

## EQUATIONS OF MOTION

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### 5.1 Introduction

We turn now to the formulation of the basic equations—the starting point of the theories that will lead, one hopes, to an understanding of phenomena such as those described in Chapters 2 to 4. These equations are formulations appropriate to a fluid in motion of the usual laws of mechanics—conservation of mass and Newton's laws of motion. In some situations other physical processes may be present, thermodynamic processes for instance, and equations for these are similarly formulated, as we shall examine more fully in Chapter 14.

### 5.2 Fluid particles and continuum mechanics

Before we can proceed with this formulation we need certain preliminary ideas, the most important being the concept of a fluid particle.

The equations concern physical and mechanical quantities, such as velocity, density, pressure, temperature, which will be supposed to vary continuously from point to point throughout the fluid. How do we define these quantities at a point? To do so we have to make what is known as the assumption of the applicability of continuum mechanics or the continuum hypothesis. We suppose that we can associate with any volume of fluid, no matter how small, those macroscopic properties that we associate with the fluid in bulk. We can then say that at each point there is a particle of fluid and that a large volume of fluid consists of a continuous aggregate of such particles, each having a certain velocity, temperature, etc.

Now we know that this assumption is not correct if we go right down to molecular scales. We have to consider why it is nonetheless plausible to formulate the equations on the basis of the continuum hypothesis. It is simplest to think of a gas, although the considerations for a liquid are very similar.

The various macroscopic properties are defined by averaging over a large number of molecules. Consider velocity for example. The molecules of a gas have high speeds associated with their Brownian motion, but these do not result in a bulk transfer of gas from one place to another. The flow velocity is thus defined as the average velocity of many

molecules. Similarly, the temperature is defined by the average energy of the Brownian motion. The density is defined by the mass of the average number of molecules to be found in a given volume. Other macroscopic properties, such as pressure and viscosity, likewise result from the average action of many molecules.

None of these averaging processes is meaningful unless the averaging is carried out over a large number of molecules. A fluid particle must thus be large enough to contain many molecules. It must still be effectively at a point with respect to the flow as a whole. Thus the continuum hypothesis can be valid only if there is a length scale,  $L_2$ , which we can think of as the size of a fluid particle, such that

$$L_1 \ll L_2 \ll L_3 \quad (5.1)$$

where the meanings of  $L_1$  and  $L_3$  are illustrated by Fig. 5.1. This figure uses the example of temperature, rather than the natural first choice of velocity, because it is easier to discuss a scalar. It shows schematically the average Brownian energy of the molecules in a volume  $L^3$  plotted against the length scale  $L$  (on a logarithmic scale). The centre of the volume may be supposed fixed as its size is varied. When the volume is so small that it contains only a few molecules, there are large random fluctuations; the change produced by increasing the volume depends on the particular speeds of the new molecules then included. As the volume becomes large enough to contain many molecules, the fluctuations become negligibly small. A temperature can then meaningfully be defined.  $L_1$  is proportional to, but an order of magnitude or so larger than, the average

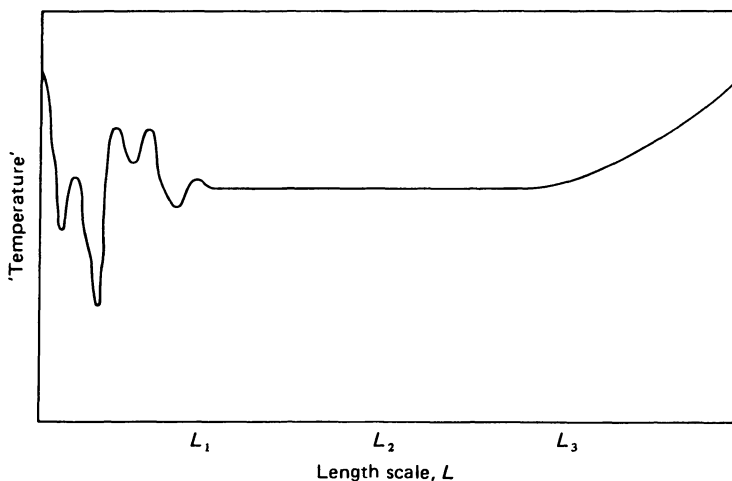


FIG. 5.1 Schematic variation of average energy of molecules with length scale. See text.

distance from a molecule to its nearest neighbour. At the other extreme, the volume may become so large that it extends into regions where the temperature is significantly different. This will result in an increase or decrease in the average.  $L_3$  is a typical length scale associated with the flow; that is, a typical distance over which the macroscopic properties vary appreciably.

The applicability of the continuum hypothesis depends on there being a significant plateau between  $L_1$  and  $L_3$  as shown. One may regard  $L_2$  as being an infinitesimal distance so far as macroscopic effects are concerned, and formulate the equations (as differential equations implicitly involving the limit of small separations) ignoring the behaviour on still smaller length scales.

The same fluid particle does not consist of just the same molecules at all times. The interchange of molecules between fluid particles is taken into account in the macroscopic equations by assigning to the fluid diffusive properties such as viscosity and thermal conductivity. For example (again considering a gas for simplicity) the physical process by which the velocity distribution of Fig. 1.1 generates the stress shown is the Brownian movement of molecules across AB; those crossing in, say, the  $+y$ -direction have on average less  $x$ -momentum and so tend to reduce the momentum of the fluid above AB. The same fluid particle may be identified at different times, once the continuum hypothesis is accepted, through the macroscopic formulation. This specifies (in principle) a trajectory for every particle and thus provides meaning to the statement that the fluid at one point at one time is the same as that at another point at another time. For example, for a fluid macroscopically at rest, it is obviously sensible to say that the same fluid particle is always in the same place—even though, because of the Brownian motion, the same molecules will not always be at that place.

However, for the continuum hypothesis to be plausible, it is evidently necessary for the molecules within a fluid particle to be strongly interacting with one another. If each molecule acted just as if the others were not there, there would be little point in identifying the aggregate as a particle. Thus, if  $\lambda$  is the molecular-mean-free path, continuum mechanics can be applied only if

$$\lambda \ll L_2 \quad (5.2)$$

so that each molecule undergoes many collisions whilst traversing a distance that can still be regarded as infinitesimal. Since  $\lambda$  can be large compared with  $L_1$  as defined above, this is an additional requirement to (5.1).

Once the continuum hypothesis has been introduced, we can formulate the equations of motion on a continuum basis, and the molecular

structure of the fluid need not be mentioned any more. Hence, although the concepts developed above underlie the whole formulation, we shall not have much occasion to refer back to them. Velocity, henceforth, will be either a mathematical quantity or something (hopefully equivalent) that one measures experimentally. So will all the other parameters. Their definitions as averages over molecules provide answers to the implicit, but rarely explicit, questions: ‘What is the real physical meaning of this mathematical quantity?’ ‘What quantity does one ideally wish to measure?’

The continuum hypothesis is only a hypothesis. The above discussion suggests that it is plausible, but nothing more. The real justification for it comes subsequently, through the experimental verification of predictions of the equations developed on the basis of the hypothesis.

### 5.3 Eulerian and Lagrangian coordinates

In setting up the equations governing the dynamics of a fluid particle, we evidently need to decide whether we should use coordinates fixed in space or coordinates that move with the particle. These two procedures are known respectively as the Eulerian and Lagrangian specifications. The equations are much more readily formulated using the former because the Lagrangian specification does not immediately indicate the instantaneous velocity field on which depend the stresses acting between fluid particles. Throughout this book we use only the Eulerian specification; i.e. we write the velocity

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) \quad (5.3)$$

where  $\mathbf{r}$  is the position in an inertial frame of reference and  $t$  is time. Values of  $\mathbf{u}$  at the same  $\mathbf{r}$  but different  $t$  do not, of course, correspond to the same fluid particle.

It is not always easy to relate Lagrangian aspects, such as the trajectories of fluid particles, to an Eulerian specification. In the context of this book, this is particularly relevant to the interpretation of flow visualization experiments in which dye marks certain fluid particles. The relationship of the observed patterns to the corresponding Eulerian velocity field may not be simple.

### 5.4 Continuity equation

We are now ready to start on the actual formulation of the equations. We consider first the representation of mass conservation, often called continuity.