Controllability of a 1-D tank containing a fluid modeled by a Boussinesq system

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Joint work with
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The Boussinesq system


\[
\begin{align*}
\eta_t + w_x + (\eta w)_x + aw_{xxx} - b\eta_{xxt} &= 0 \\
w_t + \eta_x + \eta w_x + c\eta_{xxx} - d\eta_{xxt} &= 0,
\end{align*}
\]

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The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

\( \eta \) is the elevation of the fluid surface from the equilibrium position; 
\( w = w_\theta \) is the horizontal velocity in the flow at height \( \theta h \), where \( h \) is the undisturbed depth of the liquid;

\( a, b, c, d \), are parameters required to fulfill the relations

\[
a + b = \frac{1}{2} \left( \theta^2 - \frac{1}{3} \right), \quad c + d = \frac{1}{2} (1 - \theta^2) \geq 0
\]

where \( \theta \in [0, 1] \).
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then
\[ a + b + c + d = \frac{1}{3}. \]

If we assume that
\[ a \neq 0, \quad c \neq 0 \quad \text{and} \quad b = d = 0, \]
due to global well-posedness restrictions
\[ a \leq 0 \quad \text{and} \quad c \leq 0 \quad \text{or} \quad a = c > 0. \]

Since \( a + c = \frac{1}{3} \), this leads to
\[ a = c = \frac{1}{6}, \quad \theta = \sqrt{\frac{2}{3}}. \]
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A tank containing a fluid

The tank is filled with liquid and should be moved to different steady-state workbenches as fast as possible.

Figure: Fluid in the 1-D tank
\begin{itemize}
  \item $L$ is the length of the tank and, for simplicity, we assume that $L = \pi$;
  \item $\eta(t, x)$ is the elevation of the fluid surface at time $t$ and at the position $x \in (0, \pi)$;
  \item $\omega(t, x)$ is the horizontal fluid velocity (for some parameter $\theta \in [0, 1]$) \textit{in a referential attached to the tank} at time $t$ and at the position $x \in (0, \pi)$;
  \item $D = D(t)$ is the horizontal displacement of the tank;
  \item $s = s(t)$ is the horizontal velocity of the tank;
  \item $u = u(t)$ is the horizontal acceleration of the tank.
\end{itemize}

\[ \frac{dD}{dt} = s, \quad \frac{ds}{dt} = u \quad \text{and} \quad \frac{d^2D}{dt^2} = u. \]
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The full dynamics

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\begin{align*}
\eta_t + \omega_x + a\omega_{xxx} &= 0 \\
\omega_t + \eta_x + c\eta_{xxx} &= -u(t) \\
\frac{ds}{dt} &= u \\
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\end{align*}
\]

where \(0 < x < L\) and \(t > 0\), with the boundary conditions

\[
\begin{align*}
\eta_x(t, 0) &= \eta_x(t, L) = -u(t) \\
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\end{align*}
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Since \(\ddot{D}(t) = u(t)\), we have the following initial conditions

\[
\begin{align*}
\eta(0, x) &= \eta^0(x), & \omega(0, x) &= \omega^0(x), & D(0) &= D^0, & \dot{D}(t) &= D^1.
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A control problem

Can we move the tank, continuously, from any steady state to any other steady state?

- F. Dubois, N. Petit and P. Rouchon, Motion planning and nonlinear simulations for a tank containing a fluid, European Control Conference, Karlsruhe, Germany (1999).
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Exact controllability

The system is exactly controllable in some appropriate Hilbert space $\mathcal{H}$ when $u(\cdot) \in L^2(0, T)$.

More precisely, given $T > 0$, the initial state $(\eta^0, \omega^0, D^0, D^1)$ and the terminal state $(\eta^T, \omega^T, D^{0,T}, D^{1,T})$ in $\mathcal{H}$, we can find a control $u \in L^2(0, T)$ such that the system admits a solution satisfying $(\eta(T), \omega(T), D(T), D'(T)) = (\eta^T, \omega^T, D^{0,T}, D^{1,T})$.

A classical duality approach:


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A classical duality approach:

⇒ Proof of an observability inequality for the solutions of the adjoint system.

- Fourier expansion of the solutions of the adjoint system.
- Splitting into high/small frequencies.
- Ingham’s inequality and
- reduction to a spectral problem.
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- reduction to a spectral problem.
Global well-posedness for the Boussinesq system

We first consider the following system

\begin{align*}
\eta_t + \omega_x + a\omega_{xxx} &= f, \\
\omega_t + \eta_x + c\eta_{xxx} &= g, \\
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\eta(0, x) &= \eta^0(x), \quad \omega(0, x) = \omega^0(x),
\end{align*}

where $0 < x < \pi$ and $t > 0$. At least, formally,

\[(\eta, \omega)(t, x) = \sum_{k \geq 1} (\hat{\eta}_k(t) \cos(kx), \hat{\omega}_k(t) \sin(kx)),\]

where

\begin{align*}
(\hat{\eta}_k)_t + k\hat{\omega}_k - ak^3\hat{\omega}_k &= \hat{f}_k, \quad 0 < t < T, \\
(\hat{\omega}_k)_t - k\hat{\eta}_k + ck^3\hat{\eta}_k &= \hat{g}_k, \quad 0 < t < T, \\
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\end{align*}
\]
If we set

\[ A(k) = \begin{pmatrix} 0 & 1 - ak^2 \\ -1 + ck^2 & 0 \end{pmatrix}, \]

it is easy to see that (3) is equivalent to

\[ \begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}_t + kA(k) \begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix} = \begin{pmatrix} \hat{f}_k \\ \hat{g}_k \end{pmatrix}, \quad \begin{pmatrix} \hat{\eta}_k \\ \hat{\omega}_k \end{pmatrix}(0) = \begin{pmatrix} \hat{\eta}_k^0 \\ \hat{\omega}_k^0 \end{pmatrix}. \]

Note that the eigenvalues of the matrix \( A(k) \) are with

\[ \sigma(k) = \pm \sqrt{(1 - ak^2)(-1 + ck^2)}, \]

and that they are purely imaginary.

We introduce the notations

\[ w_1(k) = 1 - ak^2, \quad w_2(k) = 1 - ck^2. \]
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We introduce the notations

$$w_1(k) = 1 - ak^2, \quad w_2(k) = 1 - ck^2.$$
We introduce the space \( V^s = H^s_{\text{even}}(0, \pi) \times H^s_{\text{odd}}(0, \pi) \), where

\[
H^s_{\text{odd}}(0, \pi) = \left\{ u = \sum_{k \geq 1} \hat{u}_k \sin(kx); \quad \|u\|^2_{H^s_{\text{odd}}(0, \pi)} := \sum_{k \geq 1} k^{2s} |\hat{u}_k|^2 < \infty \right\}
\]

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\]

endowed with the norm

\[
\| (\eta, \omega) \|^2_{V^s} := \|\eta\|^2_{H^s_{\text{even}}} + \|\mathcal{H}\omega\|^2_{H^s_{\text{odd}}}. \tag{4}
\]

The operator \( \mathcal{H} \) that appears in (4) is defined in the following way:

\[
\mathcal{H} \left( \sum_{k \geq 1} \hat{\omega}_k \sin(kx) \right) = \sum_{k \geq 1} \sqrt{\frac{w_1(k)}{w_2(k)}} \hat{\omega}_k \sin(kx).
\]
Theorem

The family of linear operators \( \{S(t)\}_{t \in \mathbb{R}} \) defined by

\[
S(t)(\eta^0, \omega^0) = \sum_{k \geq 1} (\hat{\eta}_k(t) \cos(kx), \hat{\omega}_k(t) \sin(kx)),
\]

where the Fourier coefficients of \((\eta(t), \omega(t))\) are obtained from those of \((\eta^0, \omega^0)\) by

\[
\begin{align*}
\hat{\eta}_k(t) &= \cos(k\lambda(k)t)\hat{\eta}_k^0 - \sqrt{\frac{w_1(k)}{w_2(k)}} \sin(k\lambda(k)t)\hat{\omega}_k^0, \\
\hat{\omega}_k(t) &= \sqrt{\frac{w_2(k)}{w_1(k)}} \sin(k\lambda(k)t)\hat{\eta}_k^0 + \cos(k\lambda(k)t)\hat{\omega}_k^0,
\end{align*}
\]

is a group of isometries in \( V^s \), for any \( s \in \mathbb{R} \).
Theorem

The infinitesimal generator of the group \( \{S(t)\}_{t \in \mathbb{R}} \) is the unbounded operator \((D(A), A)\), where \( D(A) = V^{s+3} \) and

\[
A(\eta, \omega) = (-\omega_x - a\omega_{xxx}, -\eta_x - c\eta_{xxx}), \quad \forall (\eta, \omega) \in D(A).
\]

Theorem

Let \( T > 0 \) and \( s \in \mathbb{R} \) be given. If \((\eta^0, \omega^0) \in V^s \) and \((f, g) \in C^1([0, T], V^{s-3})\), then the problem admits a unique solution \((\eta, \omega) \in C([0, T], V^s) \cap C^1([0, T], V^{s-3})\). Moreover, there exists a positive constant \( C > 0 \), such that

\[
\|(\eta, \omega)\|_{C([0, T], V^s)} + \|(\eta, \omega)\|_{C^1([0, T], V^{s-3})} \\ \leq C \left[ \|(f, g)\|_{C^1([0, T]; V^{s-3})} + \|(\eta^0, \omega^0)\|_{V^s} \right].
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\| (\eta, \omega) \|_{C([0, T], V^s)} + \| (\eta, \omega) \|_{C^1([0, T], V^{s-3})} \leq C \left[ \| (f, g) \|_{C^1([0, T]; V^{s-3})} + \| (\eta^0, \omega^0) \|_{V^s} \right].
\]
Global well-posedness for the tank problem

Now consider the following system

\[
\begin{align*}
\eta_t + \omega_x + a\omega_{xxx} &= 0 \\
\omega_t + \eta_x + c\eta_{xxx} &= -u(t) \\
\dot{D}(t) &= u(t),
\end{align*}
\]

where \(0 < x < L\) and \(t > 0\), with the boundary conditions

\[
\begin{align*}
\eta_x(t, 0) &= \eta_x(t, L) = -u(t) \\
\omega(t, 0) &= \omega(t, L) = 0 \\
\omega_{xx}(t, 0) &= \omega_{xx}(t, L) = 0,
\end{align*}
\]

and the following initial conditions

\[
\begin{align*}
\eta(0, x) &= \eta^0(x), & \omega(0, x) &= \omega^0(x), & D(0) &= D^0, & \dot{D}(t) &= D^1.
\end{align*}
\]
The change of functions

\[(\varphi(t,x), \psi(t,x)) = (\eta(t,x), \omega(t,x)) - S(t)(\eta^0, \omega^0) + (u(t)\phi(x), 0)\]

transforms the previous system into

\[
\begin{align*}
\varphi_t + \psi_x + a\psi_{xxx} &= f := u'(t)\phi(x), \\
\psi_t + \varphi_x + c\varphi_{xxx} &= g := u(t)(-1 + \phi'(x) + c\phi'''(x)), \\
\ddot{D}(t) &= u(t), \\
\varphi_x(t,0) &= \varphi_x(t,\pi) = 0, \\
\psi(t,0) &= \psi(t,\pi) = 0, \\
\psi_{xx}(t,0) &= \psi_{xx}(t,\pi) = 0, \\
\varphi(0,x) &= 0, \quad \psi(0,x) = 0, \quad D(0) = D^0, \quad \dot{D}(0) = D^1,
\end{align*}
\]

for a convenient \(\phi = \phi(x)\) and \(u \in C^2([0, T], \mathbb{R})\); \(u(0) = 0\).
Exact controllability

For each $s$, we introduce the spaces

$$
\hat{H}^s_{\text{odd}}(0, \pi) = \left\{ u \in H^s_{\text{odd}}(0, \pi); \sum_{n \geq 1} |c_n|^2 n^{2s} < \infty \text{ e } c_n = 0 \text{ for } n \in 2\mathbb{Z} \right\};
$$

$$
\hat{H}^s_{\text{even}}(0, \pi) = \left\{ u \in H^s_{\text{even}}(0, \pi); \sum_{n \geq 1} |c_n|^2 n^{2s} < \infty \text{ e } c_n = 0 \text{ for } n \in 2\mathbb{Z} \right\};
$$

$$
\mathcal{H} = \hat{H}^1_{\text{even}} \times \hat{H}^1_{\text{odd}} \times \mathbb{R} \times \mathbb{R} \text{ and } \mathcal{H}' = \hat{H}^{-1}_{\text{even}} \times \hat{H}^{-1}_{\text{odd}} \times \mathbb{R} \times \mathbb{R}.
$$

Theorem

Let $T > 0$. Then, for any $(\eta^0, \omega^0, D^0, D^1) \in \mathcal{H}'$ and any $(\eta^T, \omega^T, D^0,T, D^1,T) \in \mathcal{H}'$, there exists a control input $u \in L^2(0, T)$ such that the solution $(\eta, \omega, D)$ of the system satisfies $(\eta(T), \omega(T), D(T), \dot{D}(T)) = (\eta^T, \omega^T, D^0,T, D^1,T)$. 
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$$

$$
\hat{H}^s_{even}(0, \pi) = \{ u \in H^s_{even}(0, \pi); \sum_{n \geq 1} |c_n|^2 n^{2s} < \infty \text{ e } c_n = 0 \text{ for } n \in 2\mathbb{Z} \};
$$

$$
\mathcal{H} = \hat{H}^1_{even} \times \hat{H}^1_{odd} \times \mathbb{R} \times \mathbb{R} \text{ and } \mathcal{H}' = \hat{H}^{-1}_{even} \times \hat{H}^{-1}_{odd} \times \mathbb{R} \times \mathbb{R}.
$$

Theorem

Let $T > 0$. Then, for any $(\eta^0, \omega^0, D^0, D^1) \in \mathcal{H}'$ and any $(\eta^T, \omega^T, D^{0,T}, D^{1,T}) \in \mathcal{H}'$, there exists a control input $u \in L^2(0, T)$ such that the solution $(\eta, \omega, D)$ of the system satisfies $(\eta(T), \omega(T), D(T), \dot{D}(T)) = (\eta^T, \omega^T, D^{0,T}, D^{1,T})$. 
The adjoint system

We consider \((p, q, E)\), solution of

\[
\begin{align*}
    p_t + q_x + cq_{xxx} &= 0, \\
    q_t + p_x + ap_{xxx} &= 0, \\
    \ddot{E}(t) &= 0,
\end{align*}
\]

satisfying the boundary conditions

\[
\begin{align*}
    p_x(t, 0) &= p_x(t, \pi) = 0, \\
    q(t, 0) &= q(t, \pi) = 0, \\
    q_{xx}(t, 0) &= q_{xx}(t, \pi) = 0,
\end{align*}
\]

and initial conditions

\[
p(T, x) = p^T(x), \quad q(T, x) = q^T(x), \quad E(T) = E^0, T, \quad \dot{E}(T) = E^1, T,
\]

where \(0 < x < \pi\) and \(t > 0\).

Observe that \(E(t) = \beta t + \alpha\), where \(\alpha = E^0, \beta = E^1\).
The adjoint system

We consider \((p, q, E)\), solution of

\[
\begin{aligned}
    p_t + q_x + cq_{xxx} &= 0, \\
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\end{aligned}
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\[
\begin{aligned}
    p_x(t, 0) &= p_x(t, \pi) = 0, \\
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    q_{xx}(t, 0) &= q_{xx}(t, \pi) = 0,
\end{aligned}
\]

and initial conditions

\[
p(T, x) = p^T(x), \quad q(T, x) = q^T(x), \quad E(T) = E^0, T, \quad \dot{E}(T) = E^1, T,
\]

where \(0 < x < \pi\) and \(t > 0\).

Observe that \(E(t) = \beta t + \alpha\), where \(\alpha = E^0, \beta = E^1\).
Definition of the solution by transposition

Multiply the first equation of the system by \(p\), the second one by \(q\) and the third one by \(E(t)\). Integrating by parts over \((0, T) \times (0, \pi)\) and assuming that the functions \((\eta, \omega, D)\) and \((p, q, E)\) are sufficiently regular, we obtain

\[
- \int_0^T \int_0^\pi \eta (p_t + q_x + cq_{xxx}) \, dx \, dt - \int_0^T \int_0^\pi \omega (q_t + p_x + ap_{xxx}) \, dx \, dt \\
+ \int_0^\pi \{ \ldots \} \, dx + \int_0^T \{ \ldots \} \, dt = - \int_0^T \int_0^\pi u(t) q \, dx \, dt,
\]

\[
\left[ \dot{D}(t) E(t) \right]_0^T - \left[ D(t) \dot{E}(t) \right]_0^T + \int_0^T D(t) \ddot{E}(t) \, dt = \int_0^T u(t) E(t) \, dt.
\]

If \((p, q, E)\) is a solution of (6), then we obtain

\[
\left[ \dot{D}(t) E(t) \right]_0^T - \left[ D(t) \dot{E}(t) \right]_0^T + \int_0^T D(t) \ddot{E}(t) \, dt = \int_0^T u(t) E(t) \, dt.
\]
\[
\int_0^\pi \left[ \eta p \right]_0^T + \int_0^\pi \left[ \omega q \right]_0^T - c \int_0^T \left[ \eta_x q_x \right]_0^\pi = -\int_0^T \int_0^\pi u(t)q dx dt \\
\left[ D(t)E(t) \right]_0^T - \left[ D(t)\dot{E}(t) \right]_0^T = \int_0^T u(t)E(t) dt.
\]

**Definition**

A function \( (\eta, \omega, D) \in \mathcal{H}^\prime := \mathcal{H}^{-1}_{\text{even}} \times \mathcal{H}^{-1}_{\text{odd}} \times \mathbb{R} \times \mathbb{R} \), such that

\[
\left\langle (\eta(t), \omega(t), -\dot{D}(t), D(t)), (p(t), q(t), E(t), \dot{E}(t)) \right\rangle_{\mathcal{H}', \mathcal{H}}
= -\int_0^t u(\tau) \left\{ \int_0^\pi (q + c q_{xx}) dx + E(\tau) \right\} d\tau
+ \left\langle (\eta^0, \omega^0, -D^1, D^0), (p^0, q^0, E(0), \dot{E}(0)) \right\rangle_{\mathcal{H}', \mathcal{H}}
\]

solution by transposition of the tank model.
\[
\int_0^\pi [\eta p]_0^T + \int_0^\pi [\omega q]_0^T - c \int_0^T [\eta x q x]_0^\pi = - \int_0^T \int_0^\pi u(t) q dx dt
\]

\[
\left[ \dot{D}(t) E(t) \right]_0^T - \left[ D(t) \dot{E}(t) \right]_0^T = \int_0^T u(t) E(t) dt.
\]

**Definition**

A function \((\eta, \omega, D) \in \mathcal{H}' := \hat{H}^{-1}_{\text{even}} \times \hat{H}^{-1}_{\text{odd}} \times \mathbb{R} \times \mathbb{R}\), such that

\[
\left\langle (\eta(t), \omega(t), -\dot{D}(t), D(t)), (p(t), q(t), E(t), \dot{E}(t)) \right\rangle_{\mathcal{H}', \mathcal{H}}
\]

\[
= - \int_0^t u(\tau) \left\{ \int_0^\pi (q + c q_{xx}) dx + E(\tau) \right\} d\tau
\]

\[
+ \left\langle (\eta^0, \omega^0, -D^1, D^0), (p^0, q^0, E(0), \dot{E}(0)) \right\rangle_{\mathcal{H}', \mathcal{H}}
\]

**solution by transposition** of the tank model.
Identity (6) defines \((\eta(t), \omega(t), -\dot{D}(t), D(t)) \in \mathcal{H}'\) in a unique way and \((\eta, \omega, -\dot{D}, D) \in C([0, T]; \mathcal{H}')\).

We can assume that \(D^0 = D^1 = 0\) and \(\eta^0 = \omega^0 = 0\). Then, the following equivalent condition for the controllability holds:

\[
-\dot{D}(T)E^{0,T} + D(T)E^{1,T} + \left< \eta(T), p^T \right>_{\hat{H}_{even}^{-1}, \hat{H}_{even}^1} + \left< \omega(T), q^T \right>_{\hat{H}_{odd}^{-1}, \hat{H}_{odd}^1} + \int_0^T u(t) \left\{ \int_0^\pi (q + cq_{xx})dx + E(t) \right\} dt = 0.
\]

**Observability Inequality:** For some \(C > 0\),

\[
\left| E^{0,T} \right|^2 + \left| E^{1,T} \right|^2 + \left\| p^T \right\|_1^2 + \left\| q^T \right\|_1^2 \leq C \int_0^T \left\| \int_0^\pi (q + cq_{xx})dx + E(t) \right\|^2 dt.
\]
Identity (6) defines \((\eta(t), \omega(t), -\dot{D}(t), D(t)) \in \mathcal{H}'\) in a unique way and \((\eta, \omega, -\dot{D}, D) \in C([0, T]; \mathcal{H}')\).

We can assume that \(D^0 = D^1 = 0\) and \(\eta^0 = \omega^0 = 0\). Then, the following equivalent condition for the controllability holds:

\[
-\dot{D}(T)E^{0,T} + D(T)E^{1,T} + \left\langle \eta(T), p^T \right\rangle_{\mathcal{H}^{1}_{even}, \mathcal{H}^{-1}_{even}} + \left\langle \omega(T), q^T \right\rangle_{\mathcal{H}^{1}_{odd}, \mathcal{H}^{-1}_{odd}} + \int_{0}^{T} u(t) \left\{ \int_{0}^{\pi} (q + cq_{xx})dx + E(t) \right\} dt = 0.
\]

**Observability Inequality:** For some \(C > 0\),

\[
|E^{0,T}|^2 + |E^{1,T}|^2 + \|p^T\|_{1}^2 + \|q^T\|_{1}^2 \leq C \int_{0}^{T} \int_{0}^{\pi} (q + cq_{xx})dx + E(t) dt.
\]
Identity (6) defines $(\eta(t), \omega(t), -\dot{D}(t), D(t)) \in \mathcal{H}'$ in a unique way and $(\eta, \omega, -\dot{D}, D) \in C([0, T]; \mathcal{H}')$.

We can assume that $D^0 = D^1 = 0$ and $\eta^0 = \omega^0 = 0$. Then, the following equivalent condition for the controllability holds:

$$-\dot{D}(T)E^{0,T} + D(T)E^{1,T} + \left\langle \eta(T), p^T \right\rangle_{\hat{\mathcal{H}}_{\text{even}}^{-1}, \hat{\mathcal{H}}_{\text{even}}^1} + \left\langle \omega(T), q^T \right\rangle_{\hat{\mathcal{H}}_{\text{odd}}^{-1}, \hat{\mathcal{H}}_{\text{odd}}^1} + \int_0^T u(t) \left\{ \int_0^\pi (q + cq_{xx}) dx + E(t) \right\} dt = 0.$$ 

**Observability Inequality:** For some $C > 0$,

$$\left| E^{0,T} \right|^2 + \left| E^{1,T} \right|^2 + \left\| p^T \right\|^2_1 + \left\| q^T \right\|^2_1 \leq C \int_0^T \left( \int_0^\pi (q + cq_{xx}) dx + E(t) \right)^2 dt.$$
The change of variables $t \rightarrow T - t$ and $x \rightarrow L - x$ give us the following "initial conditions"

\[ p^0(x) = p^T(\pi - x), \quad q^0(x) = q^T(\pi - x), \quad E^0, = E^{0,T}, \quad E^1 = -E^{1,T}. \]

Therefore, the above observability inequality is equivalent to the following one:

\[
\left| E^0 \right|^2 + \left| E^1 \right|^2 + \left\| p^0 \right\|_1^2 + \left\| q^0 \right\|_1^2 \leq C \int_0^T \left| \int_0^\pi (q + cq_{xx}) dx + E(t) \right|^2 dt,
\]

for some constant $C > 0$ and any $(p^0, q^0, E^0, E^1) \in \mathcal{H}$ corresponding solution $(p, q, E)$ of the "new" adjoint system.
Proof of the observability inequality

- First case: $E \equiv 0$:

There exists $C > 0$, such that

$$\|p^0\|_1^2 + \|q^0\|_1^2 \leq C \int_0^T \left| \int_0^\pi (q + cq_{xx})dx \right|^2 dt.$$ 

Observe that

$$(p, q) = \sum_{k \geq 1} (\hat{p}_k(t) \cos(kx), \hat{q}_k(t) \sin(kx)),$$

where

$$\hat{p}_k = \cos[k\lambda(k)t] \hat{p}_0^k - \sqrt{\tilde{w}_1 \tilde{w}_2} \sin[k\lambda(k)t] \hat{q}_0^k,$$

$$\hat{q}_k = \sqrt{\tilde{w}_2 \tilde{w}_1} \sin[k\lambda(k)t] \hat{p}_0^k + \cos[k\lambda(k)t] \hat{q}_0^k,$$

with $\tilde{w}_1(k) = 1 - ck^2$ and $\tilde{w}_2(k) = 1 - ak^2.$
First case: $E \equiv 0$:

There exists $C > 0$, such that

$$\|p^0\|_1^2 + \|q^0\|_1^2 \leq C \int_0^T \left| \int_0^{\pi} (q + cq_{xx}) dx \right|^2 dt.$$  

Observe that

$$(p, q) = \sum_{k \geq 1} (\hat{p}_k(t) \cos(kx), \hat{q}_k(t) \sin(kx)),$$

where

$$\hat{p}_k = \cos[k\lambda(k)t]\hat{p}_k^0 - \sqrt{\tilde{w}_1\tilde{w}_2} \sin[k\lambda(k)t]\hat{q}_k^0$$

$$\hat{q}_k = \sqrt{\tilde{w}_2\tilde{w}_1} \sin[k\lambda(k)t]\hat{p}_k^0 + \cos[k\lambda(k)t]\hat{q}_k^0$$

with $\tilde{w}_1(k) = 1 - ck^2$ and $\tilde{w}_2(k) = 1 - ak^2$. 
We have that

\[
\int_0^T \left| \int_0^\pi (q + cq_{xx}) \, dx \right|^2 \, dt = \int_0^T \left| \sum_{k \in \mathbb{Z}, \ |k| \text{ odd}} a_k e^{i\mu_k t} \right|^2 \, dt,
\]

and

\[
\| p^0 \|_1^2 + \| q^0 \|_1^2 \leq \sum_{k \in \mathbb{Z}, \ |k| \text{ odd}} |a_k|^2.
\]

where \( a_k \) and \( \mu_k \) can be computed explicitly.

From Ingham’s inequality,

\[
\sum_{k \in \mathbb{Z}} |a_k|^2 \leq C^T \int_0^T \left| \sum_{k \in \mathbb{Z}} a_k e^{i\mu_k t} \right|^2 \, dt \leq D^T \sum_{k \in \mathbb{Z}} |a_k|^2,
\]

for some positive constants \( C^T \) and \( D^T \).
We have that

$$\int_0^T \left| \int_0^{\pi} \left( q + cq_{xx} \right) dx \right|^2 dt = \int_0^T \left| \sum_{k \in \mathbb{Z} \mid k \text{ odd}} a_k e^{i\mu_k t} \right|^2 dt,$$

and

$$\| p^0 \|_1^2 + \| q^0 \|_1^2 \leq \sum_{k \in \mathbb{Z} \mid k \text{ odd}} |a_k|^2.$$

where $a_k$ and $\mu_k$ can be computed explicitly.

From Ingham’s inequality,

$$\sum_{k \in \mathbb{Z}} |a_k|^2 \leq C^T \int_0^T \left| \sum_{k \in \mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq D^T \sum_{k \in \mathbb{Z}} |a_k|^2,$$

for some positive constants $C^T$ and $D^T$. 
\[ \left\| (p^0, q^0) \right\|_{V_1}^2 \leq C \sum_{k \in \mathbb{N}, k \text{ odd}} \frac{(1 - ck^2)^2}{k^2} \left( \frac{\tilde{w}_2}{\tilde{w}_1} |\hat{p}^0_k|^2 + |\hat{q}^0_k|^2 \right) \]
\[ \leq \sum_{k \in \mathbb{Z}, |k| \text{ odd}} |a_k|^2 \]
\[ \leq C C^T \int_0^T \left\| \sum_{k \in \mathbb{Z}, |k| \text{ odd}} a_k e^{i\mu_k t} \right\|^2 dt \]
\[ \leq C C^T \int_0^T \left\| \int_0^\pi (q + cq_{xx}) dx \right\|^2 dt. \]
Seconde case: $\ddot{E}(t) = 0$

$E(t) = \beta t + \alpha$, where $\alpha = E^0$ and $\beta = E^1$

Observability inequality: for some $C > 0$,

$$|E^0|^2 + |E^1|^2 + \|p^0\|_1^2 + \|q^0\|_1^2 \leq C \int_0^T \left| \int_0^\pi (q + cq_{xx})dx + E(t) \right|^2 dt.$$ 

Set $f(t) = \int_0^\pi (q + cq_{xx})dx$. Then,

$$\int_0^T |f(t) + E(t)|^2 dt = \int_0^T |f(t)|^2 dt + 2 \int_0^T f(t)E(t)dt + \int_0^T |E(t)|^2 dt.$$
Seconde case: $\ddot{E}(t) = 0$

$E(t) = \beta t + \alpha$, where $\alpha = E^0$ and $\beta = E^1$

**Observability inequality:** for some $C > 0$,

$$\left| E^0 \right|^2 + \left| E^1 \right|^2 + \left\| p^0 \right\|_1^2 + \left\| q^0 \right\|_1^2 \leq C \int_0^T \left| \int_0^\pi (q + cq_{xx}) \, dx + E(t) \right|^2 \, dt.$$

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$E(t) = \beta t + \alpha$, where $\alpha = E^0$ and $\beta = E^1$

**Observability inequality:** for some $C > 0$,

$$|E^0|^2 + |E^1|^2 + \|p^0\|_1^2 + \|q^0\|_1^2 \leq C \int_0^T \left| \int_0^\pi (q + cq_{xx}) dx + E(t) \right|^2 dt.$$

Set $f(t) = \int_0^\pi (q + cq_{xx}) dx$. Then,

$$\int_0^T |f(t) + E(t)|^2 dt = \int_0^T |f(t)|^2 dt + 2 \int_0^T f(t)E(t) dt + \int_0^T |E(t)|^2 dt.$$
If the statement is false, then there exists a sequence

\[(p_0^n, q_0^n, E_0^n, E_1^n)_{n \geq 0} \text{ in } \mathcal{H} := \mathcal{H}_\text{even}^1 \times \mathcal{H}_\text{odd}^1 \times \mathbb{R} \times \mathbb{R}, \]

satisfying

\[\|p_0^n\|_1^2 + \|q_0^n\|_1^2 + |E_0^n|^2 + |E_1^n|^2 = 1, \quad \forall n \geq 0,\]

and such that

\[\int_0^T |f_n(t) + E_n(t)|^2 \, dt \to 0 \text{ as } n \to \infty.\]
Extracting a subsequence, still denoted \((p_n^0, q_n^0, E_n^0, E_n^1)_{n \geq 0}\), we have that

\[
(p_n^0, q_n^0, E_n^0, E_n^1) \rightharpoonup (p^0, q^0, E^0, E^1) \quad \text{in } \mathcal{H};
\]

that is,

\[
\begin{align*}
p_n^0 & \to p^0 \quad \text{in } \hat{H}_{even}^1(0, \pi), \\
q_n^0 & \to q^0 \quad \text{in } \hat{H}_{odd}^1(0, \pi), \\
E_n^0 & \to E^0 \quad \text{in } \mathbb{R}, \\
E_n^1 & \to E^1 \quad \text{in } \mathbb{R}.
\end{align*}
\]
Since the embedding $H^1(0, \pi) \hookrightarrow L^2(0, \pi)$ is compact, we have for a subsequence still denoted $(p^0_n, q^0_n, E^0_n, E^1_n)_{n \geq 0}$

\[
p^0_n \to p^0 \quad \text{in} \quad L^2(0, \pi),
\]

\[
q^0_n \to q^0 \quad \text{in} \quad L^2(0, \pi),
\]

\[
E_n(t) = E^1_n t + E^0_n \to E(t) = E^1 t + E^0 \quad \text{in} \quad L^2(0, T).
\]

On the other hand,

\[
\|p^0_n\|_1^2 + \|q^0_n\|_1^2 + |E^0_n|^2 + |E^1_n|^2 \leq C T \left( \int_0^T |f_n(t) + E_n(t)|^2 \, dt + \|p^0_n\|_0^2 + \|q^0_n\|_0^2 \right),
\]

for all $T > 0$, i.e.,

$(p^0_n, q^0_n, E^0_n, E^1_n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{H} := \mathring{H}^1_{\text{even}} \times \mathring{H}^1_{\text{odd}} \times \mathbb{R} \times \mathbb{R}$.
Since the embedding $H^1(0, \pi) \hookrightarrow L^2(0, \pi)$ is compact, we have for a subsequence still denoted $(p^0_n, q^0_n, E^0_n, E^1_n)_{n \geq 0}$

\[ p^0_n \to p^0 \quad \text{in} \quad L^2(0, \pi), \]
\[ q^0_n \to q^0 \quad \text{in} \quad L^2(0, \pi), \]
\[ E_n(t) = E^1_n t + E^0_n \to E(t) = E^1 t + E^0 \quad \text{in} \quad L^2(0, T). \]

On the other hand,

\[
\|p^0_n\|^2_1 + \|q^0_n\|^2_1 + |E^0_n|^2 + |E^1_n|^2 \leq C^T \left( \int_0^T |f_n(t) + E_n(t)|^2 \, dt + \|p^0_n\|^2_0 + \|q^0_n\|^2_0 \right),
\]

for all $T > 0$, i. e.,

$(p^0_n, q^0_n, E^0_n, E^1_n)_{n \geq 0}$ is a Cauchy sequence in $\mathcal{H} := \mathring{H}^1_{\text{even}} \times \mathring{H}^1_{\text{odd}} \times \mathbb{R} \times \mathbb{R}$. 

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We infer that \((p_n^0, q_n^0, E_n^0, E_n^1)_{n \geq 0}\) is a Cauchy sequence in \(\mathcal{H}\), which allows us to conclude that

\[
\|p^0\|_1^2 + \|q^0\|_1^2 + |E^0|^2 + |E^1|^2 = 1 \quad (7)
\]

and

\[
\int_0^T \left( \int_0^\pi (q + cq_{xx}) \, dx + E(t) \right)^2 \, dt = 0, \quad (8)
\]

where \((p, q, E)\) is a solution of the problem.

**Claim.** For \(T > 0\), let \(N_T\) denote the space of the (initial) states \((p^0, q^0, E^0, E^1) \in \mathcal{H}\) such that the corresponding solution \((p, q, E)\) satisfies \(\int_0^\pi (q + cq_{xx}) \, dx + E(t) = 0\) in \(L^2(0, T)\). Then, \(N_T = \{0\}\) for all \(T > 0\).
We infer that \((p_0^n, q_0^n, E_0^n, E_1^n)_{n \geq 0}\) is a Cauchy sequence in \(\mathcal{H}\), which allows us to conclude that

\[
\|p^0\|_1^2 + \|q^0\|_1^2 + |E^0|^2 + |E^1|^2 = 1 \quad (7)
\]

and

\[
\int_0^T \left[ \int_0^\pi (q + cq_{xx}) \, dx + E(t) \right]^2 \, dt = 0, \quad (8)
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**Claim.** For \(T > 0\), let \(N_T\) denote the space of the (initial) states \((p^0, q^0, E^0, E^1) \in \mathcal{H}\) such that the corresponding solution \((p, q, E)\) satisfies \(\int_0^\pi (q + cq_{xx}) \, dx + E(t) = 0\) in \(L^2(0, T)\). Then, \(N_T = \{0\}\) for all \(T > 0\).
If $N_T \neq \{0\}$, the map

$$(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \rightarrow A(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T$$

(where $\mathbb{C}N_T$ denotes the complexification of $N_T$) has at least one

eigenvalue; that is, there exist $\lambda \in \mathbb{C}$ and an initial state

$(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \setminus \{(0, 0, 0, 0)\}$, such that

$$
\begin{align*}
\lambda p^0 &= -q^0 - cq^0_{xxx}, \\
\lambda q^0 &= -p^0 - ap^0_{xxx}, \\
\lambda E^0 &= E^1, \\
\lambda E^1 &= 0,
\end{align*}
$$

$p^0_x(0) = p^0_x(\pi) = 0,$

$q^0(0) = q^0(\pi) = q^0_{xx}(0) = q^0_{xx}(\pi) = 0,$

and

$$
\int_0^\pi (q + cq_{xx})dx + E(t) = 0 \text{ in } (0, T).
$$

We can prove that $(p^0, q^0, E^0, E^1) = (0, 0, 0, 0).$
If $N_T \neq \{0\}$, the map

$$(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \rightarrow A(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T$$

(where $\mathbb{C}N_T$ denotes the complexification of $N_T$) has at least one eigenvalue; that is, there exist $\lambda \in \mathbb{C}$ and an initial state $(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \setminus \{(0, 0, 0, 0)\}$, such that

$$\begin{cases}
\lambda p^0 = -q^0_x - cq^0_{xxx}, \\
\lambda q^0 = -p^0_x - ap^0_{xxx}, \\
\lambda E^0 = E^1, \\
\lambda E^1 = 0,
\end{cases}$$

$p^0_x(0) = p^0_x(\pi) = 0,$
$q^0(0) = q^0(\pi) = q^0_{xx}(0) = q^0_{xx}(\pi) = 0,$

and $\int_0^\pi (q + cq_{xx})dx + E(t) = 0$ in $(0, T)$.

We can prove that $(p^0, q^0, E^0, E^1) = (0, 0, 0, 0)$. 


The motion of the fluid was described by the so-called shallow water (or Saint-Venant) equations, obtained from the Boussinesq system by letting $a = b = c = d = 0$.

It would be interesting to prove a similar result for the full Boussinesq system (still with $b = d = 0$). This cannot be done through a simple linearization argument, since $((\eta \omega)_x, \omega \omega_x)$ is not expected to belong to $\hat{H}^r_{\text{even}} \times \hat{H}^r_{\text{odd}}$ for some $r$ when $(\eta, \omega) \in \hat{H}^s_{\text{even}} \times \hat{H}^s_{\text{odd}}$ for some $s$. 
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