, 1000\}$.
Step 2: Run the algorithm for GJR model or EGARCH model to find the $h = 1, 3, 5$ steps-ahead 95%-PI for the returns and volatilities.
Step 3: Generate 1000 sets of future values values r_{T+j}^i and σ_{T+j}^i, $j = 1, \cdots, 5$, $i = 1, \cdots, 1000$ considering that the previous observations are the same as given in Step 1. Calculated the proportions of the values $\{r_{T+h}^i, i = 1, \cdots, 1000\}$ which are inside the PI found in step 2, below the lower limit of the PI and above the PI, respectively. Similarly for the volatilities.

RESULTS

The analysis of the results shows that the performance of the proposed bootstrap procedure to obtain PI for the returns and volatilities for all the models was very good, with the empirical coverage close to the nominal values. However, while for the returns we have approximately the same proportion of observations below and above the PI, for the volatilities the proportion of values above the PI is larger than below the PI. We ran further analysis to study the effect of additive outliers in the PI. We found out that when an additive outlier is inserted in the series, the effect can be very large when the outlier is near the end of the series. Thus, the presence of an additive outlier must be treated with care.

Table 1: Summary of the Simulations of the prediction intervals for returns and volatilities of EGARCH(1,1) and GJR-GARCH(1,1) models. h-steps-ahead prediction. Nominal coverage 95%.

Horizon	Innovations distrib.	Average coverage	Standard error	Av. below coverage	Av. above coverage	Average length
			EGARCH			
			Returns			
h=1	*SN*	0.94801	0.01521	0.02548	0.02652	0.13060
	ST	0.94842	0.01593	0.02517	0.02641	0.13720
h=3	*SN*	0.94839	0.01463	0.02547	0.02614	0.13315
	ST	0.94891	0.01554	0.02492	0.02618	0.13998
h=5	*SN*	0.94802	0.01454	0.02556	0.02642	0.13531
	ST	0.94842	0.01608	0.02527	0.02631	0.14163
			Volatilities			
h=1	*SN*	0.94700	0.22415	0.02300	0.03000	0.00581
	ST	0.94000	0.23761	0.02700	0.03300	0.00774
h=3	*SN*	0.95122	0.06246	0.02019	0.02859	0.02210
	ST	0.94815	0.07980	0.02370	0.02815	0.02501
h=5	*SN*	0.95116	0.04329	0.01980	0.02904	0.02915
	ST	0.94795	0.05861	0.02338	0.02868	0.03303
			GJR-GARCH			
			Returns			
h=1	SN	0.94792	0.01445	0.02547	0.02661	1.87202
	ST	0.94756	0.01492	0.02551	0.02693	1.82846
h=3	SN	0.94797	0.01451	0.02552	0.02651	1.90349
	ST	0.94756	0.01508	0.02585	0.02660	1.85562
h=5	SN	0.94846	0.01451	0.02503	0.02651	1.93542
	ST	0.94792	0.01496	0.02552	0.02656	1.88851
			Volatilities			
h=1	SN	0.94500	0.22809	0.02300	0.03200	0.08118
	ST	0.94500	0.22809	0.01700	0.03800	0.10075
h=3	SN	0.95052	0.07615	0.02163	0.02785	0.30929
	ST	0.95229	0.07738	0.01834	0.02937	0.32566
h=5	SN	0.94821	0.05833	0.02361	0.02818	0.40440
	ST	0.94804	0.06531	0.02171	0.03025	0.42995

REFERENCES

[1] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31, 307–327, 1986.

[2] C. Fernandez and M. Steel, On bayesian modeling of fat tails and skewness. Journal of the American Statistical Association 93, 359–371, 1998.

[3] L. Glosten and R. Jagannathan and D. Runkle, On the relation be-tween the expected value and the volatility of the nominal excess return on stocks. The Journal of Finance 48, 1779–1801, 1993.

[4] D. Nelson, Autoregressive conditional heteroskedasticity: A new approach. Econometrica 59, 347–370, 1991.

[5] L. Pascual and J. Romo and E. Ruiz, Bootstrap prediction for returns and volatilities in garch models. Computational Statistics and Data Analysis 50, 2293–2312, 2006

ACKNOWLEDGEMENTS

The authors acknowledge support from CAPES, FAPESP and Laboratório EPIFISMA