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Global solution for the Combustion Model in a Porous Medium with two Layers

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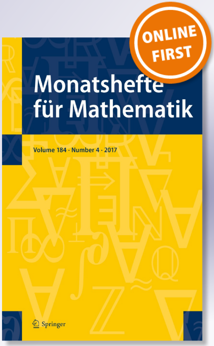

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The Cauchy problem for a combustion model in a porous medium with two layers

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Abstract We prove the existence of local and global in time solutions of the Cauchy problem for a combustion model in a porous medium with two layers. The model is a system of four equations, consisting of two nonlinear reaction–convection–diffusion equations coupled with two ordinary differential equations, with the coupling occurring in both the reaction functions and in the differential operator coefficients. To obtain the local solution, we first construct an iteration scheme of approximate solutions to the system. Using the continuous dependence of solutions for parabolic equations with respect to the coefficients of the equations, we show that the constructed iteration scheme contains a sequence which converges to a local solution of the system, under the assumption that the initial data are Lipschitz continuous, bounded and non

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Outline:

- The model and the main result
- Local solution
- Global solution

The model

We consider the following system of differential equations:

$$\begin{cases} (u_i)_t - \alpha_i(y_i)(u_i)_{xx} + \beta_i(y_i)(u_i)_x = f_i(y_i, u_1, u_2) \\ (y_i)_t = -A_i y_i f(u_i) \end{cases} \tag{1}$$
$$i = 1, 2$$

$x \in \mathbb{R}, t > 0$

A_i is a positive constant, and $\alpha_i(y_i)$, $\beta_i(y_i)$, $f_i(y_i, u_1, u_2)$, and $f(u_i)$ are the following functions:

$f(u_i)$ is the "Arrhenius function"

$$f(u_i) = e^{-\frac{E}{u_i}},$$

where E is a positive constant*;

*We note that the function $f(s) = e^{-\frac{E}{s}}$ can be extended smoothly by zero to $s \leq 0$.

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The model, cont'ed

$$\alpha_i(y_i) = \frac{\lambda_i}{a_i + b_i y_i}$$

$$\beta_i(y_i) = \frac{c_i}{a_i + b_i y_i}$$

$$f_i(y_i, u_1, u_2) = \frac{b_i A_i u_i + d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \frac{u_1 - u_2}{a_i + b_i y_i}$$

$\lambda_i, a_i, b_i, c_i, d_i, A_i, q$ are positive constants.

The model, cont'ed

The system (1) models the propagation of a combustion front in a porous medium with two layers. The unknowns u_1 and y_1 denote the temperature and the fuel concentration, respectively, in one layer, u_2 and y_2 denote the same in the other layer, and the constants λ_i, a_i, etc. are parameters related to the medium.
J. da Mota and S. Schecter, Jr. Dyn. Diff. Eq., **18** *(3), (2006)*

For the particular case when the fuel concentrations are known functions, the Cauchy problem for the system (1) was solved by J. da Mota and M. Santos (next slide):

J. da Mota – M. Santos: Reduced system

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An application of the monotone iterative method to a combustion problem in porous media☆

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ABSTRACT

Problems related to combustion fronts in porous media have been studied by many authors recently, see e.g. [Y. Akkutlu, Y.C. Yortsos, The dynamics of in-situ combustion fronts in porous media, Combust. Flame 134 (2003) 239–247; J.C. da Mota, W. Dantas, D. Marchesin, Combustion fronts in porous media, SIAM J. Appl. Math. 62 (2002) 2175–2198; D.A. Schult, B.J. Matkowsky, V.A. Volpert, A.C. Fernandez-Pello, Forced forward smolder combustion, Combust. Flame 104 (1996) 1–26]. Most of this interest is due to the combustion process for oil recovery.

In this paper we construct monotone iteractions for a Cauchy problem arising from a combustion model in a porous medium derived in [J.C. da Mota, S. Schecter, Combustion fronts in a porous medium with two layers, J. Dynam. Differential Equations 18 (3) (2006) 615–665]. We conclude that the monotone iteractions converge to a unique solution of this Cauchy problem, globally in time.

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1. Introduction

The model derived in [1] takes into account a combustion of oxygen and a solid fuel, such as coke, in a porous medium with two parallel layers. It consists of a nonlinear reaction–diffusion–convection system, derived from balance equations and Darcy's law. The model is motivated by the fact that oil reservoirs often contain different layers, characterized by different porosity, density, thermal conductivity, etc. A summarized description of this model is given in Appendix, for details see [1]. After some simplifications, the dependent variables are only temperature and unburned fuel concentration in each layer. Only the incompressible case is considered, the gas density is taken to be an average value not depending on temperature or pressure, the thermal conductivity is assumed constant and the effects of mass diffusivity are neglected. The model includes heat transfer between the layers.

Using dimensionless variables, where the temperature and fuel concentration in one of the layers are denoted by u and y, respectively, and by v and z in the other layer, the model is given by the following four nonlinear partial differential equations governing temperature and unburned fuel concentration in each layer:

$$\Big((a_1 + b_1 y)u\Big)_t + (c_1 u)_x = d_1 f(u, y) - q(u - v) + \lambda_1 u_{xx}, \tag{1}$$

$$y_t = -A_1 f(u, y), \tag{2}$$

The model, cont'ed

We note that in system (1) all equations are coupled in both the reaction functions and in the differential operator coefficients. Besides, the differential operator coefficients are not constant.

Existence results for systems of reactions-diffusion equations coupled with ODEs can be found in *F. Rothe, Global solutions of reaction–diffusion systems, Lecture Notes in Mathematics, Ed. by A. Dold and B. Eckmann, Springer-Verlag Berlin Heidelberg New York Tokyo, (1984)*, but with the coupling occurring only in the reaction functions and with the differential operators having constant coefficients.

The Cauchy problem

The system (1) can be considered in the unknowns u_1, u_2 only, since the equations

$$(y_i)_t = -A_i y_i f(u_i)$$

can be solved in terms of u_i:

$$y_i = y_{i,0} e^{-A_i \int_0^t f(u_i)\, d\tau}.$$

Thus we consider the following Cauchy problem in the unknowns u_1, u_2:

$$\begin{cases} (u_i)_t - \alpha_i(y_i)(u_i)_{xx} + \beta_i(y_i)(u_i)_x = f_i(y_i, u_1, u_2) \\ y_i = y_{i,0} e^{-A_i \int_0^t f(u_i)\, d\tau} \\ u_i\big|_{t=0} = u_{i,0} \end{cases} \tag{2}$$

for given $y_{i,0}$, $i = 1, 2$.

Upper solution

Assuming that y_1, y_2 are given non negative and bounded functions, the function

$$\varphi(t) = (M + \beta)e^{\alpha t} - \beta$$

where * $M = \max_{i=1,2} \|u_{i,0}\|_\infty$, $\alpha = \max_{i=1,2}\{\frac{A_i b_i \|y_i\|_\infty}{a_i}\}$ and $\beta = \max_{i=1,2}\{\frac{d_i}{A_i b_i}\}$, is an upper solution for the Cauchy problem

$$\begin{cases} (u_i)_t - \alpha_i(y_i)(u_i)_{xx} + \beta_i(y_i)(u_i)_x = f_i(y_i, u_1, u_2) \\ u_i|_{t=0} = u_{i,0}. \end{cases}$$

It is a straightforward computation to verify this statement.

*If g is a bounded function defined in $\mathbb{R}$, $\|g\|_\infty := \sup_{x\in\mathbb{R}} |g(x)|$.

http://www.ime.unicamp.br/~msantos/XIEMED.pdf

Main result

Theorem (J. da Mota, M. Santos, R. Santos; Monastshefte fur Mathematik, 2017)

Assume that $u_{i,0}$ *and* $y_{i,0}$ *are non-negative, Lipschitz continuous, and bounded functions in* $\mathbb{R}$, $u_{i,0} \in L^p(\mathbb{R})$ *for some* $p \in (1, \infty)$ *and* $y_{i,0} \in C^2(\mathbb{R})$ *and* $(y_{i,0})'$ *is bounded. Then the Cauchy problem (2) has a (classical) solution* $u = (u_1, u_2)$ *in* $C^{2,1}(\mathbb{R} \times (0, \infty)) \cap C^{1,\frac{1}{2}}_{loc}(\mathbb{R} \times [0, \infty)) \cap L^\infty_{loc}((0, \infty); L^p(\mathbb{R}))^*$ *satisfying* $0 \leq u_i(x, t) \leq \varphi(t)$, *for all* $(x, t) \in \mathbb{R} \times [0, \infty)$.

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*Here the term "*loc*" denotes "locally" in time, i.e. a function $u \in C^{1,\frac{1}{2}}_{\text{loc}}(\mathbb{R} \times [0, \infty)) \cap L^\infty_{\text{loc}}((0, \infty); L^p(\mathbb{R}))$ if $u|\mathbb{R} \times [0, T] \in C^{1,\frac{1}{2}}(\mathbb{R} \times [0, T]) \cap L^\infty((0, T); L^p(\mathbb{R}))$, for any $T > 0$.

Proof; main ideas

Iterative scheme

$$\begin{cases} (u_i^n)_t - \alpha_i(y_i^{n-1})(u_i^n)_{xx} + \beta_i(y_i^{n-1})(u_i^n)_x = \tilde{f}_i(y_i^{n-1}, u_1^{n-1}, u_2^{n-1}) \\ (y_i^{n-1})_t = -A_i y_i^{n-1} \tilde{f}(u_i^{n-1}) \\ (u_i^n, y_i^{n-1})|_{t=0} = (u_{i,0}, y_{i,0}), \end{cases} \tag{3}$$

$n = 1, 2, \ldots$, starting from an initial function (u_1^0, u_2^0) in $C^{1,\frac{1}{2}}(\mathbb{R} \times [0, T])$, for some sufficiently small time $T > 0$, where $\tilde{f}$ is the function that coincides with the Arrhenius function $f(s) = e^{-\frac{E}{s}}$ for $s > 0$ and it is equal to zero for $s \leq 0$, and $\tilde{f}_i$ is the function f_i except for the Arrhenius function f which is replaced by $\tilde{f}$.

Proof; main ideas: Local solution

Corresponding operator:

$$\mathcal{A}(u_1, u_2) = (w_1, w_2), \tag{4}$$

where (w_1, w_2) solves

$$\begin{cases} (w_i)_t - \alpha_i(y_i)(w_i)_{xx} + \beta_i(y_i)(w_i)_x = \tilde{f}_i(y_i, u_1, u_2) \\ (y_i)_t = -A_i y_i \tilde{f}(u_i) \\ (w_i, y_i)|_{t=0} = (u_{i,0}, y_{i,0}), \end{cases} \tag{5}$$

We show that there is a positive time T, depending on the initial data $u_{i,0}, y_{i,0}$ and on the parameters in the equations (i.e. on λ_i, a_i, etc.), such that

$$\mathcal{A}(u) \in \Sigma \text{ for all } u \in \Sigma.$$

where

$$\Sigma := \{u = (u_1, u_2) \in C^{1,\frac{1}{2}}(\mathbb{R} \times [0, T]); \|u_i\|_{C^{1,\frac{1}{2}}(\mathbb{R}\times[0,T])} \leq R, \ i = 1, 2\}$$

$R > 0$.

Proof; main ideas: Local solution, cont'ed

It follows that the sequence $\{u^n\} = \{(u_1^n, u_2^n)\}$ given by $\mathcal{A}(u^n) = \mathcal{A}(u^{n-1})$, starting from any $u^0 \in \Sigma$, is bounded in the norm $\|\cdot\|_{1,1/2}$. Therefore, by Arzelà–Ascoli's theorem, there exists a function $u = (u_1, u_2) \in \Sigma$ and a subsequence of $\{u^n\}$, which we still denote by $\{u^n\}$, that converges to u, uniformly on bounded sets in $\mathbb{R} \times [0, T]$.

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Proof; main ideas: Local solution, cont'ed

To show that the limit u is a solution of (2), we use the integral representation

$$\begin{aligned} u_i^n(x,t) = \quad & \int_{\mathbb{R}} \Gamma_{i,n}(x,t,\xi,0)u_{i,0}(\xi)\,d\xi \\ & + \int_0^t \int_{\mathbb{R}} \Gamma_{i,n}(x,t,\xi,\tau)\tilde{f}_i(y_i^{n-1},u_1^{n-1},u_2^{n-1})(\xi,\tau)\,d\xi\,d\tau, \end{aligned} \tag{6}$$

for the solution u_i^n of the parabolic equation

$$(u_i^n)_t - \alpha_i(y_i^{n-1})(u_i^n)_{xx} + \beta_i(y_i^{n-1})(u_i^n)_x = \tilde{f}_i(y_i^{n-1},u_1^{n-1},u_2^{n-1}) \tag{7}$$

occurring in (3).

$\Gamma_{i,n}$ is the fundamental solution of the associated operator

$$\mathcal{L}_{i,n} := \partial_t - \alpha_i(y_i^{n-1})\partial_{xx} + \beta_i(y_i^{n-1})\partial_x.$$

Proof; main ideas: Local solution, cont'ed

Using that the sequence of fundamental solutions $\{\Gamma_{i,n}\}$ converges pointwise to the fundamental solution Γ_i of the parabolic operator $\mathcal{L}_i := \partial_t - \alpha_i(y_i)\partial_{xx} + \beta_i(y_i)\partial_x$, when n tends to infinity, where

$$y_i = y_i(x,t) = y_{i,0}(x)\mathrm{e}^{-A_i \int_0^t f(u_i(x,s))\, ds}, \tag{8}$$

having that the sequence $\{u_i^n\}$ is bounded in $\mathbb{R} \times [0,T]$, for some positive T, and that it converges uniformly to $u_i \in C^{1,1/2}(\mathbb{R} \times [0,T])$ in bounded sets in $\mathbb{R} \times [0,T]$, we can show from (6) that u_i satisfies the integral equation

$$\begin{aligned} u_i(x,t) = \quad & \int_{\mathbb{R}} \Gamma_i(x,t,\xi,0) u_{i,0}(\xi)\, d\xi \\ & + \int_0^t \int_{\mathbb{R}} \Gamma_i(x,t,\xi,\tau) \tilde{f}_i(y_i,u_1,u_2)(\xi,\tau)\, d\xi\, d\tau, \end{aligned} \tag{9}$$

for all $(x,t) \in \mathbb{R} \times [0,T]$. Thus, $u_i \in C^{2,1}(\mathbb{R} \times (0,T]) \cap C^{1,\frac{1}{2}}(\mathbb{R} \times [0,T])$ and it is a solution of (2).

Estimate in L^p

Let the initial data $u_{i,0}$ be $\in L^p$, for some $p > 1$.

Then, using the "generalized Young's inequality" ($\|\int K(\cdot, y)f(y)dy\|_p \leq C\|f\|_p$) and considering that the fundamental solution $\Gamma_{i,n}$ is a "regular kernel", uniformly with respect to n, we can show that the sequence $\{u_i^n\}$ remains in L^p for all $t \in (0, T)$, with $\|u_i^n(\cdot, t)\|_{L^p}$ uniformly bounded with respect to t and n, if T is sufficiently small.

Then, applying the Banach–Alaoglu's theorem to the sequence $\{u_i^n\}$ in L^p, we obtain that the limit u_i is in L^p.

Upper solution bound

To show that the obtained solution $u = (u_1, u_2)$ satisfies

$$0 \leq u_i \leq \varphi$$

we observe that $u = (u_1, u_2)$ is a solution of the Cauchy problem

$$\left\{ \begin{array}{l} (w_i)_t - \alpha_i(y_i)(w_i)_{xx} + \beta_i(y_i)(w_i)_x = \tilde{f}_i(y_i, w_1, w_2) \\ w_i(x,0) = u_{i,0}(x), \end{array} \right.$$
$$i = 1, 2$$

for y_i given by (8), and the reaction function $\tilde{f}_i$ is increasing with respect to w_j $(i, j = 1, 2,\ j \neq i)$, since $\partial \tilde{f}_i / \partial w_j = q/(a_i + b_i y_i)$.

Then, the result follows by comparison.

Global solution

Let $[0, T^*)$, $0 < T^* \le \infty$, be a maximal interval in which there exists a solution u^* to the problem (2) in the space

$$X_{T^*} := C^{2,1}(\mathbb{R}\times(0,T^*))\cap C^{1,\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}\times[0,T^*))\cap L^\infty_{\mathrm{loc}}((0,T^*);L^p(\mathbb{R}))\cap\langle 0,\varphi\rangle_{T^*}.$$

We shall show that if $T^* < \infty$, then we have a contradiction, by proving that, in this case, the maximal solution u^* has a continuous extension up to the time T^*, and that this extension is Lipschitz continuous and it is in L^p, as a function of $x \in \mathbb{R}$, for $t = T^*$, thus u^* can be extended to a larger time in X_{T^*}, accordingly with the local existence.

Global solution, cont'ed

To extend u^* up to the time $T^* < \infty$, we use again to use the integral representation (9), for $u_i = u_i^*$, $(x,t) \in \mathbb{R} \times [0, T^*)$, with Γ_i being the fundamental solution of the equation

$$\partial_t - \alpha_i(y_i^*)\partial_{xx} + \beta_i(y_i^*)\partial_x$$

where

$$y_i^* = y_{i,0}(x)e^{-A_i \int_0^t f(u_i^*)\,ds}, \quad f_i(y_i, u_1, u_2) = f_i(y_i^*, u_1^*, u_2^*).$$

To accomplish this, we need to prove that the derivatives $\partial_x u_i^*$ are bounded in $\mathbb{R} \times (0, T^*)$.

Estimate of $\partial_x u_i^*$

We first show the following lemma.

Lemma

Let $\epsilon \in (0, T^*)$. There is a constant K with the following property: let $a < b$ in $\mathbb{R}$, $T \in (\epsilon, T^*)$ and

$$q_i = \max_{[a,b]\times[\epsilon,T]} |\partial_x u_i|.$$

If

(i) *for $i = 1, 2$, q_i is attained in a point in $(a, b) \times (\epsilon, T]$ or*

(ii) *for some $i \in \{1, 2\}$, q_i is attained in a point in $(a, b) \times (\epsilon, T]$ and for $j \neq i$, $j \in \{1, 2\}$, we have that $\partial_x u_j$ is bounded in $\mathbb{R} \times (0, T^*)$,*

then $q_i \leq K$.

We prove this lemma by the *method of auxiliary functions*.

The method of auxiliary functions

This terminology was used by R. Finn in the MathSciNet review #MR0064286 (16,259b). In this review he also points out that this method was "developed by Picard [see, e.g., Courant and Hilbert, Methoden der mathematischen Physik, Bd II, Springer, Berlin, 1937, pp. 274–276], Bernstein [Math. Ann. 69, 82–136 (1910); Doklady Akad. Nauk SSSR (N.S.) 18, 385–388 (1938)] and others".

Here, we employ this method as used by

- [Oleinik-Kruzhkov] O. A. Oleinik and S. N. Kruzhkov, *Quasi-linear second-order parabolic equations with many independent variables*, Russian Mathematical Surveys, **16** (5) (1961), 105–146)

or, cf. [Oleinik-Kruzhkov] p. 107,

- O. A. Oleinik and T. D. Venttsel', *Cauchy's problem and the first boundary problem for a quasilinear equation of parabolic type.* (Russian) Doklady Akad. Nauk SSSR (N.S.) 97, (1954). 605–608
- O. A. Oleinik and T. D. Venttsel', *The first boundary problem and the Cauchy problem for quasi-linear equations of parabolic type.* (Russian) Mat. Sb. N.S. 41(83) (1957), 105–128.

The method of auxiliary functions, cont'ed

The method:

We make a substitution $u_i^* = h_i(v_i)$ for an appropriate function h_i (in particular, such that h_i' is positive and bounded) and estimate $|\partial_x v_i|$ (instead of trying to estimate $|\partial_x u_i^*|$) at a maximum point, by looking for the equation satisfied by v_i.

This leads to some technical estimates where we use the explicit forms for the functions $\alpha(y_i)$, $\beta_i(y_i)$, and $f_i(y_i, u_1, u_2)$.

The method of auxiliary functions: proof of the above lemma

See our paper.

Thank you!