Conditional quenched CLTs for random walks among random conductances

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One-dimensional random walks with unbounded jumps

Many-dimensional random walks (nearest-neighbor and uniformly elliptic)
Initial motivation: gas of particles in a finite random tube (Comets, Popov, Schütz, Vachkovskaia, JSP–2010):

Figure: Particles are injected at the left boundary, and killed at both boundaries

Technical difficulty: prove that $P_\omega[\text{time } \leq \varepsilon H^2 | \text{ cross the tube}]$ is small. This would be a consequence of a conditional CLT!
The model:

- in $\mathbb{Z}$, to any pair $(x, y)$ attach a positive number $\omega_{x,y}$ (conductance between $x$ and $y$).
- $\mathbb{P}$ stands for the law of this field of conductances. We assume that $\mathbb{P}$ is stationary and ergodic.
- define $\pi_x = \sum_y \omega_{x,y}$, and let the transition probabilities be

$$q_\omega(x, y) = \frac{\omega_{x,y}}{\pi_x}$$

- $P^x_\omega$ is the quenched law of the random walk starting from $x$, so that

$$P^x_\omega[X(0) = x] = 1, \quad P^x_\omega[X(k+1) = z \mid X(k) = y] = q_\omega(y, z).$$
We assume “local uniform ellipticity” and polynomial tails of jumps:

**Condition E.**

(i) There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $q_\omega(0, \pm 1) \geq \kappa$.

(ii) Also, there exists $\hat{\kappa} > 0$ such that $\hat{\kappa} \leq \sum_{y \in \mathbb{Z}} \omega_{0,y} \leq \hat{\kappa}^{-1}$, $\mathbb{P}$-a.s.

**Condition K.** There exist constants $K, \beta > 0$ such that $\mathbb{P}$-a.s., $\omega_{0,y} \leq K|y|^{-(3+\beta)}$, for all $y \in \mathbb{Z} \setminus \{0\}$.

(observe that this implies that the second moment of the jump is uniformly bounded)
Brownian Meander:

Let $W$ be the Brownian Motion starting from 0, and define $\tau_1 = \sup\{s \in [0, 1] : W(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$.

Then, the Brownian Meander $W^+$ is defined in this way:

$$W^+(s) := \Delta_1^{-1/2} |W_1(\tau_1 + s\Delta_1)|, \quad 0 \leq s \leq 1.$$  

Informally, the Brownian Meander is the Brownian Motion conditioned on staying positive on the time interval $(0, 1]$.

Example: simple random walk $S$, conditioned on $\{S_1 > 0, \ldots, S_n > 0\}$, after usual scaling converges to the Brownian Meander.
Let 
\[ \Lambda_n := \{ X(k) > 0 \text{ for all } k = 1, \ldots, n \} \]

Consider the conditional quenched probability measure 
\[ Q_n^\omega[\cdot] := P_\omega[\cdot | \Lambda_n]. \]

Define the continuous map \( Z^n(t), t \in [0, 1] \) as the natural polygonal interpolation of the map \( k/n \mapsto \sigma^{-1} n^{-1/2} X(k) \) (with \( \sigma \) from the quenched CLT).

For each \( n \), the random map \( Z^n \) induces a probability measure \( \mu_n^\omega \) on \( (C[0,1], \mathcal{B}_1) \): for any \( A \in \mathcal{B}_1 \),
\[ \mu_n^\omega(A) := Q_n^\omega[Z^n \in A]. \]
Main result:

**Theorem**

Under Conditions E and K, we have that, \( \mathbb{P}\text{-a.s.}, \mu^n_\omega \) tends weakly to \( P_{W^+} \) as \( n \to \infty \), where \( P_{W^+} \) is the law of the Brownian meander \( W^+ \) on \( C[0, 1] \).

As a corollary of Theorem 1.1, we obtain a limit theorem for the process conditioned on crossing a large interval. Define

\[
\hat{\tau}_n = \inf\{k \geq 0 : X_k \in [n, \infty)\} \quad \text{and} \quad \Lambda'_n = \{\hat{\tau}_n < \hat{\tau}\}.
\]

**Corollary**

Assume Conditions E and K. Then, conditioned on \( \Lambda'_n \), the process converges to the “Brownian crossing”.
strategy of the proof: force the walk a bit away from the origin, and use the (unconditional) quenched invariance principle.

in fact, one needs even the “uniform” version of the quenched invariance principle (i.e., at time $t$ the rescaled RW is “close” to BM uniformly with respect to the starting point chosen in the interval of length $O(\sqrt{t})$ around the origin)

the main difficulty: control the (both conditional and unconditional) exit measure from large intervals

observe that is $\xi$ has only polynomial tail, then 

$$\frac{P[x<\xi\leq x+a]}{P[\xi>x]} \to 0 \text{ as } x \to \infty$$
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Many-dimensional random walks (nearest-neighbor and uniformly elliptic)
The model:

- in $\mathbb{Z}^d$, to any unordered pair of neighbors attach a positive number $\omega_{x,y}$ (conductance between $x$ and $y$).
- $\mathbb{P}$ stands for the law of this field of conductances. We assume that $\mathbb{P}$ is stationary and ergodic.
- define $\pi_x = \sum_{y \sim x} \omega_{x,y}$, and let the transition probabilities be
  $$q_{\omega}(x, y) = \begin{cases} \frac{\omega_{x,y}}{\pi_x}, & \text{if } y \sim x, \\ 0, & \text{otherwise}, \end{cases}$$
- $P^x_\omega$ is the quenched law of the random walk starting from $x$, so that
  $$P^x_\omega[X(0) = x] = 1, \quad P^x_\omega[X(k+1) = z \mid X(k) = y] = q_{\omega}(y, z).$$
(many recent papers) $\Rightarrow$ under mild conditions on the law of $\omega$-s, the **Quenched Invariance Principle** holds:

*For almost every environment $\omega$, suitably rescaled trajectories of the random walk converge to the Brownian Motion (with nonrandom diffusion constant $\sigma$) in a suitable sense.*

Main method of the proof: the “corrector approach”, i.e., find a “stationary deformation” of the lattice such that the random walk becomes martingale.

The corrector is shown to exist, but usually no explicit formula is known for it.
Let
\[ Λ_n := \{X_1(k) > 0 \text{ for all } k = 1, \ldots, n\} \]

(\(X_1\) is the first coordinate of \(X\)).

Consider the conditional quenched probability measure
\[ Q^n_ω[·] := P_ω[· | Λ_n]. \]

Define the continuous map \(Z^n(t), t \in [0, 1]\) as the natural polygonal interpolation of the map \(k/n \mapsto σ^{-1} n^{-1/2} X(k)\) (with \(σ\) from the quenched CLT).

For each \(n\), the random map \(Z^n\) induces a probability measure \(μ^n_ω\) on \((C[0, 1], B_1)\): for any \(A \in B_1\),
\[ μ^n_ω(A) := Q^n_ω[Z^n \in A]. \]
Condition E’. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\kappa < \omega_{0,x} < \kappa^{-1}$ for $x \sim 0$.

Denote by $P_{W^+} \otimes P_{W^{(d-1)}}$ the product law of Brownian meander and $(d-1)$-dimensional standard Brownian motion on the time interval $[0, 1]$.

Now, we formulate our main result:

**Theorem**

*Under Condition E’, we have that, $\mathbb{P}$-a.s., $\mu^n_\omega$ tends weakly to $P_{W^+} \otimes P_{W^{(d-1)}}$ as $n \to \infty$ (as probability measures on $C[0, 1]$).*
Strategy of the proof: “go away a little bit from the forbidden area in a controlled way”

(we need to control the time and the vertical displacement), and then use unconditional CLT (in fact, again, the *uniform* version of the CLT makes life easier)
control of time:

\[ N = \varepsilon \sqrt{n} \]

\[ \alpha \in \left( \frac{1}{4}, 1 \right) \]

\[ P_\omega[\tau_N > n | \Lambda_n] \approx \text{small} \]

\[ P_\omega[\tau_N > n | \Lambda_n] \leq P_\omega[\tau_{N/2} > \alpha n | \Lambda_n] + \text{something small,} \]

then iterate:

\[ P_\omega[\tau_{2^{-j}N} > \alpha^j n | \Lambda_n] \leq P_\omega[\tau_{2^{-(j+1)}N} > \alpha^{j+1} n | \Lambda_n] + \text{smth very small} \]
control of “vertical” displacement:

$$N = \varepsilon \sqrt{n}$$

$$\alpha \in (\frac{1}{2}, 1)$$

$$P_\omega \left[ \sup_{j \leq \tau_N} |X_2(j)| > \varepsilon' N \mid \Lambda_n \right] \approx \text{small}$$

$$G_k = \left\{ \sup_{j \in (\tau_{2-kN}, \tau_{2-k+1N})} |X_2(j) - X_2(\tau_{2-kN})| \leq \varepsilon'' \alpha^k N \right\}$$

observe that, for $G_k$, \( \frac{\text{vertical size}}{\text{horizontal size}} \sim (2\alpha)^k \)
Open questions:

- not uniformly bounded conductances, RWs on percolation clusters, . . . ?
- other types of conditioning?
- $P_{\omega}[\Lambda_n] \sim$ ?
- in particular, can one prove that $\frac{C_1}{n} \leq P_{\omega}[\text{cross the strip of width } n] \leq \frac{C_2}{n}$?