Point processes

1 Introduction

The origins of the theory of point processes are certainly ancient, since way back in the past man have been counting stars in regions of the sky, recording natural events such as floodings, earthquakes and appearances of comets. In more recent times, point processes have been used in life tables, counting problems, particle physics, population processes, communication engineering, etc.

A point process is a model of indistinguishable points distributed randomly in some space. This space can be quite general, but usually the space is taken as $\mathbf{R^d}$.

One of the simplest processes and most interesting is the so called **Poisson Process**. The Poisson processes is used usually as a model for counting problems. Suppose that we are counting the number of events falling into a certain region of the space. Assume that: the number of events in distinct regions of the space are independent and the number of points in a certain region of the space has a Poisson distribution, then the process obtained is called a Poisson process.

In the study of processes subject to randomness, one is interested in doing statistical inference over such process. One classical approach is likelihood estimation of parameters of interest. Note that inference for stochastic processes, in particular, for point processes differs clearly from the approach of classical inference based on a sample of size n, i.e. n i.i.d. random variables. In the case of point processes we do not have a sequence of realizations of the point process on which to base the inference, but only a unique realization of the process.

One of the goals of this work is to make likelihood estimation of parameters of the Poisson point process. With this objective in mind, this chapter outlines some of the theory of point processes and the Poisson point processes will be always used to exemplify the concepts.

2 Basic Definitions and Notations

E = locally compact Hausdorff space whose topology has a countable basis (usually $\mathbf{R}^d$)

$\mathcal{E}$ = Borel σ-algebra on E

$\mathcal{B}$ = Borel σ-algebra on $\mathbf{R}$

M = set of boundedly finite measures ($\mu \in M if \mu(A) < \infty for A \in \mathcal{B}$)

$$
\begin{aligned}
M_p &= \{\mu \in M; \mu(A) \in N \text{ for } A \in \mathcal{B}\} \\
M_s &= \{\mu \in M_p; \mu(x) \leq 1 \text{ for all } x \in E\} \\
M_a &= \{\mu \in M; \mu \text{ is purely atomic}\} \\
M_d &= \{\mu \in M; \mu \text{ is diffuse}\}
\end{aligned}
$$

Let $\mathcal{M}$ be the σ-algebra on M generated by the coordinate mappings $\mu :\to \mu(f) = \int f d\mu$ for $f \in C_k$ = set of continuous functions on E whose support is compact. Also, $\mathcal{M}_p, \mathcal{M}_s, \mathcal{M}_a, \mathcal{M}_d$ are the trace of $\mathcal{M}$ on M_p, M_s, M_a, M_d, respectively.

Definitions: Let $(\Omega, \mathcal{F}, P)$ be a probability space.
(a)A *random measure* on E is a measurable mapping $M : (\Omega, \mathcal{F}) \to (M, \mathcal{M})$.
(b) A *point process* on E is a measurable mapping $N : (\Omega, \mathcal{F}) \to (M_p, \mathcal{M}_p)$.

Proposition: Let ξ be a mapping from a probability space $(\Omega, \mathcal{F}, P)$ into M and $\mathcal{A}$ a semiring of bounded Borel sets generating $\mathcal{E}$. Then ξ is a random measure if $\xi(A)$ is a random variable for each $A \in \mathcal{A}$.

A point process N is a random distribution of points in E; $N(A)$ is the number of points in A.

There exists a representation

$$N = \sum_{i=1}^{K} \delta_{X_i} \tag{2.1}$$

where δ_x is the point mass at x, X_i are E-values random variables (points or atoms of N) and K is a random variable with values in $\{0, 1, \ldots, \infty\}$.

Then,

$$N(f) = \int f dN = \sum_{i=1}^{K} f(X_i)$$

Definition: A point process N is *simple* if $\mathbf{P}\{N \in M_s\} = 1$. In this case the points X_i in (2.1) are distinct almost surely.

Example: Let μ be a boundedly finite measure ($\mu \in M$). A point process N on E is a *Poisson process* with mean measure μ if:

(i) Whenever $A_1, A_2, \dots, A_k \in \mathcal{E}$ are disjoint, the random variables $N(A_1), N(A_2), \dots, N(A_k)$ are independent;
(ii) For each $A \in \mathcal{E}$ and $k \geq 0$

$$\mathbf{P}(N(A) = k) = \frac{e^{-\mu(A)}\mu(A)^k}{k!} \tag{2.2}$$

When $E = \mathbf{R^d}$ and $\mu(A) = \lambda m(A)$ (m is the Lebesgue measure), the process defined above is called a *homogeneous Poisson process* and λ is known as the *intensity* of the process. Otherwise, the process is called *inhomogeneous.*

A κ-*homogeneous Poisson process* is a process with $\nu = \kappa\, m_d$, where κ is a constant and m_d the Lebesgue measure on $\mathbb{R}^d$.

Algorithm: The simulation of a κ-homogeneous Poisson process is simple:

- For each finite window W, generate $R \sim \text{Poisson}\Big(\kappa\, m_d(W)\Big)$;
- Given $R = r$ generate $U_1, \dots, U_r$ independently distributed according to the the uniform distribution in W.
- Repeat independently for disjoint windows.

3 Laplace Functional, probability generating function and avoidance function

The distribution of a point process can be specified in several forms, for example, Laplace functionals, probability generating functions and avoidance functions among others.

Definition: Given a point process defined over the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the *distribution* of N is the probability measure $\mathcal{L}_N = \mathbf{P}N^{-1}$ on $(M_p, \mathcal{M}_p)$.

Definition: The point process N_1 and N_2 are *identically distributed* if $\mathcal{L}_{N_1} = \mathcal{L}_{N_2}$.
Notation:$N_1 \stackrel{d}{=} N_2$.

Definition: The *Laplace functional* of a point process N is defined by

$$L_N(f) = \mathbf{E}[e^{-N(f)}] = \mathbf{E}[\exp(-\sum f(X_i))] \tag{3.1}$$

where f is nonnegative and $\mathcal{E}$-measurable.

Definition: The *avoidance function* of N is defined by

$$z_{N_1}(A) = \mathbf{P}\{N(A) = 0\}, \qquad A \in \mathcal{E} \tag{3.2}$$

Definition: Let V(E) be the class of all real-valued Borel functions defined on E with h=1 outside some bounded set and $0 \leq h \leq 1$. The *probability generating functional (p.g.fl.)* of N is defined by

$$\begin{aligned} G_N(h) &= \mathbf{E}[\exp(\int_E \log h(x)N(dx))], h \in V(E) \\ &= \mathbf{E}[\prod h(X_i)] \\ &= \mathbf{E}[\exp(\sum \log h(X_i))] \end{aligned} \tag{3.3}$$

Note: $G_N(h) = L_N(-\log h)$.

Theorem: For the point processes N_1 and N_2, the following assertions are equivalent:
(i)$N_1 \stackrel{d}{=} N_2$;
(ii)$(N_1(A_1), \ldots, N_1(A_k)) \stackrel{d}{=} (N_2(A_1), \ldots, N_2(A_k))$;
(iii) $L_{N_1}(f) = L_{N_2}(f)$, for all f nonnegative and $\mathcal{E}$-measurable;
If in addition N_1 and N_2 are simple, then each of (i)-(iii) is equivalent to
(iv) $z_{N_1}(A) = z_{N_2}(A)$.

Example: Let N be a Poisson process with mean measure μ. Then,

$$L_N(f) = \exp[-\int_E (1 - e^{-f(x)})\mu(dx)] \tag{3.4}$$

$$z_N(A) = e^{-\mu(A)} \tag{3.5}$$

$$G_N(h) = \exp[-\int_E (1 - h(x))\mu(dx)] \tag{3.6}$$

4 Moment Measures

Definition: The *mean measure* or first moment measure of a point process N is defined by $\mu_N(A) = \mathbf{E}[N(A)]$.

Now consider the k-fold product of N with itself, that is, the measure defined a.s. for Borel rectangles $A_1 \times \ldots \times A_k$ by

$$N^k(A_1 \times \ldots \times A_k) = \prod_{i=1}^{k} N(A_i)$$

and extended to a measure on the product Borel σ-algebra $\mathcal{B}^k$.

Definition: The *moment measure of order k* of a point process N is defined as the mean measure of N^k whenever the expectation measure exists. For $A \in \mathcal{B}^k$

$$\mu_N^k(A) = \mathbf{E}[N^k(A)] \tag{4.1}$$

Definition: The *covariance measure* of a point process N is the signed measure satisfying:

$$\begin{aligned} \rho_N(A \times B) &= \operatorname{cov}(N(A), N(B)) \\ &= \mu_N^2(A \times B) - \mu_N(A)\mu_N(B) \end{aligned} \tag{4.2}$$

In many situations, the moment measures are not satisfactory. Define the process $N^{(k)}$ on E^k by

$$N^{(k)}(dx_1)(N - \delta_{x_1})(dx_2) \ldots (N - \sum_{i=1}^{k-1} \delta_{x_i})(dx_k)$$

(N^k is the process that consists of k-tuples of points whose coordinates are distinct).

Then,

$$N^k = \sum_{i_1} \ldots \sum_{i_k} \delta_{(x_{i_1}, \ldots, x_{i_k})}$$

$$N^{(k)} = \sum_{i_1} \ldots \sum_{i_k}^{i_1 \neq \ldots \neq i_k} \delta_{(x_{i_1}, \ldots, x_{i_k})}$$

Definition: The *factorial moment measure of order k* of a point process N is defined as, for $A \in \mathcal{B}^k$

$$\mu_N^{(k)}(A) = \mathbf{E}[N^{(k)}(A)] \tag{4.3}$$

Note: If $k_1 + \ldots + k_r = k$, and A_i are disjoint sets of $\mathcal{B}$ then

$$\mu_N^k(A_1^{(k_1)} \times \ldots \times A_r^{(k_r)}) = \mathbf{E}[N(A_1)^{k_1} \ldots N(A_r^{k_r})] \tag{4.4}$$

$$\mu_N^{(k)}(A_1^{(k_1)} \times \ldots \times A_r^{(k_r)}) = \mathbf{E}[N(A_1)^{[k_1]} \ldots N(A_r^{[k_r]})] \tag{4.5}$$

where $n^{[p]} = n(n-1)\ldots(n-p+1)$.

Proposition: If f is bounded $\mathcal{E}$-measurable function

(i) $\mathbf{E}(\int f dN) = \int f d\mu_N$

(ii) $\mathbf{E}[(\int f dN)^k] = \int_{E^k} f(x_1)\ldots, f(x_n)\mu_N^k(dx_1 \times \ldots \times dx_n)$.

The moment measures can be computed from the Laplace functional:

$$
\begin{aligned}
\mu_N(f) &= -\frac{d}{d\alpha}L_N(\alpha f)\mid_{\alpha=0} \\
\mathbf{E}[N(f)^2] &= \frac{d^2}{d\alpha^2}L_N(\alpha f)\mid_{\alpha=0} \\
\mathbf{E}[N(f)N(g)] &= \frac{1}{2}\{\mathbf{E}[N(f+g)^2] - \mathbf{E}[N(f)^2] - \mathbf{E}[N(g)^2]\}
\end{aligned}
\tag{4.6}
$$

from which the covariance measure can be calculated.

Example: Let N be a Poisson process with mean measure μ. Then,

$$\mu_N(A) = \mu(A)$$

$$\rho_N(A \times B) = \mu(A \cap B)$$

$$\mu_N^2(f) = \mathbf{E}[\sum_{i,j} f(X_i, X_j)] = \int f(x,x)\mu(dx) + \int\int f(x,y)\mu(dx)\mu(dy)$$

$$\mu_N^{(2)}(f) = \mathbf{E}[\sum_{i \neq j} f(X_i, X_j)] = \int\int f(x,y)\mu(dx)\mu(dy)$$

5 Janossy Measures

5.1 Finite point processes

Basic conditions:

1. The points are located in a complete separable metric space (c.s.m.s.) E;
2. A distribution of the total number of points in the population is given $\{p_n\}_{n=1}^{\infty}$, $\sum p_n = 1$;
3. For each integer $n \geq 1$, the joint distribution of positions of points, given that there are n points is given, $\Pi_n(.)$, i.e. $\Pi_n \in \mathcal{P}(E^n)$.

Definition: The *Janossy measures* for the point process N are defined as the measures satisfying:

$$J_n(A_1 \times \ldots \times A_n) = p_n \sum_{perm} \Pi_n(A_{i_1} \times \ldots \times A_{i_n}) \tag{5.1}$$

Interpretation: If E = $\mathbf{R^d}$ and $j_n(x_1, \ldots, x_n)$ denotes the density of $J_n(.)$ with respect to the Lebesgue measure on $(\mathbf{R^d})^\mathbf{n}$ with $x_i \neq x_j$ if $i \neq j$, then $j_n(x_1, \ldots, x_n)dx_1 \ldots dx_n = \mathbf{P}$ {there are exactly n points in the process, one in each of the n distinct infinitesimal regions $(x_i, x_i + dx_i)$}.

For the rest of this chapter we are going to make extensive use of the Janossy measures. In this formulation, the condition $\sum p_n = 1$, takes the form:

$$\sum_{n=0}^{\infty}(n!)^{-1}J_n(E^{(n)}) = 1$$

since we may interpret $J_0(E^{(0)}) = p_0$ and, for $n \geq 1$,

$$J_n(E^{(n)}) = p_n \sum_{perm} \Pi_n(E^{(n)}) = p_n n!$$

Proposition: Let E be a complete separable metric space and let $\mathcal{E}^{(n)}$ be the product σ - field on $E^{(n)}$. Then, the following specifications are equivalent, and each suffices to define a finite point process on E:

(i) A probability distribution $\{p_n\}$ on the nonnegative integers and a family of symmetric probability distributions $\Pi_n^{sym}(.)$ on $\mathcal{E}^{(n)}, n \geq 1$, with the added convention $\Pi_0^{sym}(E^{(0)}) = p_0$ and the set $E^{(0)}$ denotes an ideal point such that $E^{(0)} \times E = E = E \times E^{(0)}$.

(ii) A family of nonnegative symmetric measures $J_n(.)$ on $\mathcal{E}^{(n)}, n \geq 1$, satisfying,

$$\sum_{n=0}^{\infty}(n!)^{-1}J_n(E^{(n)}) = 1$$

and with $J_0(E^{(0)}) = p_0$, a nonnegative scalar.

(iii) A probability measure $\mathcal{P}$ on the Borel sets of the countable union, $E^{\cup} = E^{(0)} \cup E^{(1)} \cup E^{(2)} \cup \ldots$

We can write the probability distribution of the variables $N(A_i)$ in terms of the Janossy measures. If $(A_1, \ldots, A_r)$ is a finite partition of E, then if $n_1 + n_2 + \ldots + n_r = n$.

$$\begin{aligned} \mathbf{P}\{N(A_i) = n_i, i = 1, \ldots, r\} &= \mathbf{P}(A_1, \ldots, A_r; n_1, \ldots, n_r) \\ &= (n_1! \ldots n_r!)^{-1} J_n(A_1^{(n_1)} \times \ldots \times A_r^{(n_r)}) \end{aligned}$$

$$= \quad p_n \left(\begin{array}{c} n \\ n_1 \ldots n_r \end{array} \right) \frac{1}{n!} \sum_{perm} \Pi_n(A_1^{(n_1)} \times \ldots \times A_r^{(n_r)}) \tag{5.2}$$

where $\left(\begin{array}{c} n \\ n_1 \ldots n_r \end{array} \right)$ can be interpreted as the number of ways of grouping n points so that n_i of them lie in A_i.

Similarly, if $A_1, \ldots, A_k$ are disjoint and $C = (A_1 \cap \ldots \cap A_k)^c$ and $n_1 + \ldots + n_k = n$, then

$$n_1! \ldots n_k! \mathbf{P}_k(A_1, \ldots, A_k; n_1, \ldots, n_k) = \sum_{r=0}^{\infty} (r!)^{-1} J_{n+r}(A_1^{(n_1)} \times \ldots \times A_k^{(n_k)} \times C^{(r)})$$

Note that the factorial moment measure

$$\mu_N^{(k)}(A_1^{(k_1)} \times \ldots \times A_k^{(k_r)}) = \mathbf{E}(N(A_1)^{[k_1]} \ldots N(A_r)^{[k_r]})$$

is simply the factorial moment of a finite dimensional distribution, which can be expressed in terms of the Janossy measures, we can then obtain an expression for $\mu_N^{(k)}(.)$ in terms of $J_n(.)$.

Observation: The finite dimensional distributions for a point process N are $\mathbf{P}_k(A_1, \ldots, A_k; n_1, \ldots, n_k) = \mathbf{P}(N(A_i) = n_i, i = 1, \ldots, k)$ for bounded Borel sets $A_1, A_2, \ldots$ and nonnegative integers $n_1, n_2, \ldots$.

If $(A_1, \ldots, A_r)$ is a partition of E and $\mathbf{E}(N(E)^{[k]}) < \infty$, where $k = k_1 + \ldots + k_r$

$$\begin{aligned}
\mu_N^{(k)}(A_1^{(k_1)} \times \ldots \times A_k^{(k_r)}) &= \mathbf{E}(N(A_1)^{[k_1]} \ldots N(A_r)^{[k_r]}) \\
&= \sum_{j_i \geq k_i, i=1,\ldots,r} j_1^{[k_1]} \ldots j_r^{[k_r]} \mathbf{P}_r(A_1, \ldots, A_r; j_1, \ldots, j_r) \\
&= \sum_{j_i \geq k_i} (\prod_{i=1}^{r} (j_i - k_i)!)^{-1} J_{j_1 + \ldots + j_r}(A_1^{(j_1)} \times \ldots \times A_r^{(j_r)}) \\
n_i = j_i - k_i, n = n_1 + \ldots + n_r \quad &= \sum_{n=0}^{\infty} (n!)^{-1} \sum_{\sum n_i = n} \left(\begin{array}{c} n \\ n_1 \ldots n_r \end{array} \right) J_{k+n}(A_1^{(k_1+n_1)} \times \ldots \times A_r^{(k_r+n_r)}) \\
&= \sum_{n=0}^{\infty} (n!)^{-1} J_{k+n}(A_1^{(k_1)} \times \ldots \times A_r^{(k_r)} \times E^{(n)})
\end{aligned} \tag{5.3}$$

In particular, if $B \in \mathcal{E}^{(k)}$,

$$\mu_N^{(k)}(B) = \sum_{n=0}^{\infty} \frac{J_{k+n}(B \times E^{(n)})}{n!} \tag{5.4}$$

Similarly, we can show that

$$J_n(A_1^{(k_1)} \times \ldots \times A_r^{(k_r)}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_N^{(n+k)}(A_1^{(k_1)} \times \ldots \times A_r^{(k_r)} \times E^{(k)}) \tag{5.5}$$

So, if $B \in \mathcal{E}^{(n)}$,

$$J_n(B) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_N^{(k+n)}(B \times E^{(k)}) \tag{5.6}$$

Example: *Avoidance function* By restricting E to be A in the formula above we obtain:

$$\begin{aligned} z_N(A) &= \mathbf{P}(N(A) = 0) = \mathbf{P}_1(A; 0) \\ &= J_0(A) = \sum_{k=0}^{\infty} (-1)^k \frac{\mu_N^{(k)}(A^{(k)})}{k!} \end{aligned}$$

If $\mathcal{U} : E \to C$ are Borel measurable functions with $\mid \xi(x) \mid \leq 1$. Then, for a finite point process the *probability generating functional* can be written as:

$$G(\xi) = \mathbf{E}(\Pi_{i=1}^{N} \xi(X_i)), \qquad \xi \in \mathcal{U} \tag{5.7}$$

($G(\xi) = 1$ if $N = 0$ or $\xi(X_i) = 0$, for all i).

Expanding G about $\xi \equiv 0$ we have

$$\begin{aligned} G(\xi) &= p_0 + \sum_{n=1}^{\infty} p_n \int \dots \int_{E^{(n)}} \xi(x_1) \dots \xi(x_n) \Pi_n(dx_1 \times \dots \times dx_n) \\ &= J_0 + \sum_{n=1}^{\infty} (n!)^{-1} \int \dots \int_{E^{(n)}} \xi(x_1) \dots \xi(x_n) J_n(dx_1 \times \dots \times dx_n) \end{aligned} \tag{5.8}$$

Expanding G about $\xi \equiv 1$, we have

$$G(1 + \eta) = 1 + \sum_{k=1}^{\infty} (k!)^{-1} \int \dots \int_{E^{(k)}} \eta(x_1) \dots \eta(x_k) \mu_N^{(k)}(dx_1 \times \dots \times dx_k) \tag{5.9}$$

The result is true for simple functions $\eta(x) = \sum_{i=1}^{r} (z_i - 1) I_{A_i}(x)$. Since any Borel measurable function can be approximated by simple functions as η, the general result follow by familiar continuity arguments, using the dominated convergence theorem and the assumed convergence of $\sum \mu_{[k]} z^k in \mid z \mid < \epsilon$ (where $\mu_{[k]} = \int_{E^{(k)}} \mu_N^{(k)}(dx_1 \times \dots \times dx_k)$), supposing that $\mid \eta(x) \mid < \epsilon$ for $x \in E$.

By taking logarithms of the expression above (under the same conditions for convergence), we obtain

$$\log G(\xi) = -K_0 + \sum_{n=1}^{\infty} (n!)^{-1} \int \dots \int_{E^{(n)}} \xi(x_1) \dots \xi(x_n) K_n(dx_1 \times \dots \times dx_n) \tag{5.10}$$

where $J_0 = \exp(-K_0)$, and $K_n(.)$, $n = 1, 2, \dots$ are symmetric signed measures called *Khintchin measures*.

Let $\tau = \{S_1(\tau), \ldots, S_j(\tau)\}$ denote a partition of the set $\{1, \ldots, k\}$ into j nonempty subsets and $\mid S_i(\tau) \mid =$ number of elements in $S_i(\tau)$.

If the measures K_n and J_n have densities k_n and j_n respectively, then

$$j_n(x_1, \ldots, x_n) = J_0(\sum_{r=1}^{n} \sum_{\tau \in \mathcal{P}_{rn}} \Pi_{i=1}^{r} k_{|S_i(\tau)|}(x_{i_1}, \ldots, x_{i_{|S_i(\tau)|}}) \tag{5.11}$$

where $\mathcal{P}_{rn} =$ set of all partitions of $\{1, \ldots, n\}$ into r non-empty subsets.

5.2 General Case:

Since we require the point process N to be finite a.s. on bounded Borel sets, the definitions and results for finite point processes extend easily to the general case.

Let A be a bounded Borel set, within A N is a finite point process a.s. and the conditions given in the section before hold. Therefore, within A we can express the distribution of the process in terms of some family of local probability distribution or Janossy measures. However, such measures do not exist, in general, for the process defined on E.

We can define *local Janossy measures* for a bounded set A. Denote by $J_n(. \mid A)$ the Janossy measures for the process N restricted to A, that is
$J_n(dx_1 \times \ldots \times dx_n \mid A) = \mathbf{P}$ {exactly n points in A at locations $dx_1, \ldots, dx_n$}.

Then, for $B \subset A$, (compare with (5.6))

$$J_n(B \mid A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_N^{(k+n)}(B \times A^{(k)}) \tag{5.12}$$

provided the moment measures exist and the sum converges.

On the other hand, the local Janossy measures can be obtained, as in the finite case, from the expansion of the p.g.fl. $G(\xi)$.

Let $\mathcal{U}(A)$ be the set of measurable functions h, $0 \leq h \leq 1$ and for $h \in \mathcal{U}(A)$ extend h to E by

$$h^*(x) = \begin{cases} h(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Then the p.g.fl. $G_A(h)$, for $h \in \mathcal{U}(A)$,of the process restricted to A is given by

$$G_A(h) = G[1 - I_A + h^*] \tag{5.13}$$

Thus, the local Janossy measures can be obtained from the expansion of the p.g.fl. G about $1 - I_A(.)$.

$$\begin{aligned} G_A(\rho h) &= G(1 - I_A + \rho h^*) \\ &= J_0(A) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \int \ldots \int_{A^{(n)}} h(x_1) \ldots h(x_n) J_n(dx_1 \times \ldots \times dx_n \mid A) \end{aligned} \tag{5.14}$$

Similarly, we can obtain the *local Khintchin measures* from the expansion of $\log G_A$

$$\begin{aligned} \log G_A(\rho h) &= \log G(1 - I_A + \rho h^*) \\ &= -K_0(A) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \int \ldots \int_{A^{(n)}} h(x_1) \ldots h(x_n) K_n(dx_1 \times \ldots \times dx_n \mid A) \end{aligned} \tag{5.15}$$

Example: Let N be a Poisson process with mean measure with density μ, then the p.g.fl. for N is

$$G_N(h) = \exp(-\int_E (1 - h(x))\mu(x)dx)$$

Then the p.g.fl. $G_A(.)$ for the Poisson process restricted to A is given by

$$\begin{aligned} G_A(h) &= G(1 - I_A + h^*) \\ &= \exp(-\int_E (I_A - h^*)\mu(x)dx) \\ &= \exp(-\int_A (1 - h)\mu(x)dx) \end{aligned} \tag{5.16}$$

Note that the process restricted to A is unaffected by the process outside A and is again a Poisson process, then

$$\log G_A(h) = -\int_A (1 - h(x))\mu(x)dx \tag{5.17}$$

and only the first Khintchin measure is non-zero, so

$$K_0 = -\log z(A) = \mu(A) \tag{5.18}$$

and $k_1(x \mid A) = \mu(x)$ is the density of K_1. Moreover,

$$\begin{aligned} j_n(x_1, \ldots, x_n) &= J_0(A)\Pi_{i=1}^n k_1(x_i) \\ &= \exp(-\int_A \mu(y)dy)\Pi_{i=1}^n \mu(x_i) \end{aligned} \tag{5.19}$$

6 Likelihoods

Suppose there is a realization $\{x_1, \ldots, x_N\}$ of a point process over a bounded set A in $\mathbf{R^d}$.

Definition: A point process on $E = \mathbf{R^d}$ is *regular on A* for a bounded Borel set $A \subseteq \mathbf{R^d}$ if for all integers $k \geq 1$, the local Janossy measures $J_k(dx_1 \times \ldots \times dx_n \mid A)$ are absolutely continuous with respect to the Lebesgue measure m on A. It is called *regular* if it is regular on A for all bounded Borel sets A.

Example: The Poisson process with mean measure having density μ is a regular process.

Definition: The *likelihood* of a realization $x_1, \ldots, x_N$ of a regular point process on a bounded Borel set $A \subseteq \mathbf{R^d}$ is the local Janossy density

$$L_A(x_1, \ldots, x_N) \quad = \quad j_N(x_1, \ldots, x_N \mid A) \tag{6.1}$$

Example: Poisson process with mean measure having density μ we have

$$L_A(x_1, \ldots, x_n) = j_n(x_1, \ldots, x_n \mid A) = [\exp(-\int_A \mu(y)dy)]\Pi_{i=1}^n \mu(x_i) \tag{6.2}$$

and

$$\log L_A(x_1, \ldots, x_n) = -\int_A \mu(y)dy + \sum_{i=1}^n \log \mu(x_i) \tag{6.3}$$

We are interested in the concept of *likelihood ratio*. That is, we want to compare our process with a standard process. In the context of simple point process in $\mathbf{R^d}$, the natural standard process is the Poisson process with parameter measure equal to the Lebesgue measure on $\mathbf{R^d}$.

Proposition: Let N be a point process on $\mathbf{R^d}$, N_A its restriction to the bounded set $A \in \mathcal{B}(\mathbf{R}^d)$ and $\mathcal{P}_A$ the distribution of N_A. Let $\mathcal{P}_A^*$ be the distribution of the Poisson process with Lebesgue measure in $\mathbf{R^d}$ as mean measure. Then $\mathcal{P}_A \ll \mathcal{P}_A^*$ if N is regular on A.

The *likelihood ratio* is given by

$$\frac{L_A}{L_A^*} = e^{m(A)} j_N(x_1, \ldots, x_N \mid A) \tag{6.4}$$

Example: Let $\mathcal{P}_A$ be the distribution of a Poisson process with mean measure having density μ, then

$$\log \frac{L_A}{L_A^*} = \sum_{i=1}^{N} \log \mu(x_i) - \int_A (\mu(x) - 1) dx \tag{6.5}$$

Intuition: If $A_1, A_2, \ldots, A_n$ are disjoint, then

$$\begin{aligned}
\frac{dP_{\theta, N(A_1),\ldots,N(A_n)}}{dP^{\lambda}_{N(A_1),\ldots,N(A_n)}}(k_1, \ldots, k_n) &= \prod_{j=1}^{n} \frac{e^{-M_\theta(A_j)}[M_\theta(A_j)]^{k_j}/k_j!}{e^{-\lambda(A_j)}[\lambda(A_j)]^{k_j}/k_j!} \\
&= e^{-\sum_j M_\theta(A_j) + \sum_j \lambda(A_j)} \prod_{j=1}^{n} (M_\theta(A_j)/\lambda(A_j))^{k_j} \\
&= \exp\{-\sum_j M_\theta(A_j) + \sum_j \lambda(A_j) + \sum_{j=1}^{n} k_j \log(M_\theta(A_j)/\lambda(A_j))\}
\end{aligned}$$

Let $A_1^n, A_2^n, \ldots$ be sets such that $\cup_j A_j^n = A$ and $\max_j \operatorname{diam} A_j^n \to 0$, as $n \to \infty$.

$$\begin{aligned}
\frac{dP_\theta/\mathcal{F}_A}{dP^\lambda/\mathcal{F}_A} &= \lim_{n\to\infty} \frac{dP_{\theta, N(A_1^n),\ldots,N(A_n^n)}}{dP^{\lambda}_{N(A_1^n),\ldots,N(A_n^n)}} \\
&= \lim_{n\to\infty} \exp\{-\sum_j M_\theta(A_j) + \sum_j \lambda(A_j) + \sum_{j=1}^{n} N(A_j^n) \log(M_\theta(A_j)/\lambda(A_j))\} \\
&= \exp\{-M_\theta(A) + \lambda(A) + \lim_{n\to\infty} \sum_{j=1}^{n} N(A_j^n) \log(M_\theta(A_j^n)/\lambda(A_j^n))\} \\
&= \exp\{-M_\theta(A) + \lambda(A) + \sum_{j=1}^{N(A)} \log \mu_\theta(x_i, y_i)\} \\
&= \exp\{-\int_A (\mu_\theta(x,y) - 1)\, dx\, dy + \sum_{j=1}^{N(A)} \log \mu_\theta(x_i, y_i)\}
\end{aligned}$$

where $(x_i, y_i), i = 1, \ldots, N(A)$ are the points of N in the set A.

The independence property characterizes the Poisson process. Most of the applications deal with point processes having interaction between points. In this work, we are going to consider a particular case, point processes with probability law (restricted to a finite box $\Lambda \subset \mathbb{R}^d$) which are absolutely continuous with respect to the probability law of a homogeneous Poisson point process. In fact, if we call μ^0_Λ the law of the unit-homogeneous Poisson process, their distribution is characterized by the Radon-Nikodym derivative (or Gibbs measure) given by

$$\mu_\Lambda(dN) = \frac{1}{Z_\Lambda} e^{-H(N,\Lambda)} \mu^0_\Lambda(dN) \tag{6.6}$$

where $H(N, \Lambda)$ is the energy function, Z_Λ is a normalizing constant.

The Radon-Nikodym derivative can be thought as a measure of how much more likely are the configuration N in this process than in the Poisson process. That is, the set of possible configurations are the same for the interacting process and the Poisson process. However, their likelihood changes and the RN derivative measures this likelihood.

6.1 Area-interaction point processes

In these processes each point (=*germ*) has associated a *grain* formed by a copy of a fixed compact (and usually convex) set $G \subset \mathbb{R}^d$. The intersections of these grains determine a weight that corrects the otherwise Poissonian distribution of the germs. In this case, the Gibbs measure (6.6) is given by

$$\mu_\Lambda(dN) \;=\; \frac{\kappa^{N(\Lambda)}\,\phi^{-m_d(N\oplus G)}}{Z_\Lambda(\kappa,\phi)}\,\mu^0_\Lambda(dN)\;, \tag{6.7}$$

where κ and ϕ are positive parameters, $Z_\Lambda(\kappa,\phi)$ is a normalizing constant and $N\oplus G$ is the *coverage process* given by

$$N\oplus G \;:=\; \bigcup_{x\in N}\{x+G\}\;. \tag{6.8}$$

Note that when N is Poisson process, this coverage process is a Boolean model. Hence the area-interacting process defined by (6.7) can be thought as a "weighted Boolean model" with weights depending exponentially on the area of the covered region.

The parameter ϕ controls the area-interaction between the points of N: the process is *attractive* if $\phi > 1$ and *repulsive* otherwise. If $\phi = 1$ the process is just the (unweighted) Boolean model with grain G and Poissonian rate κ. The case $\phi > 1$ is related to the *penetrable sphere model* and described in Section 6.3. The case of *area-exclusion* corresponds to a suitable limit $\phi \to 0$.

6.2 Strauss Processes

A related process to the area-interaction point is the so-called Strauss process. In this case, the unit Poisson process is weighted according to an exponential of the number of pairs of points closer than a fixed threshold r. In this case, the Gibbs measure (6.6) is defined by

$$\mu_\Lambda(dN) = \frac{1}{Z_\Lambda}e^{\beta_1 N(\Lambda)+\beta_2 S(N,\Lambda)}\,\mu^0_\Lambda(dN) \tag{6.9}$$

where $S(N,\Lambda)$ is the number of unordered pairs such that $\|x_i - x_j\| < r$. The case $\beta_2 > 0$ was introduced by Strauss (1975) to model the clustering of Californian red wood seedlings around older stumps, however in this case (6.9) is not integrable.

6.3 Penetrable spheres mixture model

The penetrable sphere model was introduced by widow and Rowlison (1970) to study liquid-vapor phase transitions. It is a point process with two types of points, therefore it can be seen as a bi-dimensional point process (N, M) in the product space $\mathcal{N}\times\mathcal{N}$ which is absolutely continuous with respect to the product of two independent unit Poisson processes and Radon-Nikodym derivative given by

$$\tilde\mu_\Lambda(dN, dM) = \frac{1}{Z_\Lambda}\beta_1^{N(\Lambda)}\beta_2^{M(\Lambda)}\mathbf{1}_{\{d(N,M)>R\}}(\mu^0_\Lambda\times\mu^0_\Lambda)(dN, dM) \tag{6.10}$$

where $d(N, M) = \min\{d(x, y); x \in N, y \in M\}$ is the shortest distance between a point of N and M. That is, in this model points of different type cannot be at a distance shorter than R.

Marginal and conditional distributions:

It is easy to see that the conditional distribution of N given M is a homogeneous Poisson process with intensity β_1 on $\Lambda \setminus (M \oplus G)$. Where G is a sphere of radius R. Similarly, the conditional distribution of M given N is a homogeneous Poisson process with intensity β_2 on $\Lambda \setminus (N \oplus G)$.

The marginal distribution of N is an area-interaction point process with $\kappa = \beta_1$ and $\phi = e^{\beta_2}$. Similarly, the marginal distribution of M is an area-interaction point process with $\kappa = \beta_2$ and $\phi = e^{\beta_1}$.

7 Spatial birth-and-death processes

Non-spatial birth and death processes are continuous time Markov chains with $\{0, 1, 2, \ldots\}$ as state space and transition probabilities that are positive only to neighbors (see, for example, Feller (1968) for an overview). Preston (1977) introduced a birth and death process which takes into account the position of the individuals. He defines the process as a continuous time jump process with state space that contains all possible configurations of individuals. In our case, we are interested in point processes that are specified through a density (Radon Nikodym derivative) with respect to a unit Poisson point process, that is,

$$\mu_\Lambda(dN) = \frac{1}{Z_\Lambda} e^{-H(N,\Lambda)} \mu_\Lambda^0(dN) \tag{7.1}$$

where $H(N, \Lambda)$ is the energy function, Z_Λ is a normalizing constant.

Therefore, the state space for this process is $\mathcal{S} = \{\mathbf{n}; H(\mathbf{n}, \Lambda) < +\infty\}$, that is the set of configurations with positive density. Ripley (1977) showed that such measure μ_Λ is the invariant measure of a spatial birth and death process. We specify this process in terms of a nonnegative functions $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to [0, \infty)$ and $\delta : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to [0, \infty)$. The meaning of λ is that if the point configuration at time t is $\mathbf{n} \in \mathcal{N}(\mathbb{R}^d)$, then the probability that a point is added to the configuration in a neighborhood of the point x having area ΔA in the next interval of length Δt is approximately $\lambda(x, \mathbf{n})\Delta A \Delta t$. The interpretation for δ is similar, except that, a point can only be deleted from the configuration if already present, that is, if the point configuration at time t is $\mathbf{n} \in \mathcal{N}(\mathbb{R}^d)$, then the probability that a point $x \in \mathbf{n}$ is deleted from the configuration in the next interval of length Δt is approximately $\delta(x, \mathbf{n})\Delta t$. In fact, there is more than one process that has the same invariant measure, we can choose λ and δ in such way that they satisfy the detailed balance condition:

$$\lambda(x, \mathbf{n}) e^{-H(\mathbf{n},\Lambda)} = \delta(x, \mathbf{n}) e^{-H(\mathbf{n} \cup \{x\},\Lambda)} \qquad \text{if } \mathbf{n} \cup \{x\} \in \mathcal{S}. \tag{7.2}$$

Notice that equation (7.2) says that any pair of birth and death rates such that

$$\frac{\lambda(x, \mathbf{n})}{\delta(x, \mathbf{n})} = \exp\{-H(\mathbf{n} \cup \{x\}, \Lambda) + H(\mathbf{n}, \Lambda)\} \tag{7.3}$$

will give raise to a process with invariant measure having the density given by (7.1). We can always take $\delta(x, \mathbf{n}) = 1$, that is, whenever a point is added to the configuration it lives an exponential amount of time independently of the configuration of the process.