Performance Analysis of QAM-Like Constellations in Hyperbolic Space

Abstract

We address in this paper the performance analysis of 16 and 64-QAM signal constellations in Euclidean space and the $\{15, 3\}$ and $\{63, 3\}$ signal constellations in hyperbolic space. In order to perform the analysis we derive the probability density function of the gaussian noise in one and two dimensional hyperbolic spaces. From this, we determine the probability of error of each one of the previously mentioned signal constellations. It is shown that the $\{15, 3\}$ and $\{63, 3\}$ signal constellations in hyperbolic space have coding gains of 5.36 and 4.49 dBs when compared to the 16 and 64-QAM signal constellations. These coding gains can be achieved in Euclidean space by use of Ungerboeck codes with memory $\mu = 6$ and $\mu = 5$, respectively.

1 Introduction

The signal constellations often used in communication systems are the M-PSK and the M-QAM. It is well known that, under certain conditions, to achieve a given bit error probability the M-QAM constellation needs less signal tp noise ratio than the M-PSK, for a fixed M.

Signal constellations in $\mathbb{R}^2$ having interesting geometric properties are the ones considered as finite subsets of the Euclidean tessellations $\{4, 4\}$ and $\{6, 3\}$ ($\{6, 3\}$ denotes hexagons where 3 hexagons meet at each vertex), that is, signal constellations derived from the $\mathbb{Z}^2$ and $\mathbb{A}_2$ lattices. Unfortunately, these subsets can not be reproduced in the hyperbolic plane. Since our primary interest is in the performance analysis of a communication system using signal constellations derived from the $\mathbb{Z}^2$ lattice, then for a specified number of signal points in $\mathbb{Z}^2$ we have to choose a subset with the same number of signal points in a hyperbolic tessellation.. However, this is not a simple selection, for the following reasons: 1) there are infinite tessellations in the hyperbolic plane; 2) once p and q are fixed, so is the distance between points in the tessellation in $\mathbb{H}^2$, since in this geometry the concept of similarity does not hold.

This paper is organized as follows. In Section 2, we review briefly the basic concepts and elements of hyperbolic geometry. In Section 3, we establish the gaussian probability density function in $\mathbb{H}^2$. In Section 4, the performance analysis is realized.

2 Overview of Hyperbolic Geometry

Historically, hyperbolic geometry was developed first as an axiomatic geometry, with a system of axiom similar to the one adopted in the Euclidean Geometry, except for the Parallel Postulate: *"Given a line and a point not on it, there is a unique line passing through the given point and parallel to the given line ".* In hyperbolic geometry, this axiom is replaced by the following: *"Given a line and a point not on it, there are at least two lines passingthrough the given point and parallel to the given line".* It was a long time until the consistence of this geometry was accepted, *i.e.*, the nonexistence of paradox. This was done by the introduction of models to this geometry, made by mathematicians as Poincaré, Klein and Lobatchevsky.

In this paper, we avoid the axiomatic discussion and go directly to the models. In this paper we assume the Poincaré model for the hyperbolic plane. As a point set, it is just the unit open disk $\mathbb{H}^2 = \left\{ z = x + iy \in \mathbb{C} |\, |z| = \sqrt{x^2 + y^2} < 1 \right\}$. For those that are acquainted with differential (or Riemannian) geometry, we say the *line element* in $\mathbb{H}^2$ is defined by $ds^2 = \dfrac{dx^2 + dy^2}{1 - x^2 - y^2}$. The meaning of this is that, given a parametrized curve

$$\begin{aligned} \gamma &: [a, b] \to \mathbb{H}^2 \\ &: t \mapsto x(t) + iy(t) \end{aligned}$$

we define its length by the formula

$$l(\gamma) = \int_a^b \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{1 - x^2 - y^2}} dt.$$

where $x'(t) = \frac{d}{dt} x(t)$.

Knowing how to calculate the length of curves, we can define a *geodesic* as the "shortest path " between any two given points. Beside these definitions, the model in consideration is well behaved and we can find elementary descriptions of all of its important features, namely:

1. A geodesic, the shortest path between any two of its points, is either the arc of a circumference orthogonal to the unitary circle $\{z \in \mathbb{C} |\, |z| = 1\}$ or the diameter of the disk $\mathbb{H}^2$. In particular, given two points, there is one and only one geodesic connecting them.

2. The distance between two points $z, w \in \mathbb{H}^2$ is defined by the function

   $$\rho(z, w) = \ln \left(\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|} \right).$$

   We should note that the hyperbolic plane is unbounded: take for example $z = 0$ and then it is easy to see that

   $$\rho(0, w) = \ln \left(\frac{|1| + |w|}{|1| - |w|} \right)$$

   and this goes to ∞ as $|w| \to 1$.

3. The angle between any two given curves is the same as they look to our Euclidean eyes, that is, if $\alpha(t) = a_1(t) + ia_2(t)$ and $\beta(t) = b_1(t) + ib_2(t)$ are curves meeting at $x_0 = \alpha(0) = \beta(0)$, then $\cos\theta = \dfrac{\langle \alpha'(0), \beta'(0)\rangle}{\|\alpha'(0)\| \|\beta'(0)\|}$, where θ is the angle between the curves, $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$ and $\|\cdot\|$ the usual norm.

4. An isometry of the hyperbolic Poincaré disc is a bijection $T : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\rho(z, w) = \rho(T(z), T(w))$ for every $z, w \in \mathbb{H}^2$. We have a simple description of all the (orientation preserving) isometries of $\mathbb{H}^2$: they are just the maps

$$h(z) = \frac{az + \overline{c}}{cz + \overline{a}}$$

where $a, c \in \mathbb{C}$ are any complex numbers such that $|a|^2 - |c|^2 = 1$. We observe that the group of isometries of the hyperbolic plane is as large as the group of isometries of the Euclidean plane: both are 3-dimensional, the largest possible dimension for a 2-dimensional geometric manifold. Moreover, given any two triangles with equal sizes, there is an isometry carrying one into the other (and these determines uniquely the isometry).

Although all these common properties, the existence of more then one parallel (in fact, infinitely many) is a strong difference between hyperbolic and Euclidean spaces. Another (equivalent) way to state this difference is to say that the sum of the internal angles of any triangle in $\mathbb{H}^2$ is strictly *less* than π. This is the fact that will enable us to construct many essentially different regular tessellations of $\mathbb{H}^2$.

A *regular n-gon* is a region bounded by n geodesic segments with the same length, having equal angle between any two subsequent line segments. Contrary to the Euclidean case, the angle between subsequent line segments is not determined by the number of sides of the polygon: these equal angles can be as small as we wish. This surprising fact will permit us to tessellate $\mathbb{H}^2$ in an infinitude of ways: Given two positive integers p and q such that $(p-2)(q-2) > 4$, we can construct (algorithmically) a polygon with p equal sides and having equal internal angles measuring $\dfrac{2\pi}{q}$. Hence, when gluing them side by side, all of the angles meeting at a given vertex sum up to 2π, and so, $\mathbb{H}^2$ can be tesselated by isometric copies of the given fundamental polygon, also known as Voronoi region (an illustration of such tessellation is shown in the picture **??**). To each of these tessellations we may associate a group of isometries, namely the group generated by the isometries that identify the polygons of the tessellation.

$$itbpFU2.3929in2.3929in0inAn$$

$\{8, 3\}$ tesselation of the hyperbolic plane.figuraisit1.eps

3 Establishing the Gaussian Probability Density Function in $\mathbb{H}^2$

In this section we develop a systematic approach to derive the gaussian probability density function (*pdf*) in $\mathbb{H}^2$.

This problem at first seems to be simple to solve. However, the difficulty lies in the fact that there is no isometry between $\mathbb{H}^2$ and $\mathbb{R}^2$. Therefore, we have to start by exploring judiciously some of the properties inherited by the gaussian *pdf* in the Euclidean plane. Our first assumption is that both random variables have the same variance σ^2. Thus,

$$p(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}(x^2 + y^2)\right).$$

This function may be written as $p(P) = k_1 \exp(k_2 d_E^2(P, 0))$ where $P = (x, y)$ and d_E is the Euclidean distance.

Thus, the problem is to determine the constants A and B such that

$$f(z) = B \exp(-A\rho^2(z, 0)), \qquad z = x + iy \in \mathbb{H}$$

is in fact a gaussian probability density function in $\mathbb{H}^2$. After some algebraic manipulations and computer calculations we end up with the following relationship between A and B,

$$B = \frac{\sqrt{A}}{\sqrt{\pi^3}} e^{-\frac{1}{4A}} \left[erf\left(\frac{1}{2\sqrt{A}}\right)\right]^{-1}.$$

Finally,

$$f(z) = \frac{\sqrt{A}}{\sqrt{\pi^3}} e^{-\frac{1}{4A}} \left[erf\left(\frac{1}{2\sqrt{A}}\right)\right]^{-1} \exp\left(-A \ln^2\left(\frac{1 + |z|}{1 - |z|}\right)\right).$$

Note that it is also possible to define the marginal *pdf* in the hyperbolic plane by use of line integral. We also show that the resulting gaussian *pdf* in $\mathbb{H}^2$ satisfies the following properties:

- the intersections of $f(z)$ with $z = k$ are hyperbolic circles;
- the intersections of $f(z)$ with $x = k$ or $y = k$ reproduces the *pdf* in $\mathbb{H}$;
- the projection of the intersection of two gaussian *pdf* in the hyperbolic plane, $f_1(z) \cap f_2(z)$, is a geodesic. This is a crucial fact that allows us to define the Voronoi region in the hyperbolic plane similarly as it is defined in the Euclidean plane, having the same essential property: Convexity.

From the previous statements and concepts we are able to establish the model of the communication channel in consideration. Similar to the Euclidean channel model case, that is, let $x(t)$ be the transmitted signal, and $n(t)$ the additive gaussian noise. The received signal is $y(t) = x(t) + n(t)$. Since the hyperbolic space does not possess the structure of a vector space, an interpretation of this addition has to be made. Our interpretation is as follows: let $x(t)$ and $n(t)$ be defined as $x(t) : \mathbb{R}^+ \to \mathbb{R}^2$, and $n(t) : \mathbb{R}^+ \to \mathbb{R}^2$. Hence, the noise $n(t)$ acts on the signal yielding the received signal given by $y(t) = x(t) + n(t)$. We may consider this function $y(t)$ as a one parameter family of Euclidean translations $T_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_t(x(t)) = x(t) + n(t),$$

that is, additive gaussian noise is a family of Euclidean isometries (translations) acting on the signal $x(t)$.

Let $x(t) : \mathbb{R}^+ \to \mathbb{H}^2$ be a signal in $\mathbb{H}^2$. We say that hyperbolic gaussian noise acting on $x(t)$, is a one parameter family $N_t(z) : \mathbb{H}^2 \to \mathbb{H}^2$ of hyperbolic isometries such that $N_t(x(t))$ is a random process characterized by the probability density function as defined in (3).

This definition establishes that the received signal $y(t)$ is of the form $y(t) = N_t(x(t))$.

4 Performance Analysis

From the infinite number of possible tessellations $\{p, q\}$, we consider the tessellation $\{p, 3\}$, where $p > 6$. For a fixed value of p, let P_0 be a p-sided polygon, called the *fundamental polygon* of the tessellation.. The first level of $\{p, 3\}$ consists of the set whose elements are the reflections of P_0 in each one of its p sides. Let us denote this set by L_1. In this first level there are two sets of points which we can use as signal points of a signal constellation, namely, the set whose elements are the vertices of the polygons in the set L_1, and the set whose elements are the centers of the polygons in the set L_1. The number of elements in the first set is $2p + p(p - 4)$ whereas the number of elements in the second set is $p + 1$. The latter is more interesting due to the fact that for a given value of n, $n > 6$, it is always possible to find a hyperbolic constellation with n signal points. For that, it is sufficient to consider the first level of the tessellation $\{p, 3\}$ with $p = n - 1$ signal points. Since we are interested in comparing the performance of the 16 and 64-QAM with the corresponding signal constellations in the hyperbolic plane, then the first level of the tessellations $\{15, 3\}$ and $\{63, 3\}$ have to be used, respectively.

In order to calculate the error probabilities, we consider the unity open disk model and the fundamental polygon of the tessellation $\{15, 3\}$ has its center at the origin and fifteen signal points in a hyperbolic circle with radius r_h. We assume the noise in the Euclidean plane is gaussian with zero mean and variance 1, whereas the noise in the hyperbolic plane is also gaussian with zero mean, however, with pseudo-variance 1. We show that the hyperbolic gaussian *pdf* with center in each one of the signal points in $\mathbb{H}^2$ is given by

$$f_j(z) = 0.184164 \exp\left(-\frac{1}{2}\ln^2\left(\frac{|1 - z\overline{w_j}| + |z - w_j|}{|1 - z\overline{w_j}| - |z - w_j|}\right)\right)$$

No. of Points	$P_C^h = P_C^e$	$E_M^h = r_h^2$	$E_M^e = r_e^2$	d_h	d_e	G (dB)
16	0.81799	8.71317	22.80	1.00704	3.02	5.36
64	0.99975	35.3507	457.38	1.09363	6.6	4.49

Table 1: Coding gain.

where $z = x + yi$, and $w_j = (a_j, b_j)$, $0 \leq j \leq m$ where $m = 15, 63$, are the coordinate of the signal points.

From the characterization of $f_j(x, y)$, the Voronoi region of each one of the signal points in the corresponding constellations are determined. As a consequence of the symmetry of the fifteen regions in the first level, the correct probability associated with each one of these regions will be denoted by p_1. Let p_0 be the correct probability associated with the fundamental region. Therefore, the average correct probability is given by $P_C^h = \frac{mp_1+p_0}{m+1}, \quad m = 15, 63$. To calculate p_0 we compute the integral $\int_{L_0} f_0(z) \left[\frac{2}{1-x^2-y^2}\right]^2 dydx$ where L_0 is the fundamental polygon and we use the same procedure to calculate p_1.

On the other hand, the average energy of a signal constellation with equally likely signal points $P_1 \ldots P_k$, both Euclidean and hyperbolic, with mass center in Q is given by $E_M^s = \frac{1}{k}\sum_{i=1}^{k} d_s^2(P_i, Q), \quad k = 16, 64$, where d_s^2 is the hyperbolic ($s = h$) or Euclidean ($s = e$) distances between P_i and Q, and E_M^s is the hyperbolic ($s = h$) or the Euclidean ($s = e$) average energies.

To calculate the correct probability associated with each signal point in the QAM constellations in the Euclidean space, we used the Euclidean gaussian *pdf* given by

$$g_j(x, y) = \frac{1}{2\pi} \exp(-\frac{1}{2}(x - x_i)^2 - \frac{1}{2}(y - y_i)^2),$$

where $P_i = (x_i, y_i)$ are the coordinates of the signal points associated with the 16 and 64-QAM constellations. We have used the classical procedures both to determine the Voronoi regions as well as the correct probabilities, and so they will not be presented.

Table 1, illustrates the results obtained from the comparison of the signal constellations in the Euclidean and hyperbolic planes. Note that for a fixed value of the average correct probability (error) (first column) the performance of the communication system using signal constellations in the hyperbolic plane have a coding gain of 5.36 dB e 4.49 dB when compared with the 16 and 64-QAM constellations, respectively, where the figure of merit used is $G = 10\log\left[\frac{d_h^2}{E_M^h}\frac{E_M^e}{d_e^2}\right]$.

Table 2, illustrates the correct probabilities both in the hyperbolic and in the Euclidean planes for a fixed value of the average energy.

References

[1]

No. of Points	P_C^h	P_C^e	$E_M^h = r_h^2 = E_M^e = r_e^2$
16	0.81799	0.54325	8.71317
64	0.99975	0.47050	35.3507

Table 2: Performance analysis in the hyperbolic and in the Euclidean planes.

[2] A.F. Beardon, *The Geometry of Discrete Groups*, Springer Verlag, New York, 1982.

[3] E.B. da Silva, and R. Palazzo, Jr., S.R. Costa, "Improving the performance of asymmetric M-PAM signal constellations in Euclidean space by embedding them in hyperbolic space," *Proceedings 1998 IEEE Information Theory Workshop*, Killarney, Ireland, June 22-26, pp.98-99.

[4] E.B. da Silva, and R. Palazzo, Jr., "M-PSK Signal Constellations in Hyperbolic Space Achieving Better Performance Than the M-PSK Signal Constellations in Euclidean Space," *1999 IEEE Information Theory Workshop*, Metsovo, Greece, June 27 - July 1.