Weierstrass $\wp$ traveling solutions for BBM type equations

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Nonlinear Waves
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Outline

1. Introduction
   - PDE → ODE via traveling ansatz
   - ODE → Abel’s equation
   - BBM equation

2. Non dissipative BBM
   - Elliptic functions from Abel’s equation
   - Classical solutions

3. Dissipative BBM
   - General approach
   - Closed form solutions
   - General solution using Weierstrass \( \wp \) functions

4. Nonlinear Waves Lab
to obtain traveling wave solutions to many nonlinear dispersive eq. with dissipation
we apply the derivation to BBM
via reductions to 1\textsuperscript{st} kind Abel, with polynomial nonlinearities and dissipation
we explain why such integration via $\wp$ functions can be performed via genus of curves
we show equivalence between nonlinear dissipative PDEs and classical ODE theory
we present graphs of closed form solutions of $\wp$ functions from which in limiting cases classical solutions can be obtained
certain classes of PDEs can be reduced via traveling wave reduction $\zeta = x - vt$ into the ODE

$$u_{\zeta\zeta} + f_2(u)u_\zeta + f_3(u) + f_1(u)u_\zeta^2 + f_0(u)u_\zeta^3 = 0 \quad (1)$$

Examples KdV-type

KdV-Burgers: $\delta u_{\zeta\zeta\zeta} = \nu u_{\zeta\zeta} - \alpha uu_\zeta - cu_\zeta$

Gardner: $\delta u_{\zeta\zeta\zeta} = \nu u_{\zeta\zeta} - \alpha uu_\zeta - cu_\zeta - \beta u^2u_\zeta$

Fisher: $u_\zeta = u_{\zeta\zeta} + \alpha u(1 - u)$

Ginzburg-Landau: $-cu_\zeta = \epsilon u + \nu_1 u_{\zeta\zeta} + \nu_3 u|u|^2 + \nu_5 u|u|^4$

- after reduction $f_0(u) = f_1(u) = 0$
- $f_2(u), f_3(u)$ are polynomials

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Abel

- Letting $u_\zeta = \eta(u)$, we obtain a 2\textsuperscript{nd} kind Abel’s equation

\[ \eta \frac{d\eta}{du} + f_2(u)\eta + f_3(u) + f_1(u)\eta^2 + f_0(u)\eta^3 = 0 \]  

(2)

- 2\textsuperscript{nd} kind can be transformed into a 1\textsuperscript{st} kind via $\eta = \frac{1}{y}$

\[ \frac{dy}{du} = f_0(u) + f_1(u)y + f_2(u)y^2 + f_3(u)y^3 \]  

(3)

- It is still not known how to integrate it for general $f_i(u)$, for special cases, see Kamke [3] (normal, canonical form)
The Benjamin-Bona-Mahony (1972) equation \[1\] or regularized long-wave equation (small wave amplitude, large wavelength, inviscid, incompressible flow)

\[u_t + u_x + uu_x - u_{xxt} = 0 \tag{4}\]

is a popular alternative to the Korteweg-de Vries (KdV) (1895) \[4\] for modeling long waves in a wide class of nonlinear dispersive systems.

A generalization to BBM to include a viscous term is given by

\[u_t + u_x + uu_x - u_{xxt} = \nu u_{xx} \tag{5}\]

where \(\nu > 0\) is transformed kinematic viscosity.
Benjamin *et. al* showed (5) has similar properties as KdV but solutions have better smoothness properties.

Moreover, the linearized version has dispersion relation for which $c_p, c_g$ are bounded for $\forall k$, so fine scales features tend not to propagate.

Existence and stability of the solitary waves solutions of (5) has been investigated by Benjamin *et. al*, Bona *et. al*, Pritchard, Scott, Tzvetkov, etc.

Here, we will find the general solutions of (5) using Weierstrass $℘$ functions without simplifying assumptions.
Abel for BBM

- using $\zeta = x - vt$ and integrating once (keeping the integration constant)

\[
(1 - c)u + \frac{u^2}{2} + cu_{\zeta\zeta} = \nu u_\zeta + A \tag{6}
\]

- which leads to polynomials

\[
f_2(u) = -\frac{\nu}{c}
\]

\[
f_3(u) = \frac{1}{2c}u^2 + \frac{1 - c}{c}u - \frac{A}{c} \tag{7}
\]

- in Abel’s equation

\[
\frac{dy}{du} = f_2(u)y^2 + f_3(u)y^3 \tag{8}
\]
Weierstrass no dissipation

- if \( f_2(u) = 0 \) then (8) is separable with solution given by the elliptic curve

\[
\eta^2 = q_3(u) \equiv -\frac{u^3}{3c} - \frac{1 - c}{c} u^2 + \frac{2A}{c} u - 2B
\]  

(9)

- therefore the well known solutions for nondissipative BBM are found easily from the elliptic equation

\[
u_\zeta^2 = q_3(u)
\]  

(10)

- which can be transformed in standard form

\[
\hat{u}_\zeta^2 = 4\hat{u}^3 - g_2\hat{u} - g_3
\]  

(11)

- via transformation \( u = -\sqrt[3]{12c}\hat{u} - (1 - c) \)
Classical solutions

which has solution \( \hat{u}(\zeta) = \wp(\zeta, g_2, g_3) \) with invariants

\[
g_2 = \sqrt[3]{\frac{12}{c^2}}(c - 1)^2 > 0 \quad (12)
\]

\[
g_3 = \frac{2(1 - c)^3}{3c} \quad (13)
\]

two limiting cases with assumption \( A = 0, B = 0 \):

\[
\begin{align*}
c \neq 1 \\
c > 1 & \rightarrow g_3 < 0 \text{ fast waves solitary} \\
0 < c < 1 & \rightarrow g_3 > 0 \text{ slow waves periodic}
\end{align*}
\]

\[
c = 1
\]

Jacobian elliptic functions with modulus \( k = \sin\frac{5\pi}{12} \)

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Classical solutions

\[ u(x, t) = 3(c - 1)\text{Sech}^2\left[\frac{\sqrt{(c - 1)/c}}{2}(x - ct)\right] \quad \text{if } c > 1 \quad (14) \]

\[ u(x, t) = -\frac{3c}{1 + c}\text{Sec}^2\left[\frac{\sqrt{c}}{2}\left(x - t/(1 + c)\right)\right] \quad \text{if } 0 < c < 1 \quad (15) \]

\[ u(x, t) = C\left[1 - \sqrt{3}\frac{1 \mp \text{cn}(\sqrt{C}/4\sqrt{3}(x - t), k)}{1 \pm \text{cn}(\sqrt{C}/4\sqrt{3}(x - t), k)}\right] \quad (16) \]
Graphs of classical solutions

Figure: Traveling waves $\nu = 0$, left $c = 1.5$; middle $c = 0.5$; right $c = 1$
Weierstrass for dissipative BBM

- consider (8) in the form of non-autonomous eq
  \[ F(y, y_u, u) = 0 \]
- Poincare [6] proved in 1885 that any non-autonomous eq having genus \( p = 1 \) is integrable via Weierstrass \( \wp \) functions, after linear fractional transformation
- \( \forall \) elliptic functions \( f(z) = A(\wp) + B(\wp)\wp' \), see Whittaker [7]
Lemke’s transformation

- progress of integration of (8) is based on Lemke transformation \( \nu = \int f_2(u) du = \frac{\nu}{c} u + D \) [5]

\[
\frac{dy}{dv} = y^2 + g(v)y^3, \quad (17)
\]

where \( g(v) = a_2 v^2 + a_1 v + a_0 = \frac{f_3(v)}{f_2(v)} \)

- letting \( y = -\frac{1}{z} \frac{dz}{dv} \), we obtain the 2\(^{nd}\) order non-autonomous system

\[
z^2 \frac{d^2 v}{dz^2} + g(v) = 0 \quad (18)
\]
since \( g(v) \) has no singularities \( \rightarrow \) \( v \) has only poles of \( O(2) \) 
\( \rightarrow \) solution is an elliptic function

Ince [2] proposed solutions to (18) of the type

\[
v = Ez^p \omega(z^q) + F
\]  \hspace{1cm} (19)

which by substitution give \( p = \frac{2}{5}, q = \frac{1}{5} \)

\( E, F \) are arbitrary constants to be determined next

we will show next that the function \( \omega \) satisfies an elliptic equation, while due to Lemke’s transformation \( z \) satisfies a linear equation (23)
all the above transformations can be combined into

$$u(\zeta) = \sigma - e^{-n\zeta} \omega(z(\zeta))$$  \hspace{1cm} (20)

which by substitution into (5) leads to

$$(z')^2 \ddot{\omega} + \left( z'' - \left(2n + \frac{\nu}{c}\right) z' \right) \dot{\omega} + \left( n^2 + \frac{1 - c + \sigma + n\nu}{c} \right) \omega = \frac{1}{2c} e^{-n\zeta} \omega^2$$  \hspace{1cm} (21)

The free term was eliminated by setting $A = \frac{\sigma^2}{2} + \sigma(1 - c)$. 
By letting
\[ z'' - \left(2n + \frac{\nu}{c}\right)z' = 0 \] (22)
we obtain
\[ z'(\zeta) = c_1 e^{(2n + \frac{\nu}{c})\zeta} \] (23)
We also choose \( \sigma = -(n^2c + n\nu + 1 - c) \) which cancels the linear term in (21).
We are left to solve

\[(z')^2 \ddot{\omega} - \frac{1}{2c} e^{-n\zeta} \omega^2 = 0 \quad (24)\]

subject to (23). If \( n = -\frac{2\nu}{5c} \), then

\[\sigma = \frac{14\nu^2}{25c} + c - 1 \quad (25)\]

By substituting (23) into (24), we obtain

\[\ddot{\omega} = \frac{1}{2cc_1^2} \omega^2. \quad (26)\]
Letting $c_1 = \frac{1}{2\sqrt{3}c}$, we arrive at the elliptic equation

$$\ddot{\omega} = 6\omega^2$$

(27)

which by multiplication by $\dot{\omega}$ and integration becomes

$$(\dot{\omega})^2 = 4\omega^3 - g_3.$$

(28)

Its solution is

$$\omega(z) = \wp(z + c_3, 0, g_3)$$

(29)

with invariants $g_2 = 0$, and $g_3$. 
Then, the general solution to (5) is

$$u(\zeta) = \frac{14\nu^2}{25c} \pm \sqrt{\left(\frac{14\nu^2}{25c}\right)^2 - 2A - e^{\frac{2\nu\zeta}{5c}} \wp\left(c_4 + \frac{5\sqrt{3c}}{6\nu} e^{\frac{\nu\zeta}{5c}}, 0, g_3\right)}$$

(30)

If $A = 0 \rightarrow c - 1 = \pm \frac{14\nu^2}{25c}$, and one selects the lower branch of the radical, we obtain

$$u(\zeta) = -e^{\frac{2\nu\zeta}{5c}} \wp\left(c_4 + \frac{5}{\nu} \sqrt{\frac{c}{3a}} e^{\frac{\nu\zeta}{5c}}, 0, g_3\right) \text{ if } g_3 \neq 0$$

$$u(\zeta) = -\frac{e^{\frac{2\nu\zeta}{5c}}}{\left(c_6 \pm \frac{5\sqrt{3c}}{6\nu} e^{\frac{\nu\zeta}{5c}}\right)^2} \text{ if } g_3 = 0.$$  

(31)
Graphs of \( \wp \) solutions

Figure: Weierstrass solutions \( \nu = 0.1 \) left \( c = 1.5 \); middle \( c = 0.5 \); right \( c = 1 \)
Graphs of kink solutions

Figure: Kink solutions $\nu = 1$ left $c = 1.5$; middle $c = 0.5$; right $c = 1$
Nonlinear Waves Lab

- 16' × 4' × 4'
- 3' water max
- $192ft^3 \approx 5500l$
- 1.5 yrs to build
- attracted > 120k USD
- shallow and deep water for UWV research, supercavitation
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