

The Eighth International Conference On Nonlinear
Partial Differential Equations and Their Numerical
Analysis

In honor of Professor Tai-Ping Liu's 70th birthday

*The Cauchy Problem for the Combustion Model in
a Porous Medium with two Layers*

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Overview:

- ▶ The model
- ▶ Monotone Iterative Method
- ▶ Fundamental solution
- ▶ The Cauchy problem

The model

$$(*) \quad \left\{ \begin{array}{l} ((a_1 + b_1 y_1)u_1)_t + (c_1 u_1)_x = d_1 f_1(u_1, y_1) - q(u_1 - u_2) + \lambda_1 u_{1xx} \\ y_{1t} = -A_1 f(u_1, y_1) \\ ((a_2 + b_2 y_2)u_2)_t + (c_2 u_2)_x = d_2 f(u_2, y_2) - q(u_2 - u_1) + \lambda_2 u_{2xx} \\ y_{2t} = -A_2 f_2(u_2, y_2) \end{array} \right.$$

(J.C. da Mota and S. Schecter, Jr. *Dyn. Diff. Eq.*, **18** (3), (2006))

$$x \in \mathbb{R}, \quad t > 0$$

$u_i = u_i(x, t)$ and $y_i = y_i(x, t)$, $i = 1, 2$, are the temperature and the fuel concentration in one layer; f_i is a given function defined by

$$f_i(u, y) = y e^{-\frac{E}{u}}$$

reaction rate functions / Arrhenius' law ;

$a_i, b_i, c_i, d_i, A_i, \lambda_i$, $i = 1, 2$, E and q are nonnegative parameters; q is the heat transfer coefficient.

Rewriting the system

Applying the product rule, we can write the system (*) as follows:

$$\begin{cases} u_{it} - \frac{\lambda_i}{a_i + b_i y_i} u_{i,xx} + \frac{c_i}{a_i + b_i y_i} u_{i,x} \\ \qquad \qquad \qquad = \frac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \frac{u_1 - u_2}{a_i + b_i y_i}, \\ y_{it} = -A_i y_i f(u_i), \end{cases}$$

or

$$\begin{cases} u_{it} - L_i u_i = f_i(u_1, u_2, y_i) \\ y_{it} = -A_i y_i f(u_i) \end{cases}$$

where $i = 1, 2$,

$$L_i u = \frac{\lambda_i}{a_i + b_i y_i} u_{xx} - \frac{c_i}{a_i + b_i y_i} u_x,$$

$$f_i(u_1, u_2, y_i) = \frac{1}{a_i + b_i y_i} ((A_i b_i u_i + d_i) f(u_i, y_i) + (-1)^i q (u_1 - u_2)),$$

The system (*) can also be written as a standard reaction-diffusion system

Denoting $W = (u_1, y_1, u_2, y_2)$, we have

$$W_t = DW_{xx} + MW_x + F(W)$$

where

$$D = \begin{pmatrix} \frac{\lambda_1}{a_1 + b_1 y_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{a_2 + b_2 y_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{c_1}{a_1 + b_1 y_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{c_2}{a_2 + b_2 y_2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$F(W) = \begin{pmatrix} ((b_1 A_1 u_1 f(u_1, y_1) + d_1 f(u_1, y_1) - q(u_1 - u_2))/(a_1 + b_1 y_1) - A_1 f(u_1, y_1)) \\ ((b_2 A_2 u_2 f(u_2, y_2) + d_2 f(u_2, y_2) + q(u_1 - u_2))/(a_2 + b_2 y_2) - A_2 f(u_2, y_2)) \end{pmatrix}$$

but the diffusion matrix D is degenerate.

Invariant regions

However, using the Chueh-Conley-Smoller approach¹, we can easily show that the set


$$\Sigma = \{(u_1, y_1, u_2, y_2); u_1 \geq 0, 0 \leq y_1 \leq 1, u_2 \geq 0, 0 \leq y_2 \leq 1\}$$

is an invariant region. More precisely,

Theorem. *If $W = (u_1, y_1, u_2, y_2)$ is a smooth solution for $(*)$, in the domain $x \in \mathbb{R}, t > 0$, with an initial condition $W(x, 0) = W_0(x)$ such that $W_0(x) \in \Sigma$ for all $x \in \mathbb{R}$ and, if there are positive constants c and ϵ such that*

$$W(x, t) \in \Sigma \quad \forall (x, t); |x| > c \text{ and } 0 < t < \epsilon,$$

then $W(x, t) \in \Sigma$ for all $(x, t) \in \mathbb{R} \times [0, \infty)$.

¹K. N. Chueh, C. C. Conley, J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. **26**(2) (1977), 373–392. 

Reduced system

Considering the concentrations functions y_1, y_2 as known, we obtain the system for the unknowns u_1, u_2 :

$$\{ (u_i)_t - L_i(u_i) = f_i(x, t, u_1, u_2)$$

where

$$L_i u = \frac{1}{a_i + b_i y_i(x, t)} (\lambda_1 u_{xx} - c_i u_x), \tag{1}$$

$$f_i(x, t, u_1, u_2) = \frac{1}{a_i + b_i y_i(x, t)} ((A_i b_i u_i + d_i) f(u_i, y_i) + (-1)^i q(u_1 - u_2)).$$

The reaction function $F = (f_1, f_2)$ is quasi-monotone nondecreasing:

$$\frac{\partial f_1}{\partial u_2} = \frac{q}{a_1 + b_1 y} \geq 0$$

$$\frac{\partial f_2}{\partial u_1} = \frac{q}{a_2 + b_2 z} \geq 0.$$

F is also Lipschitz continuous:

$$|f_1(U) - f_1(\tilde{U})| \leq c_1 \|y_1\|_\infty |u_1 - \tilde{u}_1| + \frac{q}{a_1} |U - \tilde{U}|,$$

$$|f_2(U) - f_2(\tilde{U})| \leq c_2 \|y_2\|_\infty |u_2 - \tilde{u}_2| + \frac{q}{a_2} |U - \tilde{U}|.$$

Besides,

$$\hat{U} = (0, 0) \text{ is a lower solution.}$$

and

$$\tilde{U} = (\varphi(t), \varphi(t))$$

is an solution, where

$$\varphi(t) = (\|u_0\|_\infty + \|v_0\|_\infty + \beta) e^{\alpha t} - \beta$$

and

$$\alpha = \max\{A_1 b_1 / a_1, A_2 b_2 / a_2\}, \quad \text{and} \quad \beta = \max\{d_1 / A_1 b_1, d_2 / A_2 b_2\}.$$

The Cauchy problem for the reduced system. The monotone iterative method

Write $U = (u_1, u_2)$, $\mathbb{L}U = U_t - (L_1 u_1, L_2 u_2)$ and consider the Cauchy problem in $\Omega_T := \mathbb{R} \times (0, T]$, $T > 0$,

$$\begin{cases} \mathbb{L}U = F(x, t, U), & (x, t) \in \Omega_T \\ U(x, 0) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

for some given function $U_0 = (u_{10}, u_{20})$.

The Monotone Iterative Method²:

Define the sequence of functions $U^{(k)}$ by

$$\begin{cases} \mathbb{L}_K U^{(k)} = F_K(x, t, U^{(k-1)}) & \text{in } \Omega_T \\ U^{(k)}|_{t=0} = U_0 & \text{in } \mathbb{R}, \end{cases}$$

where $\mathbb{L}_K = \mathbb{L} + K$ and $F_K \equiv KU + F$ and K is the Lipschitz constant of F . $U^{(k)}$ is denoted by $\overline{U}^{(k)}$ if the initial iteration $U^{(0)}$ is given by $U^{(0)} = \tilde{U}$ and by $\underline{U}^{(k)}$ if the initial iteration $U^{(0)}$ is given by $U^{(0)} = \hat{U}$.

²C.V. Pao: *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York and London, (1992); *Parabolic systems in unbounded domains I. Existence and dynamics*, Jr. Math. An. Appl. **217** (1998).

Monotone property

$$\widehat{U} \leq \underline{U}^{(k)} \leq \underline{U}^{(k+1)} \leq \overline{U}^{(k+1)} \leq \overline{U}^{(k)} \leq \widetilde{U}$$

The inequalities are understood in the componentwise sense, i.e. if $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ then $U_1 \leq U_2$ means $u_1 \leq u_2$ and $v_1 \leq v_2$. Consequently, there exist the pointwise limits

$$\overline{U}(x, t) \equiv \lim_{k \rightarrow \infty} \overline{U}^{(k)}(x, t) \quad \text{and} \quad \underline{U}(x, t) \equiv \lim_{k \rightarrow \infty} \underline{U}^{(k)}(x, t)$$

for each $(x, t) \in \overline{\Omega_T}$.

The monotone property is obtained using the “Maximum principle” and the exponential growth at infinity:

$$|\tilde{U}(x, t)| \leq A_0 \exp(b|x|^2), \quad |x| \gg 1,$$

where A_0 and b are positive constants independent of $|x| \gg 1$ and $t \in [0, T]$.

i.e. Phragman-Lindelöf principle:

Let L be a parabolic operator,

$$L = \partial_t - \sum_{j,k=1}^n a_{jk} \partial_{x_j x_k} + \sum_{j=1}^n b_j \partial_{x_j} + c.$$

If $Lw \geq 0$ in $\mathbb{R}^n \times (0, T)$, $w(x, 0) \geq 0$ in \mathbb{R}^n and there exists a $\delta > 0$ such that

$$\limsup_{R \rightarrow \infty} \left[e^{-\delta R^2} \min\{w(x, t); 0 \leq t \leq T, |x| = R\} \right]$$

then $w(x, t) \geq 0$ in $\mathbb{R}^n \times (0, T)$.

Theorem (Da Mota, -, Nonl. Anal.: Real World Application, 2010).
Let u_{10} and u_{20} be nonnegative continuous and bounded functions defined on \mathbb{R} . Then, given any $T > 0$ and nonnegative and bounded functions y_1, y_2 in $C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$, the sequences $\overline{U}^{(k)}, \underline{U}^{(k)}$ converge to the unique solution $U \equiv (u_1, u_2)$ of the above Cauchy problem in $C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ satisfying

$$0 \leq u_1, u_2 \leq \varphi$$

in $\overline{\Omega_T}$, where φ is the upper solution defined above.

The Cauchy problem for the full system

$$\left\{ \begin{array}{l} u_{it} - \frac{\lambda_i}{a_i + b_i y_i} u_{i,xx} + \frac{c_i}{a_i + b_i y_i} u_{i,x} = \\ \quad = \frac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \frac{u_1 - u_2}{a_i + b_i y_i}, \\ y_{it} = -A_i y_i f(u_i), \\ (u_i(x, 0), y_i(x, 0)) = (u_{i0}(x), y_{i0}(x)), \end{array} \right. \quad (2)$$

where $u_{i,0}$ and $y_{i,0}$ are given functions, $t > 0$, $x \in \mathbb{R}$,
 $a_i, b_i, c_i, d_i, A_i, \lambda_i, E, i = 1, 2$, are nonnegative constants and

$$f(u) = \begin{cases} e^{-\frac{E}{u}}, & \text{se } u > 0 \\ 0, & \text{se } u \leq 0 \end{cases} . \quad (3)$$

We show existence of a local solution (local in time) and global solution if the initial temperatures are in L^p , for some $1 < p < \infty$.

Our proof is strongly based on fundamental solution properties.

Remark

The system can be seen in the variables u_1, u_2 only. In fact, substituting

$$y_i(x, t) = y_{i0}(x)e^{-A_i \int_0^t f(u_i(x, s)) ds}$$

in the first equation, i.e. in the equation for u_i , we see that the system is of the form

$$\begin{cases} u_{it} - a(x, \int_0^t f(u_i(x, s)) ds) u_{ixx} + b(x, \int_0^t f(u_i(x, s)) ds) u_{ix} \\ = F_i(x, u_1, u_2, \int_0^t f(u_i(x, s)) ds) \end{cases}$$

Fundamental solution

Consider the equation

$$Lu \equiv \frac{\partial u}{\partial t} - a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t)u = 0, \quad (4)$$

where a , b and c are bounded and Hölder continuous functions in $\Omega_T = \{(x, t); x \in \mathbb{R}, 0 \leq t \leq T\}$ and such that L is uniformly parabolic in Ω_T , i.e., it satisfies

$$\lambda_0 \leq a(x, t) \leq \lambda_1, \quad \text{for all } (x, t) \in \Omega_T, \quad (5)$$

for some positive constants λ_0 , λ_1 . Then

Definition

A fundamental solution of $Lu = 0$ in Ω_T is a function $\Gamma(x, t, \xi, \tau)$, defined for all $(x, t) \in \Omega_T$, $(\xi, \tau) \in \Omega_T$, $t > \tau$, which satisfies the following conditions:

$L\Gamma = 0$ as a function of (x, t) , for each fixed $(\xi, \tau) \in \Omega_T$;

for all continuous function f such that $|f(x)| \leq Ke^{hx^2}$, $h \leq \frac{1}{4\lambda_1 T}$, one has

$$\lim_{t \rightarrow \tau^+} \int_{\mathbb{R}} \Gamma(x, t, \xi, \tau) f(\xi) d\xi = f(x).$$

The existence of a fundamental solution is obtained by the *parametrix method*, due to E. E. Levi (*Sulle equazioni totalmente ellittiche alle derivate parziali*, Rend. del Circ. Mat. Palermo, **24**, (1907), 275–317.) The method is detailed e.g. in the books A. Friedman *Partial differential equations of parabolic type*, Dover Publications, New York, (2008) and O. A. Ladyzenskaja, V. A. Solonnikov e N. N. Ural'ceva *Linear and quasi-linear equations of parabolic type*. The fundamental solution is given by

$$\Gamma(x, t, \xi, \tau) = Z(x - \xi, \xi, t, \tau) + \int_{\tau}^t \int_{\mathbb{R}} Z(x - y, y, t, \sigma) \phi(y, \xi, \sigma, \tau) dy d\sigma,$$

where for $(x, t), (\xi, \tau) \in \Omega_T, \tau < t$,

$$Z(x, t, \xi, \tau) = \frac{1}{(4\pi a(\xi, \tau)(t - \tau))^{\frac{1}{2}}} e^{-\frac{(x - \xi)^2}{4a(\xi, \tau)(t - \tau)}}, \quad \text{and}$$

$\phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} (-1)^m (LZ)_m(x, t, \xi, \tau)$, with

$$(LZ)_{m+1}(x, t, \xi, \tau) = \int_{\tau}^t \int_{\mathbb{R}} [LZ(x, t, y, \sigma)] (LZ)_m(y, \sigma, \xi, \tau) dy d\sigma,$$

and $(LZ)_1 = LZ = (a(\xi, \tau) - a(x, t)) \frac{\partial^2 Z}{\partial x^2} + b \frac{\partial Z}{\partial x} + cZ$.

Representation formula

Consider the Cauchy problem

$$\begin{cases} Lu(x, t) = f(x, t), & \text{in } \mathbb{R} \times (0, T], \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}, \end{cases} \quad (6)$$

with L , defined in (4), having Hölder continuous coefficients, and, $|f(x, t)|, |u_0(x)| \leq Ke^{hx^2}$, where K and h are positive constants, with $h < \frac{1}{4\lambda_1 T}$. Then,

Theorem

If $f(x, t)$ and $u_0(x)$ are continuous functions in Ω_T and \mathbb{R} , respectively, and, furthermore, $f(x, t)$ is locally Hölder continuous in x , uniformly with respect to t , then

$$u(x, t) = \int_{\mathbb{R}} \Gamma(x, \xi, t, 0) u_0(\xi) d\xi + \int_0^t \int_{\mathbb{R}} \Gamma(x, \xi, t, \tau) f(\xi, \tau) d\xi d\tau$$

is the unique solution of the Cauchy problem (6) in $C^{2,1}(\mathbb{R} \times (0, T]) \cap C(\mathbb{R} \times [0, T])$, with $|u(x, t)| \leq \tilde{K}e^{\tilde{h}x^2}$ for some positive constants \tilde{K}, \tilde{h} .

Continuous dependence of the fundamental solution on the coefficients

Let R , λ and α be positive real numbers, with $0 < \alpha \leq 1$ and $\lambda < R$. We define a set $B(R, \lambda, \alpha)$ for the coefficients of L as

$$B(R, \lambda, \alpha) = \{(a(x, t), b(x, t), c(x, t)) \in (C^{\alpha, \frac{\alpha}{2}}(\Omega_T))^3; a > \lambda \text{ and } a, |b|, |c| < R\}$$

and for $v, \bar{v} \in B(R, \lambda, \alpha)$, we set

$$\|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}} = \max\{\|a - \bar{a}\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)}, \|b - \bar{b}\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)}, \|c - \bar{c}\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)}\}.$$

To emphasize the dependence of the L operator on the coefficients $v \equiv (a, b, c)$, we write $L = L_{[v]}$, i.e.

$$Lu = L_{[v]}u \equiv \frac{\partial u}{\partial t} - a(x, t) \frac{\partial^2 u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t)u, \quad (7)$$

with corresponding fundamental solution

$$\Gamma_{[v]}(x, t, \xi, \tau) = Z_{[v]}(x - \xi, \xi, t, \tau) + \int_{\tau}^t \int_{\mathbb{R}} Z_{[v]}(x - y, y, t, \sigma) \phi_{[v]}(y, \xi, \sigma, \tau) dy d\sigma. \quad (8)$$

Lemma

Given $v, \bar{v} \in B(R, \lambda, \alpha)$, we have that

$$|(D_x^s Z_{[v]} - D_x^s Z_{[\bar{v}]})(x, \xi, t, \tau)| \leq K \|a - \bar{a}\|_\infty \frac{1}{(t - \tau)^{\frac{s+1}{2}}} e^{-C \frac{(x-\xi)^2}{(t-\tau)}},$$

for $s \in \mathbb{Z}_+$, where $C = C(R)$ and $K = K(\lambda)$ are positive constants.

Lemma

Let $\phi_{[v]}$ and $\phi_{[\bar{v}]}$ be defined in (16), with $v, \bar{v} \in B(R, \lambda, \alpha)$ and $0 \leq \beta \leq 1$. Then, we have the following inequalities:

$$|(\phi_{[v]} - \phi_{[\bar{v}]})(x, \xi, t, \tau)| \leq K \|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}} \frac{1}{(t - \tau)^{\frac{3-\alpha}{2}}} e^{-C \frac{(x-\xi)^2}{t-\tau}}; \quad (9)$$

and

$$\begin{aligned} & |(\phi_{[v]}(x, \xi, t, \tau) - \phi_{[\bar{v}]}(x, \xi, t, \tau)) - (\phi_{[v]}(y, \xi, t, \tau) - \phi_{[\bar{v}]}(y, \xi, t, \tau))| \\ & \leq K \|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}}^{\beta} |x - y|^{\alpha(1-\beta)} \frac{1}{(t - \tau)^{\frac{3-\beta\alpha}{2}}} (e^{-C \frac{(x-\xi)^2}{t-\tau}} + e^{-C \frac{(y-\xi)^2}{t-\tau}}); \end{aligned} \quad (10)$$

where $C = C(R)$, and $K = K(R, \lambda, \alpha, T)$ is continuous with respect to T .

Lemma

Let $v, \bar{v} \in B(R, \lambda, \alpha)$, $0 < \beta < 1$, $\Gamma_{[v]}$ and $\Gamma_{[\bar{v}]}$ fundamental solutions of $L_{[v]} = 0$ and $L_{[\bar{v}]} = 0$, respectively. Then,

$$|(D_x^s \Gamma_{[v]} - D_x^s \Gamma_{[\bar{v}]})(x, t, \xi, \tau)| \leq \frac{K \|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}}}{(t - \tau)^{\frac{s+1}{2}}} e^{-C \frac{(x-\xi)^2}{t-\tau}}, \quad (11)$$

$$|(\partial_{xx} \Gamma_{[v]} - \partial_{xx} \Gamma_{[\bar{v}]})(x, t, \xi, \tau)| \quad (12)$$

$$\leq K (\|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}} + \|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}}^\beta) \left(\frac{1}{|x-\xi|^{\frac{2-\alpha}{2}} (t-\tau)^{\frac{2-\alpha}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{2}}} \right) e^{-C \frac{(x-\xi)^2}{t-\tau}}$$

and

$$|(\partial_t \Gamma_{[v]} - \partial_t \Gamma_{[\bar{v}]})(x, t, \xi, \tau)| \quad (13)$$

$$\leq K (\|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}} + \|v - \bar{v}\|_{\alpha, \frac{\alpha}{2}}^\beta) \left(\frac{1}{|x-\xi|^{\frac{2-\alpha}{2}} (t-\tau)^{\frac{2-\alpha}{2}}} + \frac{1}{(t-\tau)^{\frac{3}{2}}} \right) e^{-C \frac{(x-\xi)^2}{t-\tau}},$$

$C = C(R)$ and $K = K(R, \lambda, \alpha, T)$ is continuous with respect to T .

Theorem

Let $f, \bar{f} \in C^{1, \frac{1}{2}}(\Omega_T)$, $T > 0$ and u_0, \bar{u}_0 be Lipschitz continuous and bounded functions. If u and \bar{u} are solutions of

$$L_{[v]}u = f, \quad \mathbb{R} \times (0, T], \quad u(x, 0) = u_0, \quad \mathbb{R}, \quad (14)$$

and

$$L_{[\bar{v}]}u = \bar{f}, \quad \mathbb{R} \times (0, T], \quad u(x, 0) = \bar{u}_0 \quad \mathbb{R}, \quad (15)$$

where $v = (a, b, 0)$, $\bar{v} = (\bar{a}, \bar{b}, 0) \in B(R, \lambda, 1)$, then

$$\begin{aligned} \|u - \bar{u}\|_{1, \frac{1}{2}} &\leq K(\|v - \bar{v}\|_{1, \frac{1}{2}} + \|v - \bar{v}\|_{1, \frac{1}{2}}^\beta + \|u_0 - \bar{u}_0\|_1 + \\ &T^{\frac{1}{2}} \max\{\|f\|_{1, \frac{1}{2}}, 1\}(\|f - \bar{f}\|_{1, \frac{1}{2}} + \|v - \bar{v}\|_{1, \frac{1}{2}} + \|v - \bar{v}\|_{1, \frac{1}{2}}^\beta)), \end{aligned} \quad (16)$$

where $K = K(R, \lambda, T, \|u_0\|_1)$ is continuous with respect to T .

Corollary

If u is a solution of (14), then

$$\|u\|_{1, \frac{1}{2}} \leq K(\|v\|_{1, \frac{1}{2}} + \|v\|_{1, \frac{1}{2}}^\beta + \|u_0\|_1 + T^{\frac{1}{2}} \max\{\|f\|_{1, \frac{1}{2}}, 1\}(\|f\|_{1, \frac{1}{2}} + \|v\|_{1, \frac{1}{2}} + \|v\|_{1, \frac{1}{2}}^\beta)),$$

where $K = K(R, \lambda, T, \|u_0\|_1)$.

Lemma

Let us consider $v_n, v \in B(R, \lambda, \alpha)$ and the corresponding fundamental solutions, $\Gamma_{[v_n]}$ and $\Gamma_{[v]}$. If v_n converges pointwise to v , then $\Gamma_{[v_n]}$ converges also pointwise to $\Gamma_{[v]}$.

Local solution

Let A be the operator given by

$$A(u_1, u_2) = (w_1, w_2), \quad (17)$$

where (w_1, w_2) is the solution of the problem

$$\begin{cases} L_{[v(u_i)]}(w_i) = F_i(u_1, u_2, y_i), & \mathbb{R} \times (0, T] \\ y_{i_t} = -A_i y_i f(u_i), & \mathbb{R} \times (0, T] \\ (w_i(x, 0), y_i(x, 0)) = (u_{i0}(x), y_{i0}(x)), & \mathbb{R}. \end{cases} \quad (18)$$

Here, u_{i0} and $y_{i0} \geq 0$ are lipschitzian and bounded functions,

$$v_i(u_i) = \left(\frac{\lambda_i}{a_i + y_i(u_i)}, \frac{c_i}{a_i + y_i(u_i)}, 0 \right),$$

$$F_i(u_1, u_2, y_i) = \frac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \frac{u_1 - u_2}{a_i + b_i y_i}$$

and $f(u)$ is the Arrhenius' function, defined in (3).

Using the integral representation for the solution (w_1, w_2) and previous estimates, we get the following lemma.

Lemma

Let $K_i = K(\max\{\frac{\lambda_i}{a_i}, \frac{c_i}{a_i}\}, \frac{\lambda_i}{a_i + b_i \|y_{i,0}\|_\infty}, 1, \|u_{i,0}\|_1)$ be the constant given in

a Corollary above, $\nu_i = 2(\max\{\frac{\lambda_i}{a_i}, \frac{c_i}{a_i}\} + \max\{\frac{\lambda_i b_i}{a_i^2}, \frac{c_i b_i}{a_i^2}\})$,

$M_i > K_i(\nu_i \|y_{i,0}\|_1 + (\nu_i \|y_{i,0}\|_1)^\beta + \|u_{i,0}\|_1)$, and,

$\Sigma = \{(u_1, u_2) \in C^{1, \frac{1}{2}}(\Omega_T); \|u_i\|_{1, \frac{1}{2}} \leq M_i\}$. Then, if T is sufficiently small, $A(\Sigma) \subset \Sigma$.

Theorem

If T is sufficiently small the Cauchy problem (2) has a solution in $C^{2,1}(\mathbb{R} \times (0, T]) \cap C^{1, \frac{1}{2}}(\Omega_T)$.

Proof:

Iterative scheme:

$$(w_1^{(n)}, w_2^{(n)}) = A(w_1^{(n-1)}, w_2^{(n-1)})$$

By Arzelà-Ascoli's theorem, there exists a continuous function (u_1, u_2) in $\mathbb{R} \times [0, T]$ and a subsequence of $(w_1^{(n)}, w_2^{(n)})$ such that $(w_1^{(n)}, w_2^{(n)})$ converges to (u_1, u_2) uniformly in compact sets.

By the representation formula, we have

$$w_i^{(n+1)}(x, t) = \int \Gamma_n(x, \xi, t, 0) u_{i,0}(\xi) d\xi + \int_0^t \int \Gamma_n(x, \xi, t, \tau) F_i(w_1^{(n)}, w_2^{(n)}, y_i(w_i^{(n)}))(\xi, \tau) d\xi d\tau, \quad (19)$$

where

$$\Gamma_n = \Gamma_{[v_i(w_i^{(n)})]}, \quad v_i(w_i^{(n)}) = \left(\frac{\lambda_i}{a_i + y_i(w_i^{(n)})}, \frac{c_i}{a_i + y_i(w_i^{(n)})}, 0 \right) \quad (20)$$

and

$$y_i(w_i^{(n)})(x, t) = y_{i,0}(x) e^{-A_i \int_0^t f(w_i^{(n)}(x, s)) ds}. \quad (21)$$

□

Global solution

Theorem. Let $u_{i,0}$, $i = 1, 2$, be bounded lipschitzian functions on \mathbb{R} and in $L^p(\mathbb{R})$ for some $p \in (1, \infty)$. Assume also $y_{i,0} \in C^2 \cap L^\infty$ and $y'_{i,0} \in L^\infty$. Then, for any $T > 0$, there exists a solution of (2) in $C^{2,1}(\mathbb{R} \times (0, T]) \cap C^{1,\frac{1}{2}}(\mathbb{R} \times [0, T]) \cap L^\infty([0, T]; L^p(\mathbb{R}))$.

On the proof:

- ▶ We define $[0, T^*)$ as the maximal interval for the local solution $U \equiv (u_1, u_2)$ and show that there exists the limit $\lim_{t \rightarrow T^*} U(\cdot, t)$ in the above space.
- ▶ To show that U is bounded in $\mathbb{R} \times [0, T^*)$ we use the upper solution mentioned earlier.
- ▶ To bound U_x we use the technique of Oleinik, O. A. and Kruzhkov, S. N. *Quasilinear second order parabolic equations with many independent variable*, Russ. Math. Surv., **16**, no.5, (1961), 105-146.