VII taller de análisis no lineal y ecuaciones diferenciales parciales

28 al 31 de mayo 2025 - UTP

On the Navier-Stokes equations for stationary incompressible fluids with a discontinuous density and the Ladyzhenskaya-Solonnikov problem for the power model

Overview

- Discontinuous density in a bounded domain of the plane
- Variable viscosity
- Leray problem
- Ladyzhenskaya-Solonnikov problem

The stationary Navier-Stokes equations for incompressible inhomogeneous fluids

$$\begin{cases} -\nu\Delta\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = \rho\mathbf{f} \\ \mathrm{div}(\mathbf{v}) = 0, \qquad \mathrm{div}(\rho\mathbf{v}) = 0 \end{cases} \tag{1}$$

$$\left.\begin{array}{lll} \rho & : & \text{density} \\ \mathbf{v} & : & \text{velocity} \\ (\mathcal{P} & : & \text{pressure}) \end{array}\right\} \quad \text{the unknowns}$$

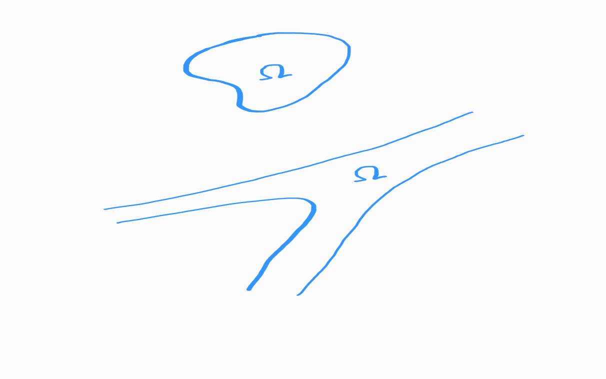
$\mathbf{f}$: external force

$\nu > 0$: constant viscosity

$$\mathbf{v} = (v_1, \cdots, v_n), \qquad (\mathbf{v}\cdot\nabla)\mathbf{v} = (\mathbf{v}\cdot\nabla v_1, \cdots, \mathbf{v}\cdot\nabla v_n)$$

Domains

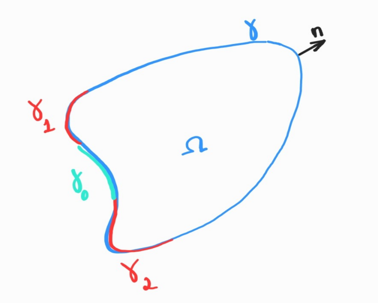
We consider these equations (system (1)) in a bounded domain and in an unbounded domain (a domain with an unbounded boundary)



The bounded domain

The bounded domain we consider only in 2D (in the plane) with the boundary being a Jordan curve γ (i.e., a smooth simple closed curve).

We take three successives arcs on γ, γ_1, γ_0, γ_2, denote by $\mathbf{n}$ the unit outward normal on γ



and pose the following boundary value problem:

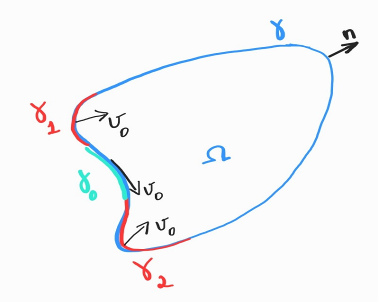
Boundary value problem (bvp) in the bounded domain

$$\left\{\begin{array}{c} \left.\begin{array}{c} -\nu\Delta\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = \rho\mathbf{f} \\ \mathrm{div}(\mathbf{v}) = 0, \qquad \mathrm{div}(\rho\mathbf{v}) = 0 \end{array}\right\} \text{ in } \Omega \\ \mathbf{v}|\gamma = \mathbf{v}_0, \qquad \rho|\gamma_i = \rho_i, \quad i = 1,2 \end{array}\right. \tag{2}$$

where $\mathbf{v}_0$ or the set $\{\gamma_1, \gamma_0, \gamma_2\}$ is such that

$$\gamma_1 \cup \gamma_2 = \{x \in \gamma\,;\, \mathbf{v}_0 \cdot \mathbf{n} < 0\},$$

and $\mathbf{v}_0 \cdot \mathbf{n} = 0$ on



$\gamma_1 \cup \gamma_2$ is the part of γ where the fluid is incoming.

(Compatibility condition: $\int_\gamma \mathbf{v}_0 \cdot \mathbf{n} = 0$.)

Theorem (–, ZAMP, 2002)

If $\rho_i \in L^\infty(\gamma_i)$, $i = 1, 2$, $\mathbf{f} \in L^2(\Omega)$ *and* $\mathbf{v}_0 \in H^{1/2}(\gamma)$ *satisfies* $(\mathbf{v}_0 \cdot \mathbf{n})|\gamma_i < 0$, $i = 1, 2$, $(\mathbf{v}_0 \cdot \mathbf{n})|\gamma_0 = 0$ *and* $\int_\gamma \mathbf{v}_0 \cdot \mathbf{n} = 0$, *then problem* (2) *has a* weak solution* $(\rho, \mathbf{v})$ *in* $L^\infty(\Omega) \times H^1(\Omega)$.

*

Definition

A pair $(\rho, \mathbf{v}) \in L^\infty(\Omega) \times H^1(\Omega)$ is said to be a *weak solution* of problem (1) if $\operatorname{div}(\mathbf{v}) = 0$ in $H^1(\Omega)$ (i.e. $\int_\Omega \mathbf{v} \cdot \nabla\theta \, dx = 0$ for all $\theta \in C_0^\infty(\Omega)$), $\mathbf{v}|\gamma = \mathbf{v}_0$ where $\mathbf{v}|\gamma$ is the trace of $\mathbf{v}$ on γ,

$$\int_\Omega \rho\mathbf{v} \cdot \nabla\varphi dx = \int_{\gamma_i} \rho_i(\mathbf{v}_0 \cdot \mathbf{n})\varphi ds, \tag{3}$$

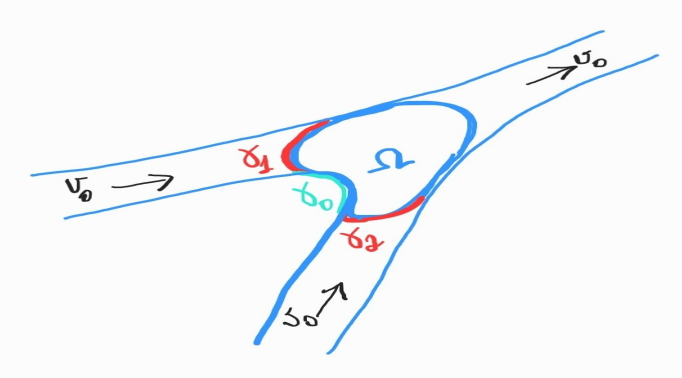
for all $\varphi \in H^1(\Omega)$ such that $\varphi|(\gamma/\gamma_i) = 0$, $i = 1, 2$, and

$$\nu \int_\Omega \nabla v_j \cdot \nabla\Phi_j dx - \int_\Omega \rho v_j \mathbf{v} \cdot \frac{\partial\Phi}{\partial x_j} dx = \int_\Omega \rho\mathbf{f} \cdot \Phi dx, \tag{4}$$

for all $\Phi = (\Phi_1, \Phi_2) \in C_0^\infty(\Omega; \mathbb{R}^2)$ such that $\operatorname{div}(\Phi) = 0$.

Physical motivation

The meeting of fluids, e.g., the junction of rivers with different densities



On the proof

N.N. Frolov (in Mat. Zametki, 1993) solved the problem (2) in the case that

$$\gamma_1 = \{x \in \gamma\, ;\, \mathbf{v}_0 \cdot \mathbf{n} < 0\},$$

$(\gamma_2 = \emptyset)$ and ρ_1 is a Hölder continuous function.

In particular, he proved the existence of a solution $(\rho, \mathbf{v})$ in $C^\alpha(\Omega) \times H^2(\Omega)$, if $\mathbf{v}_0 \in C^2(\gamma)$, $\rho_1 \in C^\alpha(\gamma_1)$ and $\mathbf{f} \in L^2(\Omega)$ $(\alpha > 0)$.

Frolov method

- Since $\text{div}(\mathbf{v}) = 0$ and Ω is simply connected, we can write

$$\mathbf{v} = \nabla^{\perp}\psi := (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})$$

for some scalar function ψ (*stream function*). Frolov looks for solutions with ρ in the form $\rho = \omega(\psi) \equiv \omega \circ \psi$, for a smooth function $\omega : \mathbb{R} \to \mathbb{R}$. Note that such a ρ automatically satisfies the equation $\text{div}(\rho\mathbf{v}) = 0$.
$(\text{div}(\rho\mathbf{v}) = \nabla\rho \cdot \mathbf{v} = \omega'(\psi)\nabla\psi \cdot \nabla^{\perp}\psi = 0)$

- The boundary condition $\rho|\gamma_i = \rho_i$ imposes the following restriction on ω:

$$\omega(\psi(x)) = \rho_i(x), \quad x \in \gamma_i. \tag{5}$$

The bvp for ψ

If $(\rho, \mathbf{v})$ is smooth, we may take the curl of the first equation in (1) and easily see that the problem (1) turns into the following boundary value problem for the unknown ψ:

$$\begin{cases} \nu\Delta^2\psi = -\mathrm{div}(\omega(\psi)(\nabla^\perp\psi \cdot \nabla)\nabla\psi) + \mathrm{div}(\omega(\psi)\mathbf{f}^\perp) \text{ in } \Omega, \\ \frac{\partial\psi}{\partial\tau} = -\mathbf{v}_0 \cdot \mathbf{n}, \quad \frac{\partial\psi}{\partial\mathbf{n}} = \mathbf{v}_0 \cdot \tau \qquad \text{on } \gamma, \end{cases} \tag{6}$$

where $\mathbf{f}^\perp = (-f_2, f_1)$, $\mathbf{f} = (f_1, f_2)$, $\tau = \mathbf{n}^\perp$.

Notice that $\mathrm{curl}\,\mathbf{f} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = -\mathrm{div}\,\mathbf{f}^\perp$, $\mathrm{curl}\nabla^\perp\psi = \Delta\psi$, and $((\mathbf{v} \cdot \nabla)\mathbf{v})^\perp = (\mathbf{v} \cdot \nabla)\mathbf{v}^\perp$.

Solving the bvp (6)

The idea is to use the *Leray-Schauder fixed point theorem*[*]:

> *Let V be a Banach space, and let $\mathcal{A} : [0,1] \times V \to V$ be a map such that $\mathcal{A}(0,\cdot)$ is a constant map. Suppose that*
>
> *i. $\mathcal{A}$ is continuous and compact, and*
>
> *ii. the set $\mathcal{S} = \{v \in V : \mathcal{A}(\mu, v) = v$ for some $\mu \in [0,1]\}$ is bounded.*
>
> *Then the map $\mathcal{A}(1,\cdot)$ has a fixed point.*

[*] M.E. Taylor, *Partial Differential Equations, vol. III*

Solving the bvp (6) – the extension of boundary values

First we define

$$\psi_0 = -\int_\gamma \mathbf{v}_0 \cdot \mathbf{n} \tag{7}$$

and fix $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\omega(\psi_0(x)) = \rho_i(x), \quad \text{for } x \in \gamma_i.$$

Second, by a converse of the *trace theorem*[†], we can take a function $\widetilde{\Phi_0}$ defined in $\overline{\Omega}$ such that

$$\widetilde{\Phi_0}|\gamma = \psi_0, \quad \frac{\partial \widetilde{\Phi_0}}{\partial \mathbf{n}}|\gamma = \mathbf{v}_0 \cdot \tau \tag{8}$$

and

$$||\widetilde{\Phi_0}||_{H^2(\Omega)} \leq c(||\psi_0||_{H^{3/2}(\gamma)} + ||\mathbf{v}_0||_{H^{1/2}(\gamma)}), \tag{9}$$

where c is a constant depending only on Ω.

[†]see e.g. Theorem 3.4, p. 48 in G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, v. I, Springer (1994)

Solving the bvp (6) – the extension of boundary values, continued

Finally, we take the following family of extensions of ψ_0 to Ω, concentrating the support of $\widetilde{\Phi_0}$ and controlling its "slope" near the boundary γ:

$$\widetilde{\psi_{0,\delta}} := \widetilde{\Phi_0}\zeta_\delta, \quad 0 < \delta \ll 1$$

that is, ζ_δ is a smooth function such that $\zeta_\delta = 1$ near γ, zero at all points of Ω with distance from γ bigger than δ and

$$|\zeta_\delta(x)| \le c, \quad |\nabla\zeta_\delta(x)| \le c/\delta, \tag{10}$$

where c is a constant independent of x an

The arbitrariness of $0 < \delta \ll 1$ above is important!

Solving the bvp (6) – the homogenization of boundary values and the variational problem

Let us consider a solution of (6) given by $\psi = \varphi + \widetilde{\psi_{0,\delta}}$, where φ is now the unknown we seek in the space $H_0^2(\Omega)$.

Multiplying the equations in (6) by test functions $\theta \in H_0^2(\Omega)$ and integrating by parts we see that φ satisfies the following variational problem:

$$\begin{aligned} & \nu \int_\Omega (\Delta\varphi)(\Delta\theta)\,dx \\ = \; & \int_\Omega \omega(\varphi + \widetilde{\psi_{0,\delta}}) \left(\nabla^\perp(\varphi + \widetilde{\psi_{0,\delta}})\right)_j \left(\nabla^\perp(\varphi + \widetilde{\psi_{0,\delta}})\right) \cdot \frac{\partial}{\partial x_j}(\nabla^\perp\theta)\,dx \\ & + \int_\Omega \omega(\varphi + \widetilde{\psi_{0,\delta}})\,\mathbf{f} \cdot (\nabla^\perp\theta)\,dx - \nu \int_\Omega (\Delta\widetilde{\psi_{0,\delta}})(\Delta\theta)\,dx. \end{aligned} \tag{11}$$

Solving the bvp (6) – the solution as a fixed point of the map associated with the right hand side of (11)

$$\mathcal{A} : [0, \nu^{-1}] \times H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$$

$$(\mathcal{A}(\mu, \varphi), \theta) :=$$
$$\mu \int_\Omega \omega(\mu\nu\widetilde{\psi}_{0,\delta} + \varphi) \left(\nabla^\perp(\mu\nu\widetilde{\psi}_{0,\delta} + \varphi)\right)_j \nabla^\perp(\mu\nu\widetilde{\psi}_{0,\delta} + \varphi) \cdot \frac{\partial}{\partial x_j}(\nabla^\perp\theta)$$
$$+ \ \mu \int_\Omega \omega(\mu\nu\widetilde{\psi}_{0,\delta} + \varphi)\, \mathbf{f} \cdot (\nabla^\perp\theta)\, dx - \mu\nu \int_\Omega (\Delta\widetilde{\psi}_{0,\delta})(\Delta\theta)\, dx, \tag{12}$$

where we endowed $H_0^2(\Omega)$ with the inner product

$$(\varphi, \theta) := \int_\Omega (\Delta\varphi)(\Delta\theta) dx.$$

A fixed point of $\mathcal{A}(\nu^{-1}, \cdot)$ is a solution of (11).
The existence of a fixed point of $\mathcal{A}(\nu^{-1}, \cdot)$ is guaranteed by the Leray-Schauder fixed point theorem because the two following claims hold true.

i. *$\mathcal{A}$ is continuous and compact.*

ii. *The set*
$\mathcal{S} := \{\varphi \in H_0^2(\Omega) \,:\, \mathcal{A}(\mu, \varphi) = \varphi$ for some $\mu \in [0, \nu^{-1}]\}$ is a bounded set in $H_0^2(\Omega)$.

The proof of claim i. uses the Hölder continuity of Ω. In fact, assuming $\omega \in C^{\alpha}(\mathbb{R})$, Frolov (in Mat. Zametki, 1993) obtains the estimates

$$||\mathcal{A}(\mu_1, \varphi_1) - \mathcal{A}(\mu_1, \varphi_2)||_{H^2} \leq c||\varphi_1 - \varphi_2||_{W^{1,4}}^{\alpha} \tag{13}$$

and

$$||\mathcal{A}(\mu_1, \varphi_1) - \mathcal{A}(\mu_2, \varphi_1)||_{H^2} \leq c|\mu_1 - \mu_2|^{\alpha} \tag{14}$$

for all φ_1, φ_2 in an arbitrary ball B of $H^2(\Omega)$ and $\mu_1, \mu_2 \in [0, \nu^{-1}]$, where c is a constant that depends only on ω, δ, B and Sobolev imbeddings. From (13) and (14) it follows claim i.

On the proof of claim ii.

For claim ii, it suffices that ω be continuous and bounded.
The arbitrariness of δ is used here!
In fact, assuming that the set $\mathcal{S}$ is not bounded for some $0 < \delta_1 \ll 1$, following an argument in the book by Ladyzhenskaya[‡], it is possible to find some limit function $\mathbf{z} \in H_0^2(\Omega)$ independent of $\delta \in (0, \delta_1]$ such that

$$\nu \leq \begin{array}{l} c||\omega||_{L^\infty(\mathbb{R})}||\widetilde{\Phi_0}||_{L^\infty(\Omega)}\left(\delta^{-1}||\mathbf{z}||_{L^2(\Omega_\delta)}\right) \\ +c||\omega||_{L^\infty(\mathbb{R})}||\nabla^\perp\widetilde{\Phi_0}||_{L^4(\Omega_\delta)}||\mathbf{z}||_{L^4(\Omega_\delta)}, \end{array} \tag{15}$$

for all $\delta \in (0, \delta_1]$. This leads to a contradiction when letting $\delta \to 0$, since $\nu > 0$ and, by a Poincaré type inequality, we have that $\delta^{-1}||\mathbf{z}||_{L^2(\Omega_\delta)} \leq c_1||\mathbf{z}||_{H^1(\Omega_\delta)}$, where c_1 is some constant independent of δ. This ends the proof of claim ii.

[‡] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, (1969), p. 123

Frolov's theorem (Mat. Zametki, 1993)

Given any bounded and Hölder continuous function $\omega : \mathbb{R} \to \mathbb{R}$*, if* $\mathbf{f} \in L^2(\Omega)$ *and* $\mathbf{v}_0 \in C^2(\gamma)$ *then, for each* $0 < \delta << 1$*, the bvp* (6)*, i.e.,*

$$\begin{cases} \nu \Delta^2 \psi = -div(\omega(\psi)(\nabla^{\perp}\psi \cdot \nabla)\nabla\psi) + div(\omega(\psi)\mathbf{f}^{\perp}) \text{ in } \Omega, \\ \frac{\partial \psi}{\partial \tau} = -\mathbf{v}_0 \cdot \mathbf{n}, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \mathbf{v}_0 \cdot \tau \qquad \text{on } \gamma, \end{cases}$$

has a weak solution ψ *of the type* $\psi = \varphi + \widetilde{\psi_{0,\delta}}$*, with* $\varphi \in H_0^2(\Omega)$.

On the proof of our theorem (in ZAMP, 2002)

If $\rho_i \in L^\infty(\gamma_i)$, $i = 1, 2$, $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{v}_0 \in H^{1/2}(\gamma)$ satisfies $(\mathbf{v}_0 \cdot \mathbf{n})|\gamma_i < 0$, $i = 1, 2$, $(\mathbf{v}_0 \cdot \mathbf{n})|\gamma_0 = 0$ and $\int_\gamma \mathbf{v}_0 \cdot \mathbf{n} = 0$, then problem (2) has a weak solution $(\rho, \mathbf{v})$ in $L^\infty(\Omega) \times H^1(\Omega)$.

We approximate all data, use Frolov's theorem and pass to the limit.

- Data approximations:

$$\gamma_i^\epsilon := (\inf \gamma_i + \epsilon, \sup \gamma_i - \epsilon), i = 1, 2, \qquad 0 < \epsilon \ll 1$$
$$(\mathbf{v}_0^\epsilon \cdot \tau)(\lambda(s)) := \left(\left((\mathbf{v}_0 \cdot \tau) \circ \lambda^{per}\right) * m^\epsilon\right)(s)$$
$$(\mathbf{v}_0^\epsilon \cdot \mathbf{n})(\lambda(s)) := \left(\left((\mathbf{v}_0 \cdot \mathbf{n}) \circ \lambda^{per}\right) * m^\epsilon\right)(s)$$
$$\mathbf{v}_0^\epsilon := (\mathbf{v}_0^\epsilon \cdot \tau)\tau + (\mathbf{v}_0^\epsilon \cdot \mathbf{n})\mathbf{n}$$
$$\rho_0^\epsilon(\lambda(s)) := \left((\rho_0 \circ \lambda^{per}) * m^\epsilon\right)(s)$$
$$\rho_i^\epsilon := \rho_0^\epsilon|\gamma_i^\epsilon$$

where

Notations in the data approximations

$\lambda : [0, \ell) \longrightarrow \mathbb{R}^2$, $\ell > 0$, is a smooth parametrization of γ with $\lambda'(s) = \tau(\lambda(s))$ (we recall that $\tau := \mathbf{n}^\perp$);

λ^{per} is the periodical extension of λ to $\mathbb{R}$ of λ with period ℓ;

$*$ stands for the convolution in $\mathbb{R}$;

ρ_0 is the extension to γ of ρ_1 and ρ_2 defined by setting $\rho_0(x) = \rho_i(x)$ for $x \in \gamma_i$, $i = 1, 2$, and $\rho_0(x) = 0$ for $x \in \gamma/(\gamma_1 \cup \gamma_2)$ and;

$\{m^\epsilon\}$ is a sequence of mollifiers in $\mathbb{R}$, i.e., $m^\epsilon(s) = \epsilon^{-1}\phi(s/\epsilon)$ for some even function $\phi : \mathbb{R} \rightarrow [0, \infty)$ such that supp $\phi \subset [-1, 1]$ and $\int_{\mathbb{R}} \phi = 1$.

Approximate solution

For $\omega^\epsilon : \mathbb{R} \to \mathbb{R}$ being a smooth and bounded function such that $\omega^\epsilon(\psi_0^\epsilon|\gamma_i^\epsilon) = \rho_i^\epsilon$, where $\psi_0^\epsilon := -\int_\gamma(\mathbf{v}_0^\epsilon \cdot \mathbf{n})$, and $0 < \delta \ll 1$, using Frolov's theorem we take a solution of the type $\psi^\epsilon = \varphi^\epsilon + \widetilde{\psi_{0,\delta}^\epsilon}$ of the bvp

$$\begin{cases} \nu\Delta^2\psi^\epsilon = -\mathrm{div}(\omega^\epsilon(\psi^\epsilon)(\nabla^\perp\psi^\epsilon\cdot\nabla)\nabla\psi^\epsilon) + \mathrm{div}(\omega(\psi^\epsilon)(\mathbf{f}^\epsilon)^\perp) & \text{in } \Omega, \\ \frac{\partial\psi^\epsilon}{\partial\tau} = -\mathbf{v}_0^\epsilon\cdot\mathbf{n}, \qquad \frac{\partial\psi^\epsilon}{\partial\mathbf{n}} = \mathbf{v}_0^\epsilon\cdot\tau & \text{on } \gamma, \end{cases} \tag{16}$$

where $\mathbf{f}^\epsilon$ is smooth and converges to $\mathbf{f}$ in $L^2(\Omega)$ (when $\epsilon \to 0$).

Approximate solution to the bvp (2):

$$(\rho^\epsilon, \mathbf{v}^\epsilon) := (\omega^\epsilon(\psi^\epsilon), \nabla^\perp\psi^\epsilon).$$

- Convergenge of $(\rho^\epsilon, \mathbf{v}^\epsilon)$: [–, ZAMP 2002, section 3].

Variable viscosity

$$\begin{cases} -\operatorname{div}(\nu D(\mathbf{v})) + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = \rho\mathbf{f} \\ \operatorname{div}(\mathbf{v}) = 0, \qquad \operatorname{div}(\rho\mathbf{v}) = 0 \end{cases} \tag{17}$$

where $D(\mathbf{v})$ is the symmetric part of velocity gradient $\nabla\mathbf{v}$, i.e.,

$$D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$$

(we notice that $\operatorname{div}(D(\mathbf{v})) = \Delta\mathbf{v}$).

Two cases:

$$\nu = \mu(\rho) \qquad\qquad \nu = |D(\mathbf{v})|^{p-2}$$

The case $\nu = \mu(\rho)$

Theorem (• He, Zihui. Doctoral dissertation, Karlsruher Institut für Technologie (KIT), Germany, (2022); • He, Zihui, and Liao, Xian. Communications in Contemporary Mathematics, (2024).)

Let $\mu,\ \omega\ :\ \mathbb{R} \to \mathbb{R}$ be bounded and continuous functions, with $\mu \geq \mu_*$ for some constant $\mu_* > 0$. If $\mathbf{f} \in L^2(\Omega)$ and $\mathbf{v}_0 \in H^{1/2}(\gamma)$ satisfies $\int_\gamma \mathbf{v}_0 \cdot \mathbf{n} = 0$ then the bvp

$$\left\{ \begin{array}{c} \left. \begin{array}{c} -div(\mu(\rho)D(\mathbf{v})) + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = \mathbf{f} \\ div(\mathbf{v}) = 0, \quad div(\rho\mathbf{v}) = 0 \end{array} \right\} \quad in \ \ \Omega \\ \mathbf{v}|\gamma = \mathbf{v}_0 \end{array} \right. \tag{18}$$

has a weak solution $(\rho, \mathbf{v})$ of the form $(\rho, \mathbf{v}) = (\omega(\psi), \nabla^\perp\psi)$, where $\psi = \varphi + \widetilde{\psi_{0,\delta}}$, with $\varphi \in H^2_0(\Omega)$, is a solution of the bvb below.

Zihui-Liao's theorem, continued – Bvp for ψ:

$$\begin{cases} L_\mu \psi = -\mathrm{div}(\omega(\psi)(\nabla^\perp \psi \cdot \nabla)\nabla\psi) + \mathrm{div}(\omega(\psi)\mathbf{f}^\perp) \text{ in } \Omega, \\ \frac{\partial \psi}{\partial \tau} = -\mathbf{v}_0 \cdot \mathbf{n}, \quad \frac{\partial \psi}{\partial \mathbf{n}} = \mathbf{v}_0 \cdot \tau \qquad \text{on } \gamma, \end{cases} \tag{19}$$

where L_μ is the following fourth order elliptic operator:

$$L_\mu = \nabla^\perp \cdot \mathrm{div}(\mu D) = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}).$$

Leray problem

The stationary Navier-Stokes equations for incompressible fluids (with constant density $\rho = 1$ and without external force, i.e., $\mathbf{f} = \mathbf{0}$)

$$\begin{cases} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \mathcal{P} = 0 \\ \text{div}(\mathbf{v}) = 0 \end{cases} \tag{20}$$

Domain with an unbounded boundary for Leray problem

(Domain with straight channels/outlets to infinity)

Let Ω be a open connect set set in $\mathbb{R}^n$ $(n = 2, 3)$ with a C^∞ boundary, such that

$$\Omega = \bigcup_{i=0}^{2} \Omega_i \ ,$$

where Ω_0 is a bounded subset of $\mathbb{R}^n$ and, in possibly different Cartesian coordinate system,

$$\Omega_1 = \{x = (x', \bar{x}) \in \mathbb{R}^n;\ x' < 0,\ |\bar{x}| < d_1\}$$

and

$$\Omega_2 = \{x = (x', \bar{x}) \in \mathbb{R}^n;\ x' > 0,\ |\bar{x}| < d_2\},$$

with $d_i > 0$, $i = 1, 2$.

We denote by $\mathbf{n}$ the unit orthonormal vector to any cross section of Ω, pointing from Ω_1 toward Ω_2.

Poiseuille flow

Let $\mathbf{v} \equiv \mathbf{v}_P$ of the form

$$\mathbf{v}_P(x) = v_P(\bar{x})\vec{e}$$

(a parallel velocity field) be a solution of (20) in a straight unbounded cylinder

$$C = \{(x', \bar{x}) \equiv (x'\vec{e}) \oplus \bar{x}; \ -\infty < x' < \infty, \ \bar{x} \in \Sigma, \ \},$$

where Σ is a connect and symply connect open set in $\mathbb{R}^{n-1}$ independent of x', such that $v_P\big|\partial\Sigma = 0$.
Then

$$\begin{cases} -\Delta v_P = c & \text{in } \Sigma \\ v_P = 0 & \text{on } \partial\Sigma\,. \end{cases} \tag{21}$$

for some constant c.

The vector field $\mathbf{v}_P$ is called *Poiseuille flow*.

Poiseuille flow, continued

The constant c can be determined by the flux $\Phi := \int_\Sigma v_P$.
Indeed, if v_1 is the solution corresponding to $c = 1$ then $v_P = cv_1$, thus, multipliyng the first equation in (21) by v_P and integrating by parts in Σ, with the help of the boundary condition $v_P|\partial\Sigma = 0$, we obtain that

$$c\Phi = \int_\Sigma cv_P = -\int_\Sigma (\Delta v_P) v_P = \int_\Sigma |\nabla v_P|^2 = c^2 \int_\Sigma |\nabla v_1|^2.$$

With this computation, we also have that

$$\int_\Sigma |\nabla v_P|^2 = c\Phi = \Phi^2 / \int_\Sigma |\nabla v_1|^2, \text{ i.e.,}$$

$$\int_\Sigma |\nabla v_P|^2 = c_1 \Phi^2,$$

where c_1 is the constant given by $c_1 = 1/\int_\Sigma |\nabla v_1|^2$ and depends only on the cross section Σ of the cylinder C.

We also notice that $\int_\Sigma |\nabla v_1|^2 = -\int_\Sigma (\Delta \nabla v_1) v_1 = \int_\Sigma v_1 =: \Phi_1$, the flux of the vector field v_1 over Σ. In particular, we also can write

$$c_1 = 1/\Phi_1.$$

Remark

Using the divergence theorem, it is easy to prove that the flux $\int_\Sigma \mathbf{v} \cdot \mathbf{n}$ of a vector field $\mathbf{v}$ over a cross section Σ of Ω, such that $\operatorname{div} \mathbf{v} = 0$ in Ω and $\mathbf{v}|\partial\Omega = 0$, is independent of the cross section Σ.

The Leray problem

Let $\mathbf{v}_P^1 = v_P^1\mathbf{n}$ *be the Poiseuille flow in* Ω_1 *and* $\mathbf{v}_P^2 = v_P^2\mathbf{n}$ *the Poiseuille flow in* Ω_2 *with equal fluxes. Find a solution* $\mathbf{v}$ *of* (20) *in* Ω *such that* $\mathbf{v}|\partial\Omega = 0$ *and*

$$\mathbf{v} \to \mathbf{v}_P^i \quad as \quad |x'| \to \infty \quad in \quad \Omega_i.$$

Theorem (Amick, 1977)

Leray problem has a solution when the fluxes of the Poiseuille flow $\mathbf{v}_P^i$ *are sufficiently small.*

Arbitrary flux: open question.

Amick's solution of Leray's problem for Newtonian fluids, with small flux

$$\mathbf{v} = \mathbf{u} + \mathbf{a}; \quad \mathbf{u} \in H_0^1(\Omega), \ \operatorname{div} \mathbf{u} = 0, \ \operatorname{div} \mathbf{a} = 0,$$
$$\mathbf{a}|\Omega_i = v_P^i, \ \mathbf{a} \in H^1_{\text{loc}}(\Omega), \ \mathbf{a}|\partial\Omega = 0.$$

Notice that the Poiseuille flows $\mathbf{v}_P^i$ are not in $H^1(\Omega)$ (they are constant with respect to x'); $\mathbf{v}_P^i \in H^1_{\text{loc}}(\Omega)$.

A divergence free vector field $\mathbf{u}$ in $H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$ carries no flux, i.e., $\int_\Sigma \mathbf{u} \cdot \mathbf{n} = 0$, for any cross section Σ of Ω. Indeed, if $\psi \in C_c^\infty(\Omega)$ then $\int_{\Sigma(x')} \psi \cdot \mathbf{n} = 0$ for all sufficiently large $|x'|$.

NS-equations become

$$-\Delta \mathbf{u} + \mathbf{u}\nabla\mathbf{u} + f(\mathbf{u}) + \nabla\mathcal{P} = 0,$$

where

$$f(\mathbf{u}) = \mathbf{a}\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{a} + \mathbf{a}\nabla\mathbf{a} - \Delta\mathbf{a}\,.$$

Method: compactness method, with Galerkin approximations.

Control of the nonlinear term $\int (\mathbf{a}\nabla\mathbf{u})\mathbf{u}$ by $\int |\nabla\mathbf{u}|^2$ (a priori estimate)

$$\int |(\mathbf{a}\nabla\mathbf{u})\mathbf{u}| \leq \left(\int |\nabla\mathbf{u}|^2\right)^{1/2} \left(\int |\mathbf{a}|^2\,|\mathbf{u}|^2\right)^{1/2}$$

$$\begin{aligned}
\int_{\Omega_i} |\mathbf{a}|^2\,|\mathbf{u}|^2 &= \int_{\Omega_i} |\mathbf{u}|^2\,|\mathbf{v}_P^i|^2 \\
&= \left|\int_0^{\pm\infty} \int_\Sigma |\mathbf{v}_P^i|^2\,|\mathbf{u}|^2\right| \\
&\leq \left|\int_0^{\pm\infty} \left(\int_\Sigma |\mathbf{v}_P^i|^4\right)^{1/2} \left(\int_\Sigma |\mathbf{u}|^4\right)^{1/2}\right| \\
&= \left|\int_0^{\pm\infty} \|\mathbf{v}_P^i\|^2_{L^4(\Sigma)}\,\|\mathbf{u}\|^2_{L^4(\Sigma)}\right| \\
&\leq c\left|\int_0^{\pm\infty} \|\nabla\mathbf{v}_P^i\|^2_{L^2(\Sigma)}\,\|\nabla\mathbf{u}\|^2_{L^2(\Sigma)}\right| \\
&= c\,\|\nabla\mathbf{v}_P^i\|^2_{L^2(\Sigma)}\left|\int_0^{\pm\infty} \|\nabla\mathbf{u}\|^2_{L^2(\Sigma)}\right| \\
&= c\,\Phi^2 \int_{\Omega_i} |\nabla\mathbf{u}|^2
\end{aligned}$$

Similarly, we can estimate $\int (\mathbf{u}\nabla\mathbf{a})\mathbf{u}$.

The terms $\int_{\Omega_i}(-\Delta\mathbf{a})\mathbf{u}$ and $\int_{\Omega_i}(\mathbf{a}\nabla\mathbf{a})\mathbf{u}$ vanish, since $\mathbf{a}\nabla\mathbf{a} = 0$, because $\mathbf{a} = v_P^i$ in Ω_i is "parallel", and $-\Delta\mathbf{a} = (-\Delta\mathbf{v}_P^i)\mathbf{n} = c\mathbf{n}$ in Ω_i, so

$$\int_{\Omega_i}(-\Delta\mathbf{a})\mathbf{u} = \left|c\int_0^{\pm\infty} \int_\Sigma \mathbf{u}\cdot\mathbf{n}\right| = 0.$$

Leray problem for inhomogeneous fluid in 2D

—, F. Ammar-Khodja

Let $\mathbf{v}_P^i$, $i = 1, 2$, *be the Poiseuille flows as above and* ρ_1 *be a given function in* $C_b(\Sigma_1)$ *(the continuous case) or in* $L^\infty(\Sigma_1)$ *(the discontinuous case), where* $\Sigma_1 = (-d_1, d_1)$. *Find a solution* $(\rho, \mathbf{v})$ *of the system*

$$\begin{cases} - \quad \nu\Delta\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = 0 \\ \quad div(\mathbf{v}) = 0, \qquad div(\rho\mathbf{v}) = 0 \end{cases} \tag{22}$$

in Ω *such that* $\mathbf{v}|\partial\Omega = 0$, $\lim_{(-1)^i x' \to \infty} \mathbf{v} = \mathbf{v}_P^i$ *and*

$$\lim_{x' \to -\infty} \rho = \rho_1.$$

Theorem (—, F. Ammar-Khodja (2004, 2006))

The above problem has a weak solution $(\rho, \mathbf{v})$ *with* $\mathbf{v} \in H^1_{loc}((\bar{\Omega}))$, $\lim_{(-1)^i x \to \infty} \|\mathbf{v}(x,\cdot) - \mathbf{v}_P^i\|_{C_b(\Sigma_i)} = 0$, $i = 1, 2$, *and:*

$$\rho \in C_b(\Omega), \quad \lim_{x \to -\infty} \|\rho(x,\cdot) - \rho_1\|_{C_b(\Sigma_1)} = 0,$$

in the continuous case;

$$\rho \in L^\infty(\Omega), \quad * - \lim_{x \to -\infty, a.e.} \rho(x,\cdot) = \rho_1,$$

in the discontinuous case, where $* - \lim_{x \to -\infty, a.e.}$ *denotes the limit in the weak-$*$ topology of* $L^\infty(\Sigma_1)$, *with* x *tending to* $-\infty$ *except for a set of zero Lebesgue measure.*

Ladyzhenskaya-Solonnikov problem

Domain with unbounded channels containing straight cylinders

Let Ω be a open connect set set in $\mathbb{R}^n$ $(n=2,3)$ with a C^∞ boundary, such that

$$\Omega = \bigcup_{i=0}^{2} \Omega_i \ ,$$

where Ω_0 is a bounded subset of $\mathbb{R}^n$ and, in possibly different Cartesian coordinate system,

$$\Omega_1 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' < 0,\ \bar{x} \in \Sigma_1(x')\}$$

and

$$\Omega_2 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' > 0,\ \bar{x} \in \Sigma_2(x')\},$$

with $\Sigma_i(x')$, $i = 1, 2$, being C^∞ simply connected domains in $\mathbb{R}^{n-1}$ such that $\sup_{x', i=1,2} \operatorname{diam} \Sigma_i(x') < \infty$ and Ω_i, $i = 1, 2$, contains some cylinder

$$C_l^i = \{x \in \mathbb{R}^n;\ (-1)^i x' > 0 \text{ e } |\bar{x}| < l\}, \quad (l > 0)$$

(in particular, $\inf_{x', i=1,2} \operatorname{diam} \Sigma_i(x') > 0$).

We will denote by $\mathbf{n}$ the unit orthonormal vector to $\Sigma(x')$, or to any cross section of Ω, pointing from Ω_1 toward Ω_2.

The Ladyzhenskaya-Solonnikov problem

Given any $\Phi \in \mathbb{R}$, find a solution $\mathbf{v}$ of the system

$$\begin{cases} - \quad \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\mathcal{P} = 0 \\ \operatorname{div}(\mathbf{v}) = 0 \end{cases}$$

in Ω such that $\mathbf{v}|\partial\Omega = 0$, $\int_{\Sigma(x')} \mathbf{v}\cdot\mathbf{n} = \Phi$ and

$$\sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla\mathbf{v}|^2 < \infty \,,$$

where $\Omega^t := \Omega_0 \cup \Omega_1^t \cup \Omega_2^t$, $\Omega_i^t := \{(\bar{x}, x') \in \Omega_i \,;\, 0 < (-1)^i x' < t\}$, $i = 1, 2$.

"*Problem 1.1*" in

[LS] O.A. Ladyzhenskaya and V.A. Solonnikov, *Determination of the Solutions of Boundary Value Problems for Steady-State Stokes and Navier-Stokes Equations in Domains Having an Unbounded Dirichlet Integral* (1980). English transl. in J. Soviet Math. **21** (1983).

Theorem (Ladyzhenskaya-Solonnikov, 1980)

Given any flux ($\Phi \in \mathbb{R}$), the above problem has a solution.

Ladyzhenskaya-Solonnikov's solution, for Newtonian fluids with arbitrary flux: $v = u + a; \quad u \in H^1_{loc}(\Omega),\ \operatorname{div} u = 0,\ u\big|\partial\Omega = 0$

and a is given by the following lemma:

Lemma [LS]. *For any $\delta > 0$ there exists a vector field a such that*

a$_1$) $a \in H^1_{loc}(\Omega)$, $\operatorname{div} a = 0$, $a|\partial\Omega = 0$,

a$_2$) $\int_\Sigma a \cdot \mathbf{n} = 1$ *for any cross section Σ of Ω,*

a$_3$) $\int_{\Omega_i^{t-1,t}} |\nabla a|^2 \leq c$ *for $i = 1, 2$ and all $t \geq 1$, where*
$\Omega_i^{t-1,t} = \{(\bar{x}, x') \in \Omega_i\, ;\ t - 1 < |x'| < t\}$,

and

a$_4$) $\int_{\Omega^t} |a|^2 |u|^2 \leq c\delta \int_{\Omega^t} |\nabla u|^2$ for all $t > 0$ and $u \in C_c^\infty(\Omega)$,

where, in a$_3$) and a$_4$), c is a constant depending only on Ω.

Remark: Given any $\Phi \in \mathbb{R}$, multiplying a by Φ, we obtain a vector field having flux Φ.

Now $a\big|\Omega_i$ might not be the Poiseuille v_P^i, but the compactness method still works, by truncating the domain and long computations:

Let u^t be a solution of the NS-equations

$$-\Delta u^t + u^t \nabla u^t + f(u^t) + \nabla \mathcal{P}^t = 0$$

in $H_0^1(\Omega^t)$ (joint with some pressure function $\mathcal{P}^t \in L^2_{loc}(\Omega^t)$).
Now, let $t' > t$. Multiplying the equation $-\Delta u^{t'} + u^{t'} \nabla u^{t'} + f(u^{t'}) + \nabla \mathcal{P}^{t'} = 0$ by $u^{t'}$ and integrating by parts in Ω^t ... [LS]

$$\int_{\Omega^t} |\nabla u^{t'}|^2 \leq ct + \int_{\Sigma(t)} (\text{bound. terms}),$$

for all $t < t'$. Integrating in t, from $\eta - 1$ to $\eta \leq t'$, we get

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta - \frac{1}{4} + \int_{\Omega^{\eta-1,\eta}} (\text{bound. terms}) .$$

Using the equation, is possible to estimate $\int_{\Omega^{\eta-1,\eta}} (\text{bound. terms})$ by a linear combinations of powers of $\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2$. But

$$\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2 = z'(\eta) \, !$$

Thus,

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta + g(z'(\eta)), \quad \forall\ \eta \leq t',$$

for some function $g : \mathbb{R} \to \mathbb{R}$. Besides,

$$z(t') \leq \int_{\Omega^{t'}} |\nabla u^{t'}|^2 \leq ct'.$$

Then, by a kind of "*reverse Gronwall lemma*" [LS], we have

$$z(\eta) \leq c\eta,$$

which implies

$$\int_{\Omega^{\eta-1}} |\nabla u^{t'}|^2 \leq c\eta, \quad \forall\ \eta \leq t'.$$

So, fixing t (arbitrary), $\{u^{t'}\}_{t'>t}$ is bounded in $H^1(\Omega^t)$, by $c(t+1)$.

Construction of a, in Ω_i and with $n = 3$:

$$a = \frac{1}{2\pi}\nabla \times (\zeta b) = \frac{1}{2\pi}\nabla\zeta \times b,$$

where

$$b(x) = \left(-\frac{x_2}{|\bar{x}|^2}, \frac{x_1}{|\bar{x}|^2}, 0\right), \quad \bar{x} = (x_1, x_2),$$

(the angle form in Σ) and ζ is the "truncating E. Hopf's function":

$$\zeta(x) = \psi\left(\varepsilon \log \frac{\sigma(|\bar{x}|)}{\rho(x)}\right);$$

$\rho(x)$:	the regularized distance to $\partial\Omega$
$\sigma, \psi : \mathbb{R} \to \mathbb{R}$:	smooth nondecreasing functions,

$$\sigma(s) = \begin{cases} \frac{l}{4}, & s \le \frac{l}{4} \\ t, & s > \frac{l}{2} \end{cases}$$

$$\psi(s) = \begin{cases} 0, & s \le 0 \\ 1, & s > 1 \end{cases}$$

$\varepsilon = \varepsilon(\delta)$.

The power model, $\nu = |D(\mathbf{v})|^{p-2}$

The equations:

$$\begin{cases} -\operatorname{div}(|D(v)|^{p-2}D(v)) + v\nabla v + \nabla \mathcal{P} = 0 \\ \operatorname{div} v = 0 \end{cases} \qquad \text{power-law fluids}$$

$$|D(v)|^{p-2}D(v): \text{ viscous stress tensor, } \mathbb{S}.$$

$$\mathbb{S} = |D(v)|^{p-2}D(v) \quad \text{or} \quad \text{viscosity} = |D(v)|^{p-2}$$

is the *power law* or *Ostwald-de Waele law* (*model*) for viscosity (1920s); see e.g. R. Bird, W. Stewart and E. Lightfoof, *Transport Phenomena*, Johh Wiley & Sons, Inc. (2007).

In the classical book by O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd ed. (1969), after the last chapter, there is a description of some models including power laws.

$|D(v)|$: shear rate

$p = 2$: Newtonian fluids (e.g. water, oil)

$p < 2$: *shear-thinning* (or plastic and pseudo-plastic, e.g. most polymer melts and solutions)
- viscosity decreases with the shear rate

$p > 2$: *shear-thickening* (or dilatant, e.g. mud, clay, cement)
- viscosity increases with the shear rate

Cf. E. Marusic-Paloka, *Steady Flow of a Non-Newtonian Fluid in Unbounded Channels and Pipes*, Mathematical Models and Methods in Applied Sciences, **10**(9) (2000).

For parallel fluids, $v(x) \equiv v(\bar{x})\vec{e}$, $x = (\bar{x}, x') \equiv \bar{x} \oplus x'\vec{e}$, the velocity field is given by a scalar function $v(\bar{x})$, the convection term $v\nabla v$ vanishes, and the Navier-Stokes equations become the *p-Laplacian equation*

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = c$$

for some constant c, related to "*pressure drop*", i.e. $\nabla\mathcal{P} = -c\vec{e}$.

The domain

Let Ω be a open connect set set in $\mathbb{R}^n$, $n = 2, 3$, with a C^∞ boundary, such that

$$\Omega = \bigcup_{i=0}^{2} \Omega_i \ ,$$

where Ω_0 is a bounded subset of $\mathbb{R}^n$ and, in possibly different Cartesian coordinate system,

$$\Omega_1 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' < 0,\ \bar{x} \in \Sigma_1(x')\}$$

and

$$\Omega_2 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' > 0,\ \bar{x} \in \Sigma_2(x')\},$$

with $\Sigma_i(x')$, $i = 1, 2$, being C^∞ simply connected domains in $\mathbb{R}^{n-1}$ such that

$$\sup_{x', i=1,2} \operatorname{diam} \Sigma_i(x') < \infty$$

and Ω_i, $i = 1, 2$, contains some cylinder

$$C_l^i = \{x \in \mathbb{R}^n;\ (-1)^i x' > 0 \text{ e } |\bar{x}| < l\}, \quad (l > 0)$$

(in particular, $\inf_{x', i=1,2} \operatorname{diam} \Sigma_i(x') > 0$).

We will denote by $\mathbf{n}$ the ortonormal vector to $\Sigma(x')$, or to any cross section of Ω, pointing from Ω_1 toward Ω_2.

The problem

"Ladyzhenskaya-Solonnikov problem"

Given any $\Phi \in \mathbb{R}$, find a solution $(v, \mathcal{P})$ of (NS) such that

$$v = 0 \quad \text{on } \partial\Omega\,,$$
$$\int_{\Sigma(x')} v \cdot \mathbf{n} = \Phi$$

and

$$\sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla v|^p < \infty\,,$$

where $\Omega^t := \Omega_0 \cup \Omega_1^t \cup \Omega_2^t$, $\Omega_i^t := \{(\bar{x}, x') \in \Omega_i\,;\, 0 < (-1)^i x' < t\}$, $i = 1, 2$.

Cf. "*Problem 1.1*" in

[LS] O.A. Ladyzhenskaya and V.A. Solonnikov, *Determination of the Solutions of Boundary Value Problems for Steady-State Stokes and Navier-Stokes Equations in Domains Having an Unbounded Dirichlet Integral* (1980). English transl. in J. Soviet Math. **21** (1983).

Theorem (—, G. Dias. Jr. Diff. Eq. (2012))

Let $p \geq 2$ and $n = 3$. Then the Ladyzhesnkaya-Solonnikov problem for power-law fluids has a weak solution, i.e. there is a $(v, \mathcal{P})$ in $W^{1,p}_{loc}(\Omega) \times L^{p'}_{loc}(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega |D(v)|^{p-2} D(v) : \nabla\psi = -\int_\Omega (v\nabla v)\cdot\psi + \int_\Omega \mathcal{P} div\psi\,, \\ \qquad \forall\ \psi \in C_c^\infty(\Omega;\mathbb{R}^3) \\ \displaystyle\int_\Omega v\cdot\nabla\psi = 0\,, \qquad \forall\ \psi \in C_c^\infty(\Omega;\mathbb{R}) \end{cases}$$

$$v|\partial\Omega = 0\,, \qquad \int_\Sigma v\cdot\mathbf{n} = \Phi$$

and

$$\sup_{t>0} t^{-1}\int_{\Omega^t} |\nabla v|^p < \infty\,.$$

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Thank you all for your attention!