Exercícios do livro do Spivak

Capítulo 7

3. Prove that there is some number x such that

(i) $x^{179} + \dfrac{163}{1 + x^2 + \sin^2 x} = 119.$

(ii) $\sin x = x - 1.$

5. Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f ?

8. Suppose that f and g are continuous, that $f^2 = g^2$, and that $f(x) \neq 0$ for all x. Prove that either $f(x) = g(x)$ for all x, or else $f(x) = -g(x)$ for all x.

10. Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$. (If your proof isn't very short, it's not the right one.)

O que está apagado na primeira linha é "f(a)<f(b)".

Capítulo 9

3. Prove that if $f(x) = \sqrt{x}$, then $f'(a) = 1/2\sqrt{a}$, for $a > 0$. (The expression you obtain for $[f(a + h) - f(a)]/h$ will require some algebraic face lifting, but the answer should suggest the right trick.)

4. For each natural number n, let $S_n(x) = x^n$. Remembering that $S_1'(x) = 1$, $S_2'(x) = 2x$, and $S_3'(x) = 3x^2$, conjecture a formula for $S_n'(x)$. Prove your conjecture. (The expression $(x + h)^n$ may be expanded by the binomial theorem.)

6. Prove, starting from the definition (and drawing a picture to illustrate):

(a) if $g(x) = f(x) + c$, then $g'(x) = f'(x)$;
(b) if $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

8. (a) Suppose $g(x) = f(x + c)$. Prove (starting from the definition) that $g'(x) = f'(x + c)$. Draw a picture to illustrate this. To do this problem you must write out the definitions of $g'(x)$ and $f'(x + c)$ correctly. The purpose of Problem 7 was to convince you that although this problem is easy, it is not an utter triviality, and there is something to prove: you cannot simply put prime marks into the equation $g(x) = f(x + a)$. To emphasize this point:

(b) Prove that if $g(x) = f(cx)$, then $g'(x) = c \cdot f'(cx)$. Try to see pictorially why this should be true, also.

(c) Suppose that f is differentiable and periodic, with period a (i.e., $f(x + a) = f(x)$ for all x). Prove that f' is also periodic.

11. (a) Prove that Galileo was wrong: if a body falls a distance $s(t)$ in t seconds, and s' is proportional to s, then s cannot be a function of the form $s(t) = ct^2$.

(b) Prove that the following facts are true about s if $s(t) = (a/2)t^2$ (the first fact will show why we switched from c to $a/2$):

(i) $s''(t) = a$ (the acceleration is constant).
(ii) $[s'(t)]^2 = 2as(t)$.

(c) If s is measured in feet, the value of a is 32. How many seconds do you have to get out of the way of a chandelier which falls from a 400-foot ceiling? If you don't make it, how fast will the chandelier be going when it hits you? Where was the chandelier when it was moving with half that speed?

16. Let $\alpha > 1$. If f satisfies $|f(x)| \leq |x|^\alpha$, prove that f is differentiable at 0.

19. (a) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$ for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$. (Begin with the definition of $g'(a)$.)

(b) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

***22.** (a) Suppose that f is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}.$$

Hint: Remember an old algebraic trick—a number is not changed if the same quantity is added to and then subtracted from it.

27. If $S_n(x) = x^n$, and $0 \leq k \leq n$, prove that

$$S_n^{(k)}(x) = \frac{n!}{(n-k)!}\, x^{n-k}$$

$$= k!\binom{n}{k} x^{n-k}.$$

Capítulo 10

1. As a warm up exercise, find $f'(x)$ for each of the following f. (Don't worry about the domain of f or f'; just get a formula for $f'(x)$ that gives the right answer when it makes sense.)

(i) $f(x) = \sin(x + x^2)$.
(ii) $f(x) = \sin x + \sin x^2$.
(iii) $f(x) = \sin(\cos x)$.
(iv) $f(x) = \sin(\sin x)$.
(v) $f(x) = \sin\left(\frac{\cos x}{x}\right)$.
(vi) $f(x) = \frac{\sin(\cos x)}{x}$.
(vii) $f(x) = \sin(x + \sin x)$.
(viii) $f(x) = \sin(\cos(\sin x))$.

2. Find $f'(x)$ for each of the following functions f. (It took the author 20 minutes to compute the derivatives for the answer section, and it should not take you much longer. Although rapid calculation is not the goal of mathematics, if you hope to treat theoretical applications of the Chain Rule with aplomb, these concrete applications should be child's play—mathematicians like to pretend that they can't even add, but most of them can when they have to.)

(i) $f(x) = \sin((x + 1)^2(x + 2))$.
(ii) $f(x) = \sin^3(x^2 + \sin x)$.
(iii) $f(x) = \sin^2((x + \sin x)^2)$.
(iv) $f(x) = \sin\left(\frac{x^3}{\cos x^3}\right)$.
(v) $f(x) = \sin(x \sin x) + \sin(\sin x^2)$.
(vi) $f(x) = (\cos x)^{31^2}$.
(vii) $f(x) = \sin^2 x \sin x^2 \sin^2 x^2$.
(viii) $f(x) = \sin^3(\sin^2(\sin x))$.
(ix) $f(x) = (x + \sin^5 x)^6$.
(x) $f(x) = \sin(\sin(\sin(\sin(\sin x))))$.
(xi) $f(x) = \sin((\sin^7 x^7 + 1)^7)$.
(xii) $f(x) = (((x^2 + x)^3 + x)^4 + x)^5$.
(xiii) $f(x) = \sin(x^2 + \sin(x^2 + \sin x^2))$.
(xiv) $f(x) = \sin(6 \cos(6 \sin(6 \cos 6x)))$.
(xv) $f(x) = \frac{\sin x^2 \sin^2 x}{1 + \sin x}$.
(xvi) $f(x) = \frac{1}{x - \frac{2}{x + \sin x}}$.

(xvii) $f(x) = \sin\left(\sin\left(\dfrac{x^3}{\sin x}\right)\right)$.

(xviii) $f(x) = \sin\left(\dfrac{x}{x - \sin\left(\dfrac{x}{x - \sin x}\right)}\right)$.

3. Find the derivatives of the functions tan, cotan, sec, and cosec. (You don't have to memorize these formulas, although they will be needed once in a while; if you express your answers in the right way, they will be simple and somewhat symmetrical.)

7. (a) A circular object is increasing in size in some unspecified manner, but it is known that when the radius is 6, the rate of change of the radius is 4. Find the rate of change of the area when the radius is 6. (If $r(t)$ and $A(t)$ represent the radius and the area at time t, then the functions r and A satisfy $A = \pi r^2$; a straightforward use of the Chain Rule is called for.)
   (b) Suppose that we are now informed that the circular object we have been watching is really the cross section of a spherical object. Find the rate of change of the *volume* when the radius is 6. (You will clearly need to know a formula for the volume of a sphere; in case you have forgotten, the volume is $\frac{4}{3}\pi$ times the cube of the radius.)

   (c) Now suppose that the rate of change of the area of the circular cross section is 5 when the radius is 3. Find the rate of change of the volume when the radius is 3. You should be able to do this problem in two ways: first, by using the formulas for the area and volume in terms of the radius; and then by expressing the volume in terms of the area (to use this method you will need Problem 9-3).

8. The area between two varying concentric circles is at all times 9π in^2. The rate of change of the area of the larger circle is 10π in^2/sec. How fast is the circumference of the smaller circle changing when it has area 16π in^2?

12. Using the derivative of $f(x) = 1/x$, as found in Problem 9-1, find $(1/g)'(x)$ by the Chain Rule.
13. (a) Using Problem 9-3, find $f'(x)$ for $-1 < x < 1$, if $f(x) = \sqrt{1 - x^2}$.
    (b) Prove that the tangent line to the graph of f at $(a, \sqrt{1 - a^2})$ intersects the graph only at that point (and thus show that the elementary geometry definition of the tangent line coincides with ours).

14. Prove similarly that the tangent lines to an ellipse or hyperbola intersect these sets only once.

15. If $f + g$ is differentiable at a, are f and g necessarily differentiable at a? If $f \cdot g$ and f are differentiable at a, what conditions on f imply that g is differentiable at a?

16. (a) Prove that if f is differentiable at a, then $|f|$ is also differentiable at a, provided that $f(a) \neq 0$.

(b) Give a counterexample if $f(a) = 0$.

(c) Prove that if f and g are differentiable at a, then the functions $\max(f, g)$ and $\min(f, g)$ are differentiable at a, provided that $f(a) \neq g(a)$.

(d) Give a counterexample if $f(a) = g(a)$.

form . . . .

20. (a) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, find a function g such that $g' = f$. Find another.

(b) If

$$f(x) = \frac{b_2}{x^2} + \frac{b_3}{x^3} + \cdots + \frac{b_m}{x^m},$$

find a function g with $g' = f$.

(c) Is there a function

$$f(x) = a_n x^n + \cdots + a_0 + \frac{b_1}{x} + \cdots + \frac{b_m}{x^m}$$

such that $f'(x) = 1/x$?

22. (a) The number a is called a **double root** of the polynomial function f if $f(x) = (x - a)^2 g(x)$ for some polynomial function g. Prove that a is a double root of f if and only if a is a root of both f and f'.

(b) When does $f(x) = ax^2 + bx + c$ $(a \neq 0)$ have a double root? What does the condition say geometrically?

28. If $f(x) = x^{-n}$ for n in $\mathbf{N}$, prove that

$$\begin{aligned} f^{(k)}(x) &= (-1)^k \frac{(n+k-1)!}{(k-1)!} x^{-n-k} \\ &= (-1)^k n! \binom{n+k-1}{k-1} x^{-n-k}, \quad \text{for } x \neq 0. \end{aligned}$$

***31.** Let $f(x) = x^{2n} \sin 1/x$ if $x \neq 0$, and let $f(0) = 0$. Prove that $f'(0), \ldots, f^{(n)}(0)$ exist, and that $f^{(n)}$ is not continuous at 0. (You will encounter the same basic difficulty as that in Problem 19.)

Capítulo 11

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.

(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.

(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.

(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.

(v) $f(x) = \dfrac{x + 1}{x^2 + 1}$ on $[-1, \frac{1}{2}]$.

(vi) $f(x) = \dfrac{x}{x^2 - 1}$ on $[0, 5]$.

2. Now sketch the graph of each of the functions in Problem 1, and find all local maximum and minimum points.

3. Sketch the graphs of the following functions.

(i) $f(x) = x + \dfrac{1}{x}$.

(ii) $f(x) = x + \dfrac{3}{x^2}$.

(iii) $f(x) = \dfrac{x^2}{x^2 - 1}$.

(iv) $f(x) = \dfrac{1}{1 + x^2}$.

5. For each of the following functions, find all local maximum and minimum points.

(i) $f(x) = \begin{cases} x, & x \neq 3, 5, 7, 9 \\ 5, & x = 3 \\ -3, & x = 5 \\ 9, & x = 7 \\ 7, & x = 9. \end{cases}$

(ii) $f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/q, & x = p/q \text{ in lowest terms.} \end{cases}$

(iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

(iv) $f(x) = \begin{cases} 1, & x = 1/n \text{ for some } n \text{ in } \mathbf{N} \\ 0, & \text{otherwise.} \end{cases}$

(v) $f(x) = \begin{cases} 1, & \text{if the decimal expansion of } x \text{ contains a 5} \\ 0, & \text{otherwise.} \end{cases}$

6. (a) Let (x_0, y_0) be a point of the plane, and let L be the graph of the function $f(x) = mx + b$. Find the point $\bar{x}$ such that the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is smallest. [Notice that minimizing this distance is the same as minimizing its square. This may simplify the computations somewhat.]

(b) Also find $\bar{x}$ by noting that the line from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$ is perpendicular to L.

(c) Find the distance from (x_0, y_0) to L, i.e., the distance from (x_0, y_0) to $(\bar{x}, f(\bar{x}))$. [It will make the computations easier if you first assume that $b = 0$; then apply the result to the graph of $f(x) - mx$ and the point $(x_0, y_0 - b)$.] Compare with Problem 4-22.

(d) Consider a straight line described by the equation $Ax + By + C = 0$ (Problem 4-7). Show that the distance from (x_0, y_0) to this line is $(Ax_0 + By_0 + C)/\sqrt{A^2 + B^2}$.

7. The previous Problem suggests the following question: What is the relationship between the critical points of f and those of f^2?

10. Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

11. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

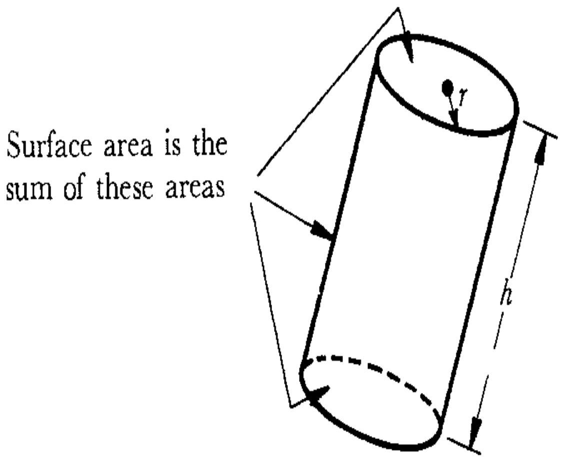

FIGURE 24

13. A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

14. Show that the sum of a number and its reciprocal is at least 2.

15. Find the trapezoid of largest area that can be inscribed in a semicircle of radius a, with one base lying along the diameter.

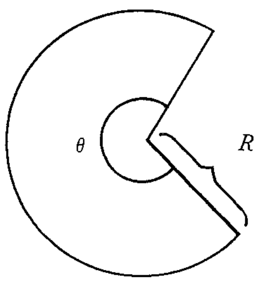

FIGURE 26

17. Ecological Ed must cross a circular lake of radius 1 mile. He can row across at 2 mph or walk around at 4 mph, or he can row part way and walk the rest (Figure 28). What route should he take so as to

(i) see as much scenery as possible?
(ii) cross as quickly as possible?

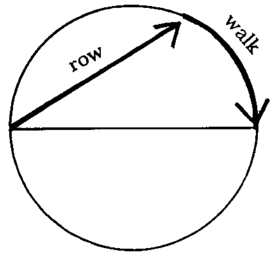

FIGURE 28

*21. (a) Suppose that the polynomial function $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has critical points $-1, 1, 2, 3$, and $f''(-1) = 0, f''(1) > 0, f''(2) < 0, f''(3) = 0$. Sketch the graph of f as accurately as possible on the basis of this information.

(b) Does there exist a polynomial function with the above properties, except that 3 is not a critical point?

22. Describe the graph of a rational function (in very general terms, similar to the text's description of the graph of a polynomial function).

28. Find all functions f such that

(a) $f'(x) = \sin x$.
(b) $f''(x) = x^3$.
(c) $f'''(x) = x + x^2$.

30. A cannon ball is shot from the ground with velocity v at an angle α (Figure 31) so that it has a vertical component of velocity $v \sin \alpha$ and a horizontal component $v \cos \alpha$. Its distance $s(t)$ above the ground obeys the law $s(t) = -16t^2 + (v \sin \alpha)t$, while its horizontal velocity remains constantly $v \cos \alpha$.

(a) Show that the path of the cannon ball is a parabola (find the position at each time t, and show that these points lie on a parabola).

(b) Find the angle α which will maximize the horizontal distance traveled by the cannon ball before striking the ground.

33. Suppose that $|f(x) - f(y)| \leq |x - y|^n$ for $n > 1$. Prove that f is constant by considering f'. Compare with Problem 3-20.

40. Suppose that f is a function such that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. Prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$. Hint: Find $g'(x)$ when $g(x) = f(xy)$.

49. Find the following limits:

(i) $\lim_{x \to 0} \frac{x}{\tan x}$.

(ii) $\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2}$.

51. Prove the following forms of l'Hôpital's Rule (none requiring any essentially new reasoning).

(a) If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, and $\lim_{x \to a^+} f'(x)/g'(x) = l$, then $\lim_{x \to a^+} f(x)/g(x) = l$ (and similarly for limits from below).

(b) If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, and $\lim_{x \to a} f'(x)/g'(x) = \infty$, then $\lim_{x \to a} f(x)/g(x) = \infty$ (and similarly for $-\infty$, or if $x \to a$ is replaced by $x \to a^+$ or $x \to a^-$).

(c) If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to \infty} f'(x)/g'(x) = l$, then $\lim_{x \to \infty} f(x)/g(x) = l$ (and similarly for $-\infty$). Hint: Consider $\lim_{x \to 0^-} f(1/x)/g(1/x)$.

(d) If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to \infty} f'(x)/g'(x) = \infty$, then $\lim_{x \to \infty} f(x)/g(x) = \infty$.

52. There is another form of l'Hôpital's Rule which requires more than algebraic manipulations: If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$, and $\lim_{x \to \infty} f'(x)/g'(x) = l$, then $\lim_{x \to \infty} f(x)/g(x) = l$. Prove this as follows.

(a) For every $\varepsilon > 0$ there is a number a such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon \quad \text{for } x > a.$$

Apply the Cauchy Mean Value Theorem to f and g on $[a, x]$ to show that

$$\left| \frac{f(x) - f(a)}{g(x) - g(a)} - l \right| < \varepsilon \quad \text{for } x > a.$$

(Why can we assume $g(x) - g(a) \neq 0$?)

(b) Now write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \cdot \frac{f(x)}{f(x) - f(a)} \cdot \frac{g(x) - g(a)}{g(x)}$$

(why can we assume that $f(x) - f(a) \neq 0$ for large x?) and conclude that

$$\left| \frac{f(x)}{g(x)} - l \right| < 2\varepsilon \quad \text{for sufficiently large } x.$$

***57.** (a) Let $y \neq 0$ and let n be even. Prove that $x^n + y^n = (x + y)^n$ only when $x = 0$. Hint: If $x_0{}^n + y^n = (x_0 + y)^n$, apply Rolle's Theorem to $f(x) = x^n + y^n - (x + y)^n$ on $[0, x_0]$.

(b) Prove that if $y \neq 0$ and n is odd, then $x^n + y^n = (x + y)^n$ only if $x = 0$ or $x = -y$.

****58.** Use the method of Problem 57 to prove that if n is even and $f(x) = x^n$, then every tangent line to f intersects f only once.

****67.** (a) A point x is called a **strict maximum point** for f on A if $f(x) > f(y)$ for all y in A with $y \neq x$ (compare with the definition of an ordinary maximum point). A **local strict maximum** point is defined in the obvious way. Find all local strict maximum points of the function

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

It seems quite unlikely that a function can have a local strict maximum at *every* point (although the above example might give one pause for thought). Prove this as follows.

(b) Suppose that every point is a local strict maximum point for f. Let x_1 be any number and choose $a_1 < x_1 < b_1$ with $b_1 - a_1 < 1$ such that $f(x_1) > f(x)$ for all x in $[a_1, b_1]$. Let $x_2 \neq x_1$ be any point in (a_1, b_1) and choose $a_1 \leq a_2 < x_2 < b_2 \leq b_1$ with $b_2 - a_2 < \frac{1}{2}$ such that $f(x_2) > f(x)$ for all x in $[a_2, b_2]$. Continue in this way, and use the Nested Interval Theorem (Problem 8-14) to obtain a contradiction.