Exercícios do livro do Spivak – Calculus Capítulo 3

1. Let $f(x) = 1/(1+x)$. What is

(i) $f(f(x))$ (for which x does this make sense?).

(ii) $f\left(\frac{1}{x}\right)$.

(iii) $f(cx)$.

(iv) $f(x+y)$.

(v) $f(x)+f(y)$.

(vi) For which numbers c is there a number x such that $f(cx) = f(x)$. Hint: There are a lot more than you might think at first glance.

(vii) For which numbers c is it true that $f(cx) = f(x)$ for two different numbers x?

2. Let $g(x) = x^2$, and let

$$h(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational.} \end{cases}$$

(i) For which y is $h(y) \leq y$?

(ii) For which y is $h(y) \leq g(y)$?

(iii) What is $g(h(z)) - h(z)$?

(iv) For which w is $g(w) \leq w$?

(v) For which ε is $g(g(\varepsilon)) = g(\varepsilon)$?

3. Find the domain of the functions defined by the following formulas.

(i) $f(x) = \sqrt{1-x^2}$.

(ii) $f(x) = \sqrt{1-\sqrt{1-x^2}}$.

(iii) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$.

(iv) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$.

(v) $f(x) = \sqrt{1-x} + \sqrt{x-2}$.

4. Let $S(x) = x^2$, let $P(x) = 2^x$, and let $s(x) = \sin x$. Find each of the following. In each case you answer should be a *number*.

(i) $(S \circ P)(y)$.

(ii) $(S \circ s)(y)$.

(iii) $(S \circ P \circ s)(t) + (s \circ P)(t)$.

(iv) $s(t^3)$.

6. (a) If $x_1, \ldots, x_n$ are distinct numbers, find a polynomial function f_i of degree $n-1$ which is 1 at x_i and 0 at x_j for $j \neq i$. Hint: the product of all $(x - x_i)$ for $j \neq i$, is 0 at x_j if $j \neq i$. (This product is usually denoted by

$$\prod_{\substack{j=1 \\ j\neq i}}^{n} (x - x_j),$$

the symbol Π (capital pi) playing the same role for products that Σ plays for sums.)

(b) Now find a polynomial function f of degree $n-1$ such that $f(x_i) = a_i$, where $a_1, \ldots, a_n$ are given numbers. (You should use the functions f_i from part (a). The formula you will obtain is called the "Lagrange interpolation formula.")

9. (a) If A is any set of real numbers, define a function C_A as follows:

$$C_A(x) = \begin{cases} 1, & x \text{ in } A \\ 0, & x \text{ not in } A. \end{cases}$$

Find expressions for $C_{A\cap B}$ and $C_{A\cup B}$ and $C_{\mathbf{R}-A}$, in terms of C_A and C_B. (The symbol $A \cap B$ was defined in this chapter, but the other two may be new to you. They can be defined as follows:

$$A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\},$$
$$\mathbf{R} - A = \{x : x \text{ is in } \mathbf{R} \text{ but } x \text{ is not in } A\}.)$$

(b) Suppose f is a function such that $f(x) = 0$ or 1 for each x. Prove that there is a set A such that $f = C_A$.

(c) Show that $f = f^2$ if and only if $f = C_A$ for some set A.

12. A function f is **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$. For example, f is even if $f(x) = x^2$ or $f(x) = |x|$ or $f(x) = \cos x$, while f is odd if $f(x) = x$ or $f(x) = \sin x$.

(a) Determine whether $f + g$ is even, odd, or not necessarily either, in the four cases obtained by choosing f even or odd, and g even or odd. (Your answers can most conveniently be displayed in a 2×2 table.)

(b) Do the same for $f \cdot g$.

(c) Do the same for $f \circ g$.

(d) Prove that every even function f can be written $f(x) = g(|x|)$, for infinitely many functions g.

13. (a) Prove that any function f with domain $\mathbf{R}$ can be written $f = E + O$, where E is even and O is odd.

(b) Prove that this way of writing f is unique. (If you try to do part (b) first, by "solving" for E and O you will probably find the solution to part (a).)

14. If f is any function, define a new function $|f|$ by $|f|(x) = |f(x)|$. If f and g are functions, define two new functions, $\max(f, g)$ and $\min(f, g)$, by

$$\max(f, g)(x) = \max(f(x), g(x)),$$
$$\min(f, g)(x) = \min(f(x), g(x)).$$

Find an expression for $\max(f, g)$ and $\min(f, g)$ in terms of $|\ \ |$.

16. Suppose f satisfies $f(x + y) = f(x) + f(y)$ for all x and y.

(a) Prove that $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$.

(b) Prove that there is some number c such that $f(x) = cx$ for all *rational* numbers x (at this point we're not trying to say anything about $f(x)$ for irrational x). Hint: First figure out what c must be. Now prove that $f(x) = cx$, first when x is a natural number, then when x is an integer, then when x is the reciprocal of an integer and, finally, for all rational x.

17. If $f(x) = 0$ for all x, then f satisfies $f(x+y) = f(x) + f(y)$ for all x and y, and also $f(x \cdot y) = f(x) \cdot f(y)$ for all x and y. Now suppose that f satisfies these two properties, but that $f(x)$ is not always 0. Prove that $f(x) = x$ for all x, as follows:

(a) Prove that $f(1) = 1$.
(b) Prove that $f(x) = x$ if x is rational.
(c) Prove that $f(x) > 0$ if $x > 0$. (This part is tricky, but if you have been paying attention to the philosophical remarks accompanying the problems in the last two chapters, you will know what to do.)
(d) Prove that $f(x) > f(y)$ if $x > y$.
(e) Prove that $f(x) = x$ for all x. Hint: Use the fact that between any two numbers there is a rational number.

21. Prove or give a counterexample for each of the following assertions:

(a) $f \circ (g + h) = f \circ g + f \circ h$.
(b) $(g + h) \circ f = g \circ f + h \circ f$.
(c) $\dfrac{1}{f \circ g} = \dfrac{1}{f} \circ g$.
(d) $\dfrac{1}{f \circ g} = f \circ \left(\dfrac{1}{g}\right)$.

22. (a) Suppose $g = h \circ f$. Prove that if $f(x) = f(y)$, then $g(x) = g(y)$.
(b) Conversely, suppose that f and g are two functions such that $g(x) = g(y)$ whenever $f(x) = f(y)$. Prove that $g = h \circ f$ for some function h. Hint: Just try to define $h(z)$ when z is of the form $z = f(x)$ (these are the only z that matter) and use the hypotheses to show that your definition will not run into trouble.

23. Suppose that $f \circ g = I$, where $I(x) = x$. Prove that

(a) if $x \neq y$, then $g(x) \neq g(y)$;
(b) every number b can be written $b = f(a)$ for some number a.

***24.** (a) Suppose g is a function with the property that $g(x) \neq g(y)$ if $x \neq y$. Prove that there is a function f such that $f \circ g = I$.
(b) Suppose that f is a function such that every number b can be written $b = f(a)$ for some number a. Prove that there is a function g such that $f \circ g = I$.

***25.** Find a function f such that $g \circ f = I$ for some g, but such that there is no function h with $f \circ h = I$.