Nome: ______________________________ RA: ____________________

Métodos Matemáticos I (F520/MS550) - Teste 2

31 de março de 2010

1. Resolva a equação de Laplace,
$$\nabla^2\psi = 0,$$
para:

   (a) (2 pontos) $\psi = \psi(\rho)$, $\rho = \sqrt{x^2+y^2}$, com a condição de contorno $\psi(\rho_0) = 0$, onde $\rho_0 > 0$. Tal solução é única?

   (b) (2 pontos) $\psi = \psi(r)$, $r = \sqrt{x^2+y^2+z^2}$, com a condição de contorno $\psi(r_0) = 0$, onde $r_0 > 0$. Tal solução é única?

   (c) (1 ponto) Em ambos os casos acima, determine a solução mais geral possível em que ψ é regular (i.e., livre de singularidades) em todo o espaço.

2. (5 pontos) Expresse $\partial/\partial x$, $\partial/\partial y$ e $\partial/\partial z$ exclusivamente em termos de coordenadas esféricas.

Fórmulas possivelmente úteis:

$$\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}$$

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix}$$

$$\boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\varphi}{\partial\varphi}\right]$$

$$\boldsymbol{\nabla}\times\boldsymbol{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ V_r & rV_\theta & r\sin\theta V_\varphi \end{vmatrix}$$

---

(1) $\nabla^2\psi = 0$

(a) $\psi = \psi(\rho) \rightarrow 0 = \nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) \Rightarrow \rho\frac{\partial\psi}{\partial\rho} = a$

$\Rightarrow \psi = a\ln\rho + b \quad , \quad a,b$ cts

$\psi(\rho_0) = 0 \Rightarrow a\ln\rho_0 + b = 0 \Rightarrow \psi(\rho) = a\ln\left(\frac{\rho}{\rho_0}\right)$

→ Infinitas soluções parametrizadas por a.

(b) $\psi = \psi(r) \rightarrow 0 = \nabla^2\psi = \frac{1}{r^2 \operatorname{sen}\theta}\left[\operatorname{sen}\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right] \rightarrow$

$\rightarrow r^2\frac{\partial\psi}{\partial r} = a \rightarrow \psi(r) = -\frac{a}{r} + b \quad , a,b$ cts

$\psi(r_0) = 0 \Rightarrow \psi(r) = -a\left(\frac{1}{r} - \frac{1}{r_0}\right)$

→ Infinitas soluções parametrizadas por a.

(c) As soluções acima são singulares para $\rho = 0$ e $r = 0$

a menos que $a = 0$.

Assim a solução regular mais geral possível nos casos acima é a trivial: $\psi = 0$.

(2) Coordenadas esféricas: $\begin{cases} x = r\,\text{sen}\,\theta\,\cos\varphi \\ y = r\,\text{sen}\,\theta\,\text{sen}\,\varphi \\ z = r\cos\theta \end{cases}$

$$\rightarrow \begin{cases} \dfrac{\partial \vec{r}}{\partial r} = \text{sen}\,\theta\cos\varphi\,\vec{i} + \text{sen}\,\theta\,\text{sen}\,\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \dfrac{\partial \vec{r}}{\partial \theta} = r\cos\theta\cos\varphi\,\vec{i} + r\cos\theta\,\text{sen}\,\varphi\,\vec{j} - r\,\text{sen}\,\theta\,\vec{k} \\ \dfrac{\partial \vec{r}}{\partial \varphi} = -r\,\text{sen}\,\theta\,\text{sen}\,\varphi\,\vec{i} + r\,\text{sen}\,\theta\cos\varphi\,\vec{j} + 0\,\vec{k} \end{cases}$$

$$\Rightarrow \begin{cases} \hat{r} = \text{sen}\,\theta\cos\varphi\,\vec{i} + \text{sen}\,\theta\,\text{sen}\,\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \hat{\theta} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\text{sen}\,\varphi\,\vec{j} - \text{sen}\,\theta\,\vec{k} \\ \hat{\varphi} = -\text{sen}\,\varphi\,\vec{i} + \cos\varphi\,\vec{j} + 0\,\vec{k} \end{cases} \qquad (*)$$

Dado um campo escalar Ψ:

$$\frac{\partial\Psi}{\partial x}\vec{i} + \frac{\partial\Psi}{\partial y}\vec{j} + \frac{\partial\Psi}{\partial z}\vec{k} = \nabla\Psi = \frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \frac{1}{r\,\text{sen}\,\theta}\frac{\partial\Psi}{\partial\varphi}\hat{\varphi} \overset{(*)}{=}$$

$$= \left(\text{sen}\,\theta\cos\varphi\frac{\partial\Psi}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial\Psi}{\partial\theta} - \frac{\text{sen}\,\varphi}{r\,\text{sen}\,\theta}\frac{\partial\Psi}{\partial\varphi}\right)\vec{i} +$$

$$+ \left(\text{sen}\,\theta\,\text{sen}\,\varphi\frac{\partial\Psi}{\partial r} + \cos\theta\,\text{sen}\,\varphi\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \frac{\cos\varphi}{r\,\text{sen}\,\theta}\frac{\partial\Psi}{\partial\varphi}\right)\vec{j} +$$

$$+ \left(\cos\theta\frac{\partial\Psi}{\partial r} - \text{sen}\,\theta\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + 0\right)\vec{k}$$

Assim

$$\begin{cases} \dfrac{\partial}{\partial x} = \text{sen}\,\theta\cos\varphi\dfrac{\partial}{\partial r} + \cos\theta\cos\varphi\dfrac{1}{r}\dfrac{\partial}{\partial\theta} - \dfrac{\text{sen}\,\varphi}{r\,\text{sen}\,\theta}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial y} = \text{sen}\,\theta\,\text{sen}\,\varphi\dfrac{\partial}{\partial r} + \cos\theta\,\text{sen}\,\varphi\dfrac{1}{r}\dfrac{\partial}{\partial\theta} + \dfrac{\cos\varphi}{r\,\text{sen}\,\theta}\dfrac{\partial}{\partial\varphi} \\ \dfrac{\partial}{\partial z} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\text{sen}\,\theta}{r}\dfrac{\partial}{\partial\theta} \end{cases}$$

$\perp$.