UIS - Bucaramanga

A family of systems including models for viscoplastic fluids

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Overview

- Introduction/Motivation to the systems
- Subdifferential (convexity, monotonicity) and some other auxiliary results
- On the solvability of an ibvp associated to the systems

The Navier-Stokes systems

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \operatorname{div}\,\mathbb{T}, \quad \operatorname{div}\mathbf{v} = 0$$

depends on the specification of the *Cauchy stress tensor*

$$\mathbb{T} = -\mathrm{p}\,\mathrm{I} + \mathbb{S}$$

where p is pressure and $\mathbb{S}$ is the *viscous part of the Cauchy stress tensor*.

There are many types of fluids.

The specification of $\mathbb{S}$ is a major issue in the mathematical modeling of fluids.

Here, we are mainly interested in viscoplastic fluids, including effects of rotating particles – *Cosserat–Bingham/Herschel-Bulkley model* (but our analysis applies to Newtonian fluids as well).

E. C. Bingham, *Fluidity and Plasticity*, McGraw-Hill, 1922.

E. Cosserat and F. *Cosserat, Théorie des Corps Déformables*, Herman, 1909.

G. Łukaszewicz, *Micropoloar Fluids*. Theory and Applications, Birkhäuser, 1999.

A. C. Eringen, *Theory of Micropolar Fluids*, J. Math. Mech. 16, 1966.

A. C. Eringen, *A Unified Theory of Thermomechanical Materials*. Int. J. Eng Sci. 4, 1966.

Wikipedia
https://en.wikipedia.org › wiki › E... · Traduzir esta página

Eugène Cosserat

Eugène-Maurice-Pierre **Cosserat** (4 March 1866 – 31 May 1931) was a French mathematician and astronomer. ... Born in Amiens, he studied at the École Normale ...

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François Cosserat

François **Cosserat** (26 October 1852 – 22 March 1914) was a French engineer and mathematician known by his theories about deformable bodies written with his ...

Scientific Research Publishing
https://www.scirp.org › reference · Traduzir esta página

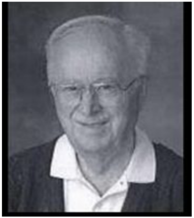
Cosserat, E. and Cosserat, F. (1909) Theories des corps ...

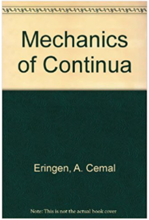
ABSTRACT: The conservation laws of continuum mechanics and of the kinetic theory with the influence of the angular momentum and associated with its rotation of ...

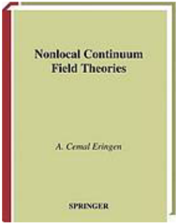
Springer
https://link.springer.com › chapter · Traduzir esta página

Cosserat Media

de H Altenbach · Citado por 30 — In this chapter we discuss the three-dimensional **Cosserat**-type theories of continua. Originally this approach belongs to **Cosserat** brothers, ...







Professor Ahmed Cemal Eringen (1921 – 2009)

See:
https://en.wikipedia.org/wiki/Ahmed_Cemal_Eringen
https://upclosed.com/people/ahmed-cemal-eringen/
http://worldcat.org/identities/lccn-n83826801/
http://www.biyografya.com/biyografi/1271
http://www.korhaber.com/haber/Cemal-Eringen-vefat-etti/10931

Biography (from Wikipedia):
Ahmed Cemal Eringen (born February 15, 1921, in Kayseri, Turkey - December 7, 2009) was a Turkish-American engineering scientist. He was a professor at Princeton University and the founder of the Society of Engineering Science. The Eringen Medal is named in his honor.

https://shellbuckling.com/cv/eringen.pdf

International Journal of Engineering Science

Editor-in-Chief A. C. ERINGEN

Volume 1
1963

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OXFORD · LONDON · NEW YORK · PARIS

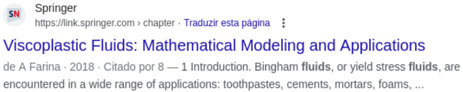
In fluids with rotating particles (more precisely, *micropolar viscoplastic fluid*), the viscous stress tensor $\mathbb{S}$ depends on both the symmetric and the antisymmetric part of $\nabla \mathbf{v}$.

V. Shelukhin, M. Růžička, *On Cosserat–Bingham Fluids.* Z. Angew. Math. Mech. 93, 2013.

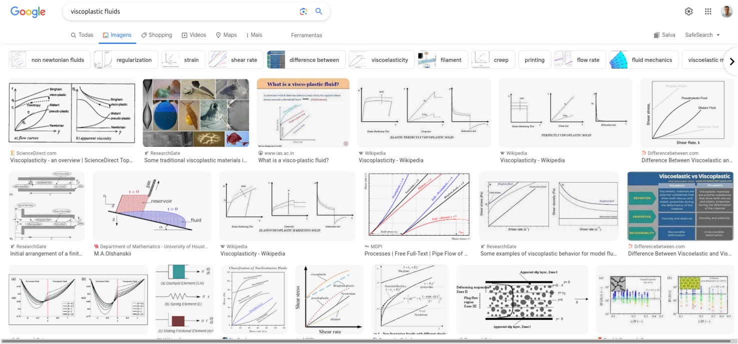
R. Teisseyre, M. Teisseyre-Jerenska, *Asymmetric continuum: Extreme processes in solids and fluids*, Springer, 2014.

Viscoplastic fluids

First entry one may find in Google searching “viscoplastic fluids”   2024/01/13



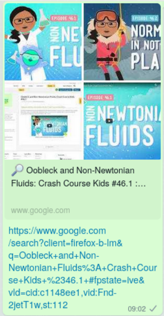
Some Google images

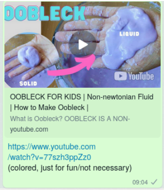


Youtube video - “What are Newtonian and Non-Newtonian Fluids?”

`https://www.youtube.com/watch?v=Yh6h-0Ppwdk`

Oobleck fluid





Kids running on oobleck `https://m.facebook.com/DiscoveryBrasil/videos/estudantes-atravessam-uma-pequena-piscina-de-oobleck/203251032051000/`

Crash Course Kids
`https://www.google.com/search?client=firefox-b-lm&q=Oobleck+and+Non-Newtonian+Fluids`

Recipes to make oobleck (it's very simple - just add water to cornstarch ("amido de milho"/"Maizena"):

1. `https://learning.sciencemuseumgroup.org.uk/resources/oozing-oobleck/`
   More things here.
2. For kids (colored just for fun/not necessary): `https://www.youtube.com/watch?v=77szh3ppZz0`

Non-Newtonian Fluids - a guide to classification

Research Gate, by Neil John Alderman https://www.researchgate.net/publication/273392367

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NON-NEWTONIAN FLUIDS : GUIDE TO CLASSIFICATION AND CHARACTERISTICS

CONTENTS

TABLE 2.1 Examples of fluids exhibiting non-Newtonian flow behaviour

Adhesives	Peat slurries
Biological fluids	Plastic melts
Cement slurries	Polymer solutions
Chalk slurries	Printing inks
Chocolate	Quicksand
Coal slurries	Rock slurries
Detergent slurries	Rubber solutions
Food sauces	Sand slurries
Greases	Sewage sludges
Hand creams	Shampoo
Margarine	Soap slurries
Mayonnaise	Starch solutions
Metal oxide slurries	Tomato paste
Oil well drilling muds	Toothpaste
Paints	Wet beach sand
Paper pulp	

Non-Newtonian Fluids - a guide to classification – continued

TABLE 5.1 FLOW CURVE MODELS

Model	Constitutive Equation	Viscosity	Rheological Parameters
Newtonian	$\tau = \mu_N\dot{\gamma}$	$\mu = \mu_N$	μ_N (Pa s)
Power law[3]	$\tau = K\dot{\gamma}^n$	$\mu = K\dot{\gamma}^{n-1}$	K (Pa s^n) n (dimensionless)
Bingham Plastic[1]	$\tau = \tau_{yB} + \mu_B\dot{\gamma}$	$\mu = \frac{\tau_{yB}}{\dot{\gamma}} + \mu_B$	τ_{yB} (Pa) μ_B (Pa s)
Herschel-Bulkley[2]	$\tau = \tau_{yHB} + K\dot{\gamma}^n$	$\mu = \frac{\tau_{yHB}}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$	τ_{yHB} (Pa) K (Pa s^n) n (dimensionless)
Casson[10]	$\sqrt{\tau} = \sqrt{\tau_{yC}} + \sqrt{\mu_C\dot{\gamma}}$	$\mu = \frac{[\sqrt{\tau_{yC}} + \sqrt{\mu_C\dot{\gamma}}]^2}{\dot{\gamma}}$	τ_{yC} (Pa) μ_C (Pa s)
Vocadlo[15]	$\tau = (\tau_{yV}^{1/n} + K\dot{\gamma})^n$	$\mu = \frac{(\tau_{yV}^{1/n} + K\dot{\gamma})^n}{\dot{\gamma}}$	τ_{yV} (Pa) K (Pa s^n) n (dimensionless)
Prandtl-Eyring[4,7]	$\tau = a\sinh^{-1}(b\dot{\gamma})$	$\mu = \frac{[a\sinh^{-1}(b\dot{\gamma})]}{\dot{\gamma}}$	a (Pa) b (s)
Powell-Eyring[8]	$\tau = c\dot{\gamma} + a\sinh^{-1}(b\dot{\gamma})$	$\mu = c + \frac{[a\sinh^{-1}(b\dot{\gamma})]}{\dot{\gamma}}$	a (Pa) b (s) c (Pa s)
Cross[13]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\dot{\gamma}^m}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\dot{\gamma}^m}$	μ_0 (Pa s) μ_∞ (Pa s) a (s^m) m (dimensionless)
Sisko[9]	$\tau = \left\{\mu_\infty + K\dot{\gamma}^{n-1}\right\}\dot{\gamma}$	$\mu = \mu_\infty + K\dot{\gamma}^{n-1}$	μ_∞ (Pa s) K (Pa s^n) n (dimensionless)
Carreau[14]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{[1 + (a\dot{\gamma})^2]^{m/2}}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{[1 + (a\dot{\gamma})^2]^{m/2}}$	μ_0 (Pa s) μ_∞ (Pa s) a (s) m (dimensionless)
Van Wazer[11]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\dot{\gamma} + b\dot{\gamma}^m}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\dot{\gamma} + b\dot{\gamma}^m}$	μ_0 (Pa s) μ_∞ (Pa s) a (s) b (s^m) m (dimensionless)
Williamson[5]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau}$	μ_0 (Pa s) μ_∞ (Pa s) a (Pa^{-1})
Reiner-Philippoff[6]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau^2}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau^2}$	μ_0 (Pa s) μ_∞ (Pa s) a (Pa^{-2})
Meter[12]	$\tau = \left\{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau^m}\right\}\dot{\gamma}$	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + a\tau^m}$	μ_0 (Pa s) μ_∞ (Pa s) a (Pa^{-m}) m (dimensionless)
Ellis[27]	$\tau = \left\{\frac{\mu_0}{1 + a\tau^{m-1}}\right\}\dot{\gamma}$	$\mu = \frac{\mu_0}{1 + a\tau^{m-1}}$	μ_0 (Pa s) a (Pa^{1-m}) m (dimensionless)

Viscoplastic fluids - continued

Viscoplastic fluids are materials that exhibit a combination of solid-like (plastic) behavior and fluid-like (viscous) behavior under different conditions. They are a type of non-Newtonian fluid that have a yield stress, which means they behave like solids at low stresses but flow like liquids once a certain threshold stress (yield stress) is exceeded. More precisely, at each time the region occupied by the fluid is divided in two parts: one where the stress surpasses the yield stress (resulting in liquid-like flow), and the other with lower stress (where the material behaves like a solid). Some examples of viscoplastic fluids are toothpaste, slurries (mixtures of solid particles suspended in a liquid), drilling fluids (see e.g. [Chapter 1, *]) and animal blood.

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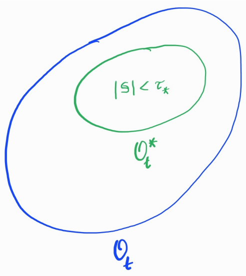
*S. Bridges, L. Robinson, *A Practical Handbook for Drilling Fluids Processing*. Elsevier, 2020

Viscoplastic fluids - continued

The parts where the fluid exhibits solid-like and liquid-like behaviors at any given time are unknown in fluid models. They can be defined in terms of the yield stress, which is a non negative number, depending (only) on the kind of viscoplastic fluid considered. We shall denoted it by τ_*. We also shall call the part where the fluid is solid-like *plug region*, denote it by $\mathcal{O}_t^*$ at a time t and define it at as being the set

$$\mathcal{O}_t^* := \{x;\ |\mathbb{S}(x,t)| \leq \tau_*\}.$$



Bingham model

In the simplest (and pionering) model for viscoplastic fluids, the tensor $\mathbb{S}$ is given by

$$\begin{cases} \mathbb{S} & = \mu_1 \mathrm{B}_s + \tau_* \frac{\mathrm{B}_s}{|\mathrm{B}_s|}, \quad \text{if } \mathrm{B}_s \neq 0 \\ |\mathbb{S}| & \leq \tau_*, \quad \text{if } \mathrm{B}_s = 0. \end{cases} \tag{1}$$

These relations are a constitutive law for the (viscoplastic) fluid model known as *Bingham fluid* or *Bingham model*. Here, and throughout the paper, we set

$$\mathrm{B} = \nabla \mathbf{v},$$

$$B_s = \frac{1}{2}(B + B^{transp}) \quad \text{and} \quad B_a = \frac{1}{2}(B - B^{transp}).$$

Bingham model - 2 remarks

1st Remark: The first relation in (??) can be written as

$$\mathbb{S} = (\mu_1 |\mathrm{B}_s| + \tau_*) \frac{\mathrm{B}_s}{|\mathrm{B}_s|},$$

thus, the plug region $\mathcal{O}_t^*$ for the Bingham model (the model (??)) is given by

$$\mathcal{O}_t^* := \{x;\ \mathrm{B}_s = 0\}.$$

2nd Remark: The tensor $\mathbb{S}$ satisfying (??) has a *potential*, i.e., the relations (??) can be achieved by the single condition

$$\mathbb{S}(x,t) \in \partial V(\mathrm{B}(x,t)),$$

for each point (x,t), where V is the convex function

$$V(\mathrm{X}) = \frac{\mu_1}{2} |\mathrm{X}_s|^2 + \tau_* |\mathrm{X}_s|$$

and $\partial V(\mathrm{X})$ denotes the subdifferential of V at the point (matrix) X.

Herschel-Bulkley model

It is the *power law model* with

$$\begin{cases} \mathbb{S} &= \mu_1 |\mathrm{B}_s|^{p-2}\mathrm{B}_s + \tau_* \frac{\mathrm{B}_s}{|\mathrm{B}_s|}, \qquad \text{if } \mathrm{B}_s \neq 0 \\ |\mathbb{S}| &\leq \tau_*, \qquad \text{if } \mathrm{B}_s = 0, \end{cases}$$

where p is a real number (the *power law index*).

Potential:

$$V(\mathrm{X}) = \frac{\mu_1}{p}|\mathrm{X}_s|^p + \tau_*|\mathrm{X}_s|$$

Cosserat-Bingham model

Fluid with rotating particles (micropolar viscoplastic fluid). Model proposed by V. Shelukhin, M. Růžička, Z. Angew. Math. Mech. (ZAMM), 2013 (asymmetric model):

$$\begin{cases} \mathbb{S} & = \mu_1 |\mathrm{B}_s|^{p-2}\mathrm{B}_s + \mu_2 |\mathrm{R}|^{p-2}\mathrm{R} + \tau_* \dfrac{\mathrm{B}_{\nu,p}}{|\mathrm{B}_{\nu,p}|}, \qquad \text{if } \mathrm{B}_\nu \neq 0 \\ |\mathbb{S}| & \leq \tau_*, \qquad \text{if } \mathrm{B}_\nu = 0, \end{cases}$$

where $\mu_1 > 0$, $\mu_2 > 0$ are (constant) viscosity coefficients, $\nu \geq 0$ (*plug parameter*),

$$\mathrm{B}_{\nu,p} = |\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu |\mathrm{R}|^{p-2}\mathrm{R}, \qquad \mathrm{B}_\nu = \mathrm{B}_{\nu,2} = \mathrm{B}_s + \nu \mathrm{R}$$

and

$$\mathrm{R} = \mathrm{B}_a - \Omega,$$

being $\Omega = \Omega(x,t)$ the *micro-rotational velocity tensor*, which is an antisymmetric matrix for each (x,t).

Cosserat-Bingham model - remarks

1st Remark: In the Cosserat-Bingham (Shelukhin-Růžička) model, Ω is an unknown (we have more equations/an additional system for the angular velocity) but, here, we consider Ω as a known function/as a given matrix function.

2nd Remark - regarding a potential: The first part of $\mathbb{S}$, i.e., $\mu_1|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \mu_2|\mathrm{R}|^{p-2}\mathrm{R}$, is the gradient of the function

$$U(X) := \frac{\mu_1}{p}|X_s|^p + \frac{\mu_2}{p}|X_a - \Omega|^p, \quad X \in \mathbb{R}^{d\times d},$$

at each matrix B, for any (x,t).

The second part $\dfrac{\mathrm{B}_{\nu,p}}{|\mathrm{B}_{\nu,p}|} = \dfrac{|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu|\mathrm{R}|^{p-2}\mathrm{R}}{\left||\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu|\mathrm{R}|^{p-2}\mathrm{R}\right|}$ is a gradient function iff $p = 2$ and $\nu = 0$ or 1. (Cf. the vector field in $\mathbb{R}^2$:

$$\frac{(|x|^{p-2}x, \nu|y+c|^{p-2}(y+c))}{|(|x|^{p-2}x, \nu|y+c|^{p-2}(y+c))|},$$

where c is a constant. It is easy to verify that this vector field is a closed vector field iff $p = 2$ and $\nu = 0$ or 1.)

Cosserat-Bingham model - Monotonicity

$$\frac{\mathrm{B}_{\nu,p}}{|\mathrm{B}_{\nu,p}|} = \frac{|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu|\mathrm{R}|^{p-2}\mathrm{R}}{\left||\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu|\mathrm{R}|^{p-2}\mathrm{R}\right|}$$

for $p = 2$ and $\nu = 0,\ 1$, i.e.,

$$\frac{\mathrm{B}_s}{|\mathrm{B}_s|}, \qquad \frac{\mathrm{B}_s + \mathrm{R}}{|\mathrm{B}_s + \mathrm{R}|} = \frac{\mathrm{B} - \Omega}{|\mathrm{B} - \Omega|}, \quad R = B_a - \Omega$$

are the gradients of the convex functions

$$W(X) = |X_s|, \qquad W(X) = |X - \Omega|.$$

In particular, the map

$$(\mathrm{B}_s, \mathrm{R}) \quad \mapsto \quad \frac{\mathrm{B}_{\nu,p}}{|\mathrm{B}_{\nu,p}|}$$

is a monotone operator! if $p = 2$ and $\nu = 0, 1$.

Question: Is this map a monotone operator for other values of p and ν?

Convexity, Monotonicity and Subdifferential - recollections

If a function $g : \mathbb{R} \to \mathbb{R}$ is the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ and f is convex and twice differentiable than g is monotone. Indeed, $g = f'$ and $f'' \geq 0$.

More generally, if a vector field G in $\mathbb{R}^m$ is the gradient of a convex and differentiable function then G is monotone, i.e.,
$(G(x) - G(y)) \cdot (x - y) \geq 0, \quad \forall\, x, y \in \mathbb{R}^m$. Indeed, a convex function $f : \mathbb{R}^m \to \mathbb{R}$ being differentiable at x is equivalent to

$$x^* \cdot (y - x) \leq f(y) - f(x), \quad \forall\, y \in \mathbb{R}^m,$$

for some $x^* \in \mathbb{R}^m$, which in the positive case[†] coincides with $\nabla f(x)$. Then if $G = \nabla f$ and f is convex and differentiable, we have that

$$\begin{aligned} (G(x) - G(y)) \cdot (x - y) &= \nabla f(x) \cdot (x - y) - \nabla f(y) \cdot (x - y) \\ &= -\nabla f(x) \cdot (y - x) - \nabla f(y) \cdot (x - y) \\ &\geq -(f(y) - f(x)) - (f(x) - f(y)) = 0. \end{aligned}$$

More generally yet, this same argument shows that if $G(x)$ is in/belongs to the subdiferential $\partial f(x)$ of a convex function f, for all x, then G is monotone.

[†]Otherwise, a vector x^* satisfying the above inequality is called a subgradient of f at x and the set of such vectors is called the subdifferential of f at x and denoted by $\partial f(x)$.

Cosserat-Bingham model - rewritting $\mathbb{S}$ and an answer to the **Question**

For $|\mathbb{S}| > \tau_*$, the tensor $\mathbb{S}$ can be written as

$$\mathbb{S} = \mathrm{B}_{\mu,p} + \tau_* \mathrm{P}(\mathrm{B}_s, \mathrm{R})$$

where $\mathrm{B}_{\mu,p} := \mu_1 |\mathrm{B}_s|^{p-2}\mathrm{B}_s + \mu_2 |\mathrm{R}|^{p-2}\mathrm{R}$ and

$$\mathrm{P}(\mathrm{B}_s, \mathrm{R}) := \frac{\mathrm{B}_{\nu,p}}{|\mathrm{B}_{\nu,p}|} = \frac{|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu |\mathrm{R}|^{p-2}\mathrm{R}}{\left||\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu |\mathrm{R}|^{p-2}\mathrm{R}\right|},$$

which is called "*plastic operator*". In the case $\mathbf{p = 2}$, i.e. $\mathrm{P} = \dfrac{\mathrm{B_s} + \nu \mathrm{R}}{|\mathrm{B_s} + \nu \mathrm{R}|}$, it was proved by **Shelukhin** that

P is monotone iff $\nu = \mathbf{0}$ or $\nu = \mathbf{1}$.

One more Remark: $\left|\mathrm{B}_{\nu,p}\right| = \sqrt{|\mathrm{B}_s|^{2(p-1)} + \nu^2 |\mathrm{R}|^{2(p-1)}}$. Thus, the plastic operator is

$$\mathrm{P} = \frac{|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu |\mathrm{R}|^{p-2}\mathrm{R}}{\sqrt{|\mathrm{B}_s|^{2(p-1)} + \nu^2 |\mathrm{R}|^{2(p-1)}}}.$$

Modified plastic operator

$$\mathcal{P} \equiv \mathcal{P}_{\nu,p}(B_s, R) = \frac{|\mathrm{B}_s|^{p-2}\mathrm{B}_s + \nu|\mathrm{R}|^{p-2}\mathrm{R}}{\sqrt[p]{(|\mathrm{B}_s|^p + \nu|\mathrm{R}|^p)^{p-1}}}.$$

The plastic operator versus the modified plastic operator:

They coincide for $p = 2$ and $\nu = 0, 1$.

The plastic operator P is the gradient of a convex function iff $p = 2$ and $\nu = 0, 1$, and, in the case $p = 2$ it is monotone iff $\nu = 0, 1$.

The modified plastic operator $\mathcal{P}$ is the gradient of a convex function (thus, it is monotone) for any values of $p \geq 1$ and $\nu \geq 0$!

Here, it is the potential for $\mathcal{P}$:

$$W(X) = \sqrt[p]{|X_s|^p + \nu|R|^p}.$$

Remark: $W(X) = \|(|X_s|, \sqrt[p]{\nu}|R|)\|_{\ell^p(\mathbb{R}^2)}$.

Our family of systems

$$\begin{cases} \mathbb{S} = \mathrm{B}_{\mu,p} + \widehat{\tau}_* \mathcal{P}_{\nu,q}(B_s, R), & \text{if } \mathrm{B}_\nu \neq 0 \\ |\mathbb{S}| \leqslant \tau_*, \quad \text{if } \mathrm{B}_\nu = 0, \end{cases} \tag{2}$$

with $\mathcal{P}_{\nu,q}(B_s, R) = \frac{\mathrm{B}_{\nu,q}}{|\widehat{\mathrm{B}_{\nu,q}}|^{\frac{2(q-1)}{q}}}$, exponents $p, q \geq 2$,

$$\widehat{\tau}_* = \frac{\tau_*}{\max(1, \sqrt[q]{\nu})}, \qquad \tau_*, \nu \geq 0, \tag{3}$$

$$\mathrm{B}_{\mu,p} = \mu_1 |\mathrm{B}_s|^{p-2} \mathrm{B}_s + \mu_2 |\mathrm{R}|^{p-2} \mathrm{R}, \tag{4}$$

$\mu = (\mu_1, \mu_2), \quad \mu_1 > 0, \quad \mu_2 \geq 0,$

$$\mathrm{B}_{\nu,q} = |\mathrm{B}_s|^{q-2} \mathrm{B}_s + \nu |\mathrm{R}|^{q-2} \mathrm{R} \tag{5}$$

and

$$\widehat{\mathrm{B}_{\nu,q}} = |\mathrm{B}_s|^{\frac{q-2}{2}} \mathrm{B}_s + \sqrt{\nu} |\mathrm{R}|^{\frac{q-2}{2}} \mathrm{R}. \tag{6}$$

Potential

The tensor given by (**??**) can be achieved by the condition $\mathbb{S}(x,t) \in \partial V(\mathrm{B}(x,t))$, if $\nu > 0$ or $\nu = \mu_2 = 0$, for the potential

$$V \equiv V_{(x,t)} = U(X) + \widehat{\tau}_* W(X) \tag{7}$$

with

$$\begin{aligned} U(X) &= \tfrac{\mu_1}{p}|X_s|^p + \tfrac{\mu_2}{p}|\mathrm{R}|^p, \\ W(X) &= \left| |\mathrm{X}_s|^{\frac{q-2}{2}}\mathrm{X}_s + \sqrt{\nu}|\mathrm{R}|^{\frac{q-2}{2}}\mathrm{R} \right|^{\frac{2}{q}} \\ &= \sqrt[q]{|\mathrm{X}_s|^q + \nu|\mathrm{R}|^q}, \end{aligned} \tag{8}$$

where $\mathrm{R} = X_a - \Omega$.

An ibvp

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary Γ, $T > 0$ and $\mathbb{S}$ given by (**??**). We consider the ibvp

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \operatorname{div}\,\mathbb{T}, \quad \operatorname{div}\mathbf{v} = 0 & \text{in} \quad \mathcal{O}_T = \mathcal{O}\times(0,T) \\ \mathbf{v}\cdot\mathbf{n} = 0, \quad (\mathbb{T}\,\mathbf{n} + \nabla g(\mathbf{v}))\cdot\boldsymbol{\tau} = 0 & \text{on} \quad \Gamma_T = \Gamma\times(0,T) \\ \mathbf{v} = \mathbf{v}_0 & \text{on} \quad \mathcal{O}\times\{t=0\}, \end{cases} \tag{9}$$

where in the boundary condition $(\mathbb{T}\,\mathbf{n} + \nabla g(\mathbf{v}))\cdot\boldsymbol{\tau} = 0$, $g = g_{(x,t)}(\mathbf{v})$ is a Caratheodory function $g : \Gamma_T \times \mathbb{R}^3 \to \mathbb{R}$, being differentiable and convex with respect to $\mathbf{v}$. We assume that g satisfies the non-negativity and boundedness conditions:

$$0 \le \nabla g_{(x,t)}(\mathbf{v})\cdot\mathbf{v}, \qquad |\nabla g_{(x,t)}(\mathbf{v})| \le \mathrm{c}|\mathbf{v}|, \tag{10}$$

for all $\mathbf{v}\in\mathbb{R}^3$ and $(x,t)\in\Gamma_T$, where c is a positive constant, independent of $\mathbf{v}$ and (x,t). The particular case $g = \frac{1}{2}\alpha|\mathbf{v}|^2$ with $\alpha : \Gamma_T \to \mathbb{R}$ being a non-negative function in $L^\infty(\Gamma_T)$, satisfies (**??**) and yields the Navier (friction/slip) boundary condition $(\mathbb{T}\,\mathbf{n} + \alpha\mathbf{v})\cdot\boldsymbol{\tau} = 0$.

Our main results

Theorem

(Characterization of the subdifferential of the potential V). *Let $p, q \geqslant 2$. Then, the potential V defined in (**??**)-(**??**) is convex, differentiable at any $X \in \mathbb{R}^{d\times d}$ such that $X_\nu \neq 0$, with*

$$\nabla V(X) = X_{\mu,p} + \widehat{\tau}_* |\widehat{X_{\nu,q}}|^{-\frac{2(q-1)}{q}} X_{\nu,q},$$

*where $X_\nu = X_s + \nu R$, $R = X_a - \Omega$, and $X_{\mu,p}$, $X_{\nu,q}$ $\widehat{X_{\nu,q}}$ are defined by (**??**)-(**??**), replacing B by X. Moreover:*

(a) $B_{r_*}(0) \subset \partial V(\Omega) \subset B_{\tau_*}(0)$, where $B_r(0)$ is the closed ball in $\mathbb{R}^{d\times d} \equiv \mathbb{R}^{d^2}$ of radius r and center X, and

$$r_* = \widehat{\tau}_* \cdot r_q, \quad r_q = \begin{cases} \sqrt[q]{\nu}/(1+\nu^{\frac{1}{q-2}})^{\frac{q-2}{2q}}, & \text{if } q > 2 \\ \min(1, \sqrt{\nu}), & \text{if } q = 2; \end{cases}$$

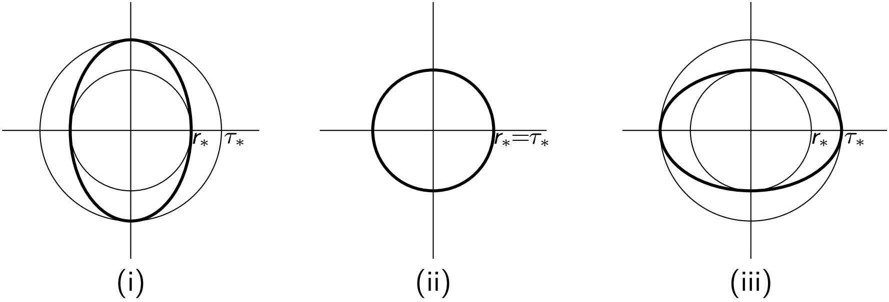
(b) The subdifferential $\partial V(\Omega)$ has an "ellipsoidal form", i.e.,

$$\partial V(\Omega) = \{X^* \in \mathbb{R}^{d\times d} : \ |X_s^*|^{q'} + \nu^{1-q'}|X_a^*|^{q'} \leqslant (\widehat{\tau}_*)^{q'}\}.$$

(c) If $\nu = 0$, the subdifferential at a matrix X such that $\mathrm{X}_\nu = 0$ (i.e., $\mathrm{X}_s = 0$ but X_a arbitrary) is given by

$$\partial V(X) = \mu_2 |\mathrm{R}|^{p-2}\mathrm{R} + \{X^* \in \mathbb{R}^{d\times d} : \ X_a^* = 0 \text{ and } |X_s^*| \leqslant \tau_*\}.$$

An illustration of the subdifferential $\partial V(\Omega)$



In the above picture we have the graphic in a cartesian plane xy of the the set (with the boundary in bold) given by the inequality

$$x^2 + \nu^{-1} y^2 \leqslant (\widehat{\tau}_*)^2,$$

which is obtained from (**??**) *with* $q = 2$ *by setting* $x = |X_s^*|$ *and* $y = |X_a^*|$, *together with the graphics of the balls* $x^2 + y^2 \leqslant r_*^2$, $x^2 + y^2 \leqslant \tau_*^2$. *We consider the three cases: (i)* $\nu > 1$; *(ii)* $\nu = 1$ $(r_* = \tau_*)$; *(iii)* $\nu < 1$. *Notice that when* $q = 2$, $\quad r_* = \widehat{\tau}_* \cdot r_2 = \tau_* \cdot \min(1, \nu)/\max(1, \nu)$.

Theorem 2

Function spaces:

$H = \{\mathbf{v} \in L^2(\mathcal{O}) : \text{ div } \mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\mathcal{O}), \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{in } H^{-1/2}(\Gamma)\},$
$V = \{\mathbf{v} \in H^1(\mathcal{O}) : \text{ div } \mathbf{v} = 0 \quad \text{a.e. in } \mathcal{O}, \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{in } H^{1/2}(\Gamma)\},$
$V_p = \{\mathbf{v} \in V : \quad |\nabla \mathbf{v}| \in L^p(\mathcal{O})\}.$

Theorem

*Let V be the potential (function) given in (**??**)-(**??**), being $\Omega \equiv \Omega(x,t)$ a matrix function in $L^p(\mathcal{O}_T; \mathbb{R}^{d\times d})$, and g as above. Then for any $\mathbf{v}_0 \in H$, $p \geqslant 2.2$ and $q \geqslant 2$, there exists a pair $(\mathbf{v}, \mathbb{S})$ of functions with $\mathbb{S}(x,t) \in \partial V_{(x,t)}(B(x,t))$ for almost all $(x,t) \in \mathcal{O}_T$, such that*

$$\mathbf{v} \in L^\infty(0,T;H) \cap L^p(0,T;V_p), \quad \mathbf{v}_t \in L^{p'}(0,T;V_p^*) \quad \textit{and} \quad \mathbb{S} \in L^{p'}(\mathcal{O}_T). \tag{11}$$

The pair $(\mathbf{v},\mathbb{S})$ satisfies the integral equality

$$\int_{\mathcal{O}_T} [\mathbf{v}\cdot\partial_t\varphi + (\mathbf{v}\otimes\mathbf{v} - \mathbb{S}) : \nabla\varphi]\, dxdt + \int_{\mathcal{O}} \mathbf{v}_0\cdot\varphi(x,0)\, dx = \int_{\Gamma_T} \nabla g(\mathbf{v})\cdot\varphi\, d\Gamma dt \tag{12}$$

for any function $\varphi \in C^1(\overline{\mathcal{O}_T})$ such that $\varphi(\cdot,T) = 0$ and $(\varphi\cdot\mathbf{n})|\Gamma_T = 0$.

Lemma: approximation of the potential and estimates

Lemma

Let Ω be any fixed $d \times d$ antisymmetric real matrix. For given $p, q \geqslant 2$, let us consider the convex potential (**??**)-(**??**) *and the approximations*

$$
\begin{aligned}
V^n(X) \;=\; & U(X) + \widehat{\tau}_* W^n(X), \\
& U(X) = \tfrac{\mu_1}{p}|X_s|^p + \tfrac{\mu_2}{p}|\mathrm{R}|^p \qquad \text{(see (??))}, \\
& W^n(X) = \sqrt[q]{|\widehat{\mathrm{X}_{\nu,q}}|^2 + n^{-1}} = \sqrt[q]{|\mathrm{X}_s|^q + \nu|\mathrm{R}|^q + n^{-1}}
\end{aligned}
$$

$$
\mathbb{S}^n = \mathrm{X}_{\mu,p} + \widehat{\tau}_* \frac{\mathrm{X}_{\nu,q}}{\sqrt[q]{(|\widehat{\mathrm{X}_{\nu,q}}|^2 + n^{-1})^{q-1}}}, \tag{13}
$$

$X \in \mathbb{R}^{d\times d}$, $n \in \mathbb{N}$. *Here,* $\mathrm{R} := \mathrm{X}_a - \Omega$ *and* $\mathrm{X}_{\mu,p}$, $\mathrm{X}_{\nu,q}$, $\widehat{\mathrm{X}_{\nu,q}}$ *are defined by* (**??**), (**??**), (**??**), *replacing* B *by* X. *Then the following statements are true.*

Lemma continued

(a) *Both the potentials V and V^n satisfy the estimates*

$$\mu_1|X_s|^p - \mu_2 2^{p-2}|\Omega|^p - \tau_*|\Omega| \;\leq\; V'(X;X) \;\leq\; c_1|X|^p + c_2|\Omega|^p + \tau_*|X| \tag{14}$$

for any matrix $X \in \mathbb{R}^d \times \mathbb{R}^d$, for any $n \in \mathbb{N}$ (we can replace V by V^n in (**??**)*), where $c_1 = \mu_1 + 2^{p-2}\mu_2(1+\frac{1}{p})$ and $c_2 = 2^{p-2}\mu_2(1-\frac{1}{p})$.*
(b) *For any given $X \in \mathbb{R}^{d\times d}$, we have that $\mathbb{S}^n = \nabla V^n(\mathrm{X})$. Since V^n is differentiable and convex, this statement is equivalent to the variational inequality $V^n(Y) - V^n(\mathrm{X}) \geqslant \mathbb{S}^n : (Y - \mathrm{X})$, for all $Y \in \mathbb{R}^{d\times d}$. Moreover,*

$$|\mathbb{S}^n| \leq \mu_1|X_s|^{p-1} + \mu_2|R|^{p-1} + \tau_*, \qquad \forall\, n,\ \forall\, X \in \mathbb{R}^{d\times d}. \tag{15}$$

Main auxiliary results

On convex functions:

A convex $f\colon \mathbb{R}^m \to \mathbb{R}$ function has one-sided directional derivative

$$f'(x;y) := \lim_{\lambda \to 0+} \frac{f(x+\lambda y) - f(x)}{\lambda} = \inf_{\lambda > 0} \frac{f(x+\lambda y) - f(x)}{\lambda}, \tag{16}$$

for all $x, y \in \mathbb{R}^m$.

If $f\colon \mathbb{R}^m \to \mathbb{R}$ is positively homogeneous of order 1, then $f'(0;y) = f(y)$, for any $y \in \mathbb{R}^m$.

Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a convex function. Then x^* is a subgradient of f at x if and only if

$$f'(x;y) \geqslant x^* \cdot y, \quad \forall y \in \mathbb{R}^m.$$

The ℓ^p norm in $\mathbb{R}^m$,

$$\|x\|_{\ell^p(\mathbb{R}^m)} = \sqrt[p]{|x_1|^p + \ldots + |x_m|^p}$$

is decreasing with respect to $p \in [1, \infty]$.

Main auxiliary results - continued

The norms

$$\|\mathbf{v}\|_{V_p} = \|\mathbf{v}\|_{L^2(\mathcal{O})} + \|\nabla \mathbf{v}\|_{L_p(\mathcal{O})}, \quad \|\mathbf{v}\|_{W^{1,p}(\mathcal{O})} = \|\mathbf{v}\|_{L^p(\mathcal{O})} + \|\nabla \mathbf{v}\|_{L^p(\mathcal{O})}$$

are equivalents for $p \geq 2$.

We have the following version of *Korn's inequality*:

For any $p \in [2, \infty)$ there exists a constant C such that

$$\|\nabla \mathbf{v}\|_{L^p(\mathcal{O})} \leqslant C(\|\mathbf{v}\|_{L^2(\mathcal{O})} + \|(\nabla \mathbf{v})_s\|_{L^p(\mathcal{O})}), \quad \forall\ \mathbf{v} \in W^{1,p}(\mathcal{O}).$$

Well-known interpolation inequality:

Let $s_1, s_2, r \in [1, +\infty]$ and $\theta_1, \theta_2 \in [0,1]$ such that $\theta_1 + \theta_2 = 1, \quad \frac{\theta_1}{s_1} + \frac{\theta_2}{s_2} = \frac{1}{r}$.
Then

$$\|\mathbf{v}\|_{L^r(\mathcal{O})} \leqslant \|\mathbf{v}\|^{\theta_1}_{L^{s_1}(\mathcal{O})} \|\mathbf{v}\|^{\theta_2}_{L^{s_2}(\mathcal{O})}, \qquad \forall \mathbf{v} \in L^{s_1}(\mathcal{O}) \cap L^{s_2}(\mathcal{O}).$$

"Sobolev imbbeding":

There exists a positive constant C, such that

$$\|\mathbf{v}\|_{L^r(\mathcal{O})} \leqslant C\|\mathbf{v}\|_{V_p}, \quad \forall \mathbf{v} \in V_p,$$

with $r = dp/(d-p)$, if $1 \leq p < d$; any $r < \infty$, if $p = d$; and $r = \infty$, if $p > d$.

Main auxiliary results - continued

The lower semicontinuity in the weak topology of functionals of the form

$$\mathcal{F}(u) = \int F(x, u(x))dx.$$

(see: E. Giusti, *Direct Methods in the Calculus of Variations*, 2003.)

The Aubin-Lions-Simon compactness result:

Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0, X_1 are reflexive, X_0 is compactly embedded in X and that X is continuously embedded in X_1. Let

$$\mathcal{Z} = \left\{ v \in L^2(0, T; X_0), \qquad \partial_t v \in L^{p'}(0, T; X_1) \right\}.$$

Then the embedding of $\mathcal{Z}$ into $L^2(0, T; X)$ is compact.

Outline of the proof of Theorem 2 - on the solution of our ibvp (**??**)

Approximation:

$$\left\{ \begin{array}{c} \mathbf{v}_t^n + (\mathbf{v}^n \cdot \nabla)\, \mathbf{v}^n = \operatorname{div}\, \mathbb{T}^n, \qquad \operatorname{div} \mathbf{v}^n = 0, \qquad \text{in} \quad \mathcal{O}_T, \\ \mathbf{v}^n \cdot \mathbf{n} = 0, \qquad (\mathbb{T}^n\, \mathbf{n} + \nabla g(\mathbf{v}^n)) \cdot \boldsymbol{\tau} = 0 \qquad \text{on} \quad \Gamma_T, \\ \mathbf{v}^n|_{t=0} = \mathbf{v}_0^n \qquad \text{in} \quad \mathcal{O}, \end{array} \right. \tag{17}$$

where $\mathbb{T}^n = -p^n\, \mathrm{I} + \mathbb{S}^n$.

''Energy equation'':

$$\frac{1}{2} \int_{\mathcal{O}} |\mathbf{v}^n(x,t)|^2 dx + \int_{\mathcal{O}_t} \mathbb{S}^n : \nabla \mathbf{v}^n dx\, dt + \int_{\Gamma_t} \nabla(g(\mathbf{v}^n) \cdot \boldsymbol{\tau})(\mathbf{v}^n \cdot \boldsymbol{\tau})\, d\Gamma\, dt = \frac{1}{2} \int_{\mathcal{O}} |\mathbf{v}_0^n|^2 dx. \tag{18}$$

The estimate (**??**) for V^n and the condition $\nabla g(\mathbf{v})\cdot\mathbf{v}\geq 0$ for every $\mathbf{v}\in\mathbb{R}^3$ imply from (**??**)

$$\frac{1}{2}\int_{\mathcal{O}}|\mathbf{v}^n|^2dx+\mu_1\int_0^t\int_{\mathcal{O}}|\mathrm{B}_s^n|^p dx\,dt\leq C+\frac{1}{2}\int_{\mathcal{O}}|\mathbf{v}_0^n|^2dx,$$

where $C=\int_{\mathcal{O}_T}[\,\mu_2 2^{p-2}|\Omega|^p-\tau_*|\Omega|\,]\,dx\,dt$. Then using the Korn's inequality, we deduce the estimates

$$\begin{gathered}\|\mathbf{v}^n\|_{L^\infty(0,T;H)}+\|\mathbf{v}^n\|_{L^p(0,T;V_p)}\leqslant C,\\ \|\mathbb{S}^n\|_{L^{p'}(\mathcal{O}_T)}\leqslant C,\end{gathered}\tag{19}$$

Then, with the help of the continuity of the trace map from $V_p \equiv W^{1,p}(\mathcal{O})$ to $L^p(\Gamma)$, the Aubin-Lions-Simon compactness result and the boundedness condition in (**??**) for the gradient of the boundary function g, we have the following convergences (up to subsequences):

$$\begin{array}{ccccl} \mathbf{v}^n & \rightharpoonup & \mathbf{v} & \text{weakly-}* \text{ in} & L^\infty(0,T;H), \\ \mathbf{v}^n & \rightharpoonup & \mathbf{v} & \text{weakly in} & L^p(0,T;V_p)\cap L^p(\Gamma_T), \\ \mathbb{S}^n & \rightharpoonup & \mathbb{S} & \text{weakly in} & L^{p'}(\mathcal{O}_T), \\ \nabla g(\mathbf{v}^n) & \rightharpoonup & \mathbf{g} & \text{weakly in} & L^p(\Gamma_T), \end{array} \tag{20}$$

for some $\mathbb{S} \in L^{p'}(\mathcal{O}_T) \equiv L^{p'}(\mathcal{O}_T;\mathbb{R}^{3\times 3})$, $\mathbf{g} \in L^p(\Gamma_T) \equiv L^p(\Gamma_T;\mathbb{R}^3)$, and

$$\mathbf{v}^n \to \mathbf{v} \quad \text{strongly in} \quad L^2(\mathcal{O}_T). \tag{21}$$

Thus, we can take $n \to \infty$ in the weak formulation of (**??**) and obtain (**??**) with $\mathbf{g}$ in place of $\nabla g(\mathbf{v})$.
At this point, it will remain to show only that $\mathbb{S} \in \partial V(\mathrm{B})$ and $\mathbf{g} = \nabla g(\mathbf{v})$.

Lemma

$p \geq 2.2 \Rightarrow$

$$\partial_t \mathbf{v} \in L^{p'}(0, T;\ V_p^*). \tag{22}$$