5ª lista de Exercícios – MA111 - Capítulo 12 do Spivak

1. Find f^{-1} for each of the following f.

(i) $f(x) = x^3 + 1$.

(ii) $f(x) = (x-1)^3$.

(iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ -x, & x \text{ irrational.} \end{cases}$

(iv) $f(x) = \begin{cases} -x^2 & x \geq 0 \\ 1 - x^3, & x < 0. \end{cases}$

(v) $f(x) = \begin{cases} x, & x \neq a_1, \dots, a_n \\ a_{i+1} & x = a_i, \quad i = 1, \dots, n-1 \\ a_1, & x = a_n. \end{cases}$

(vi) $f(x) = x + [x]$.

(vii) $f(0.a_1a_2a_3\dots) = 0.a_2a_1a_3\dots$. (Decimal representation is being used.)

(viii) $f(x) = \dfrac{x}{1-x^2}, \ -1 < x < 1$.

3. Prove that if f is increasing, then so is f^{-1}, and similarly for decreasing functions.

4. If f and g are increasing, is $f+g$? Or $f \cdot g$? Or $f \circ g$?

6. Show that $f(x) = \dfrac{ax+b}{cx+d}$ is one-one if and only if $ad - bc \neq 0$, and find f^{-1} in this case.

7. On which intervals $[a, b]$ will the following functions be one-one?

(i) $f(x) = x^3 - 3x^2$.

(ii) $f(x) = x^5 + x$.

(iii) $f(x) = (1+x^2)^{-1}$.

(iv) $f(x) = \dfrac{x+1}{x^2+1}$.

9. Suppose that f is a one-one function and that f^{-1} has a derivative which is nowhere 0. Prove that f is differentiable. Hint: There is a one-step proof.

12. (a) What are the two differentiable functions f which are defined implicitly on $(-1, 1)$ by the equation $x^2+y^2 = 1$, i.e., which satisfy $x^2+[f(x)]^2 = 1$ for all x in $(-1, 1)$? Notice that there are no solutions defined outside $[-1, 1]$.

(b) Which functions f satisfy $x^2 + [f(x)]^2 = -1$?

*(c) Which differentiable functions f satisfy $[f(x)]^3 - 3f(x) = x$? Hint: It will help to first draw the graph of the function $g(x) = x^3 - 3x$.

In general, determining on what intervals a differentiable function is defined implicitly by a particular equation may be a delicate affair, and is best discussed in the context of advanced calculus. If we *assume* that f is such a differentiable solution, however, then a formula for $f'(x)$ can be derived, exactly as in Problem 11(c), by differentiating both sides of the equation defining f (a process known as "implicit differentiation"):

15. The collection of all points (x, y) such that $3x^3 + 4x^2y - xy^2 + 2y^3 = 4$ forms a certain curve in the plane. Find the equation of the tangent line to this curve at the point $(-1, 1)$.

18. Suppose that f is a differentiable one-one function with a nowhere zero derivative and that $f = F'$. Let $G(x) = xf^{-1}(x) - F(f^{-1}(x))$. Prove that $G'(x) = f^{-1}(x)$. (Disregarding details, this problem tells us a very interesting fact: if we know a function whose derivative is f, then we also know one whose derivative is f^{-1}. But how could anyone ever guess the function G? Two different ways are outlined in Problems 14-17 and 19-15.)

***23.** (a) If f is a continuous function on $\mathbf{R}$ and $f = f^{-1}$, prove that there is at least one x such that $f(x) = x$. (What does the condition $f = f^{-1}$ mean geometrically?)

(b) Give several examples of continuous f such that $f = f^{-1}$ and $f(x) = x$ for exactly one x. Hint: Try decreasing f, and remember the geometric interpretation. One possibility is $f(x) = -x$.

(c) Prove that if f is an increasing function such that $f = f^{-1}$, then $f(x) = x$ for all x. Hint: Although the geometric interpretation will be immediately convincing, the simplest proof (about 2 lines) is to rule out the possibilities $f(x) < x$ and $f(x) > x$.

***26.** (a) Suppose that $f(x) > 0$ for all x, and that f is decreasing. Prove that there is a *continuous* decreasing function g such that $0 < g(x) \leq f(x)$ for all x.

(b) Show that we can even arrange that g will satisfy $\lim_{x \to \infty} g(x)/f(x) = 0$.