Sphere Packing: asymptotic behavior and existence of solution

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Abstract. Lattices in n-dimensional Euclidean spaces may be parameterized by the non-compact symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. We consider sphere packings determined by lattices and study the density function in the symmetric space, showing that the density function $\rho(A_k)$ decreases to 0 if A_k is a sequence of matrices in $\mathrm{SL}(n,\mathbb{R})$ with $\lim_{k\to\infty}\|A_k\| = \infty$. As a consequence, we give a simple prove that the optimal solution for the sphere packing problem is attained.

The *sphere packing problem* is one of the famous open problems in mathematics. In short, it asks about the densest way a set of equal spheres can be packed in space n-dimensional Euclidean space $\mathbb{R}^n$, without overlapping one the other. In this context, the density means the proportion between the covered and the uncovered amount of space. It has many variations: one could replace spheres of equal radii by spheres of radii $0 < a \leq r \leq b$ bounded from above and below, replace spheres by a collection of identical (preferably convex) bodies, Euclidean space may be replaced by elliptical or hyperbolic space, and many other variations.

All the above mentioned variations are generalizations, and actually, very few is known in those cases. The case we deal here is a restriction of the classical problem, because we assume the center of the spheres to form a lattice in $\mathbb{R}^n$. This condition is not as restrictive as it appears, since also in this case, not much is known.

What we do here is to parametrize all lattice sphere-packings in Euclidean space and to study the behavior of the packings as the parameters get arbitrarily large, that is, we study the asymptotic (as a function of its parameters) behavior of the packings. As a consequence, we give a proof of (the known) fact that the problem has a solution (see for example [3] for a proof of this result with much weaker conditions).

1 Lattice Sphere Packings The sphere packing problem consists of finding out the densest possible way to pack identical spheres in $\mathbb{R}^n$, when the number of sphere increase and the radius of the spheres is kept fixed. We fix some $R > 0$ and imagine an enumerable family $\mathcal{F} = \{B_k\}_{k\in\mathbb{N}}$ of disjoint open balls of equal radii R, distributed in $\mathbb{R}^n$, called a *sphere packing*. It may be misleading but, to maintain the tradition in this context, we will reffer to the balls of a packing as a sphere. We consider a ball of radius t, centered at a (fixed) point x:

$$B_t = B(x;t) = \{x \in \mathbb{R}^n |\ \|x\| < t\}.$$

The density of the distribution is the limit

$$\begin{aligned}\rho(\mathcal{F}) &= \limsup_{t\to\infty} \frac{\sum_{k=1}^{\infty} \mu(B_t \cap B_k)}{\mu(B_t)} \\ &= \limsup_{t\to\infty} \frac{\text{volume of the spheres of } \mathcal{F} \text{ intersecting } B_t}{\text{volume of } B_t}.\end{aligned}$$

Obviously, the density of a sphere packing is smaller then 1. The problem posed is to find a packing with greater density and to determine its density. To give the problem a solution is enough to tell what the centers of the spheres are. Posed in this way, in its full generality, the solution is known only for $k = 2$ (an easy instance). For $k = 3$, there is still no consensus among the mathematical community about the correctness of the proof presented by S.P. Ferguson and T.C. Hales in 1998 (see [4] for a brief discussion on this matter). An enormous and updated account of the problem is found in [1]; in [2] one can find a less encyclopedic but very readable survey on the subject.

To study the problem in its generality, we consider a discrete subset $\mathcal{A} = \{a_i \in \mathbb{R}^n | i = 1, 2, ...\}$. We also assume that

$$0 < R = \inf\{\|a_i - a_j\| \,|a_i, a_j \in \mathcal{A}, i \neq j\}.$$

Then, if we consider the spheres

$$B_i = B(a_i; R/2) = \{x \in \mathbb{R}^n |\ \|x - a_i\| < R/2\}$$

we get a family of nonintersecting spheres. Moreover, $R/2$ is maximum in the sense there are i, j distinct such that the closed balls B_i and B_j intersects. So, we have an unique optimal sphere packing $\mathcal{F} = \{B(a_i; R/2) | a_i \in \mathcal{A}\}$ determined by the subset $\mathcal{A}$. We call such R the *minimal distance* of (the distribution $\mathcal{A}$) the packing $\mathcal{F}$ and $R/2$ its *inscribed radius*.

We can estimate the density of the packing by taking a certain sum. For a given $a_i \in \mathcal{A}$, we define the *Voronoi domain of* $\mathcal{A}$ *centered at the point* a_i to be the closure of the set of points in $\mathbb{R}^n$ that are closer to a_i then to any other $a_j \in \mathcal{A}$:

$$\mathcal{D}_{a_i} = \mathcal{D}_{a_i}(\mathcal{A}) = \{x \in \mathbb{R}^n |\ \|x - a_i\| \leq \|x - a_j\|; a_i \neq a_j \in \mathcal{A}\}.$$

Each Voronoi domain is a measurable set and one can prove that

$$\begin{aligned}\rho &= \limsup_{t\to\infty} \frac{\sum_k \mu(B_t \cap B_n)}{\mu(B_t)} \\ &= \limsup_{N\to\infty} N \cdot \mu_{R/2,n} \left(\sum_{i=1}^{N} \frac{1}{\mu(\mathcal{D}_i)} \right)\end{aligned}$$

where $\mu_{R/2,n}$ is the volume of the n-dimensional sphere of radius $R/2$.

We will deal here with distributions that has some regularity properties. To understand such properties, let us examine closely the solution for $k = 2$. We consider a tessellation of $\mathbb{R}^2$ by regular hexagons and the family of spheres inscribed in the hexagons. All the hexagons and all spheres are identical (isometric). Let us suppose each inscribed sphere has radius R and volume π. Then each hexagon has sides of length $\frac{2}{\sqrt{3}}R$ and volume equal $2\sqrt{3}R^2$. Standard arguments concerning limits of sequences show that

$$\rho(\mathcal{F}) = \frac{\text{volume of a single sphere}}{\text{volume of a single hexagon}} = \frac{\pi R^2}{2\sqrt{3}R^2} = \frac{\pi}{2\sqrt{3}} \sim 0,9069.$$

The fact that the density of a packing, an asymptotic quantity, is attained by the density of a single sphere inscribed in a single hexagon is not much surprising, because of the regularity of the tiling of a plane by regular hexagons. It happens because the centers of the spheres in this packing constitute a subgroup (known as A_2) of $\mathbb{R}^2$, the subgroup of all integer linear combination of the points $(2R, 0)$ and $\left(R, \sqrt{3}R\right)$:

$$\left\{m(2R,0) + n\left(R, \sqrt{3}R\right) | m, n \in \mathbb{Z}\right\}.$$

The regular hexagons with inscribed circles of radius R are just the Voronoi domains of the those points.

Such a subgroup is what we call a *lattice* in $\mathbb{R}^n$: a discrete subgroup Γ of the (additive) group $(\mathbb{R}^n, +)$ with compact quotient $\mathbb{R}^n/\Gamma$. With the hypothesis that the spheres are centered at lattice points, the sphere packing problem is solved in a wider range of cases: for $k \leq 8$.

In order to understand the lattice packing problem, we need to introduce and explain some concepts and definitions.

The inclusion $\mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice. Actually, it is not difficult to prove that every lattice $\Gamma \subset \mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$. We can do it by induction. To make the induction step, take a single vector $v \in \Gamma$. Since Γ is discrete, there is a smaller $\lambda \in \mathbb{R}^+$ such that $\lambda v \in \Gamma$. By taking quotients, we find that $\Gamma/\mathbb{R}v$ is a discrete subgroup of the $(n-1)$-dimensional vector space $\mathbb{R}^n/\mathbb{R}v$, and thus we can use the induction hypothesis.. In other words, every lattices in $\mathbb{R}^n$ is the $\mathbb{Z}$ span

$$\{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n | \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{Z}\}$$

of some base $\{v_1, v_2, ..., v_n\}$.

The invariants of a packing defined above - minimal distance, Voronoi domain and density - are much more easy to compute in the lattice case when compared to the general case.

Since Γ is closed for the sum of vectors, the difference $a_i - a_j \in \Gamma$, whenever $a_i, a_j \in \Gamma$. So,

$$\begin{aligned} R &= \inf\{\|a_i - a_j\| \, | a_i, a_j \in \Gamma, i \neq j\} \\ &= \inf\{\|a_i\| \, | a_i \in \Gamma, a_i \neq 0\}. \end{aligned}$$

For the Voronoi domain, we have that

$$\begin{aligned} x &\in \mathcal{D}_{a_{i_0}}(\mathcal{A}) \Leftrightarrow \|x - a_{i_0}\| \leq \|x - a_i\|, \forall a_i \in \Gamma \\ &\Leftrightarrow \|(x - a_{i_0})\| \leq \|(x - a_{i_0}) - (a_i - a_{i_0})\|, \forall a_i \in \Gamma \\ &\Leftrightarrow \|(x - a_{i_0})\| \leq \|(x - a_{i_0}) - a_j\|, \forall a_j \in \Gamma \\ &\Leftrightarrow (x - a_{i_0}) \in \mathcal{D}_0(\mathcal{A}). \end{aligned}$$

that is, all Voronoi domains are nothing but a translate of any one of those domains, hence, they are isometric and in particular, have the same volume. Hence, for the density of the packing, we find that

$$\rho(\mathcal{F}) = \frac{\text{volume of a single sphere}}{\text{volume of a single Voronoi domain}} = \frac{\mu\left(B_{R/2,n}\right)}{\mu(\mathcal{D}_0)}.$$

2 Lattice Packings: Asymptotic Behavior and Existence of Solution

From here on, we consider only lattice packings. As we saw earlier, a lattice in $\mathbb{R}^n$ is defined by the choice of a basis $\{v_1, ..., v_n\}$. If we fix a base $\{e_1, ..., e_n\}$ of $\mathbb{R}^n$, there is an unique element $A \in GL(n, \mathbb{R})$ such that $A(e_i) = v_i$, $i = i, ..., n$. So, we may consider $GL(n, \mathbb{R})$ as the space of all lattice, or the space of all lattice packings. We define

$$\begin{aligned} \rho^* &: GL(n, \mathbb{R}) \to [0, 1] \\ &: A \mapsto \rho^*(A) \end{aligned} \tag{1}$$

where $\rho^*(A)$ is the density of the packing of spheres of radius

$$R = \frac{1}{2} \min\left\{\|A(\lambda_1 e_1 + ... + \lambda_n e_n)\| \,|\lambda_i \in \mathbb{Z} i = 1, ..., n\right\},$$

centered in the lattice points

$$\{\alpha_1 A(e_1) + ... + \alpha_n A(e_n) \,|\alpha_1, ..., \alpha_n \in \mathbb{Z}\}.$$

What we will do here is to understand the behavior of the density of the packing associated to linear transformations $A \in GL(n, \mathbb{R})$. From here on, we fix a base $\alpha = \{e_1, ..., e_n\}$ and the associated lattice $\mathcal{A} = \{a_i\} = \{\alpha_1 e_1 + ... + \alpha_n e_n | \alpha_1, ..., \alpha_n \in \mathbb{Z}\}$.

The center of $GL(n)$ is the subgroup $\mathrm{Z} = \{\lambda \mathrm{Id} | 0 \neq \lambda \in \mathbb{R}\}$, where Id is the identity matrix. Being the center, we find that $GL(n) = \mathrm{Z} \oplus SL(n)$. The action of Z on a lattice is not trivial, but its influence on the density of the packing is so: each element of Z acts as an homotety, so that given a lattice $\mathcal{A} = \{a_i\}$ with inscribed radius $R/2$ and Voronoi domain $\mathcal{D}_0$, we have that the inscribed radius of $\lambda\mathcal{A} = \{\lambda a_i\}$ is $\lambda R/2$, and $\lambda\mathcal{D}_0 = \{\lambda x | x \in \mathcal{D}_0\}$ is a Voronoi domain of $\lambda\mathcal{A}$. Hence we find that

$$\begin{aligned} \rho^*(\lambda \mathrm{Id}) &= \rho(\lambda\mathcal{A}) \\ &= \frac{\mu(B(0, \lambda R/2))}{\mu(\lambda\mathcal{D}_0)} \end{aligned}$$

$$\begin{aligned} &= \frac{\lambda^n \mu\left(B\left(0, R/2\right)\right)}{\lambda^n \mu\left(\mathcal{D}_0\right)} \\ &= \frac{\mu\left(B\left(0, R/2\right)\right)}{\mu\left(\mathcal{D}_0\right)} \\ &= \rho\left(\mathcal{A}\right) \\ &= \rho^*\left(\mathrm{Id}\right). \end{aligned}$$

So, in order to study the density function of a lattice packing, we may restrict ourselves to lattices obtained as an $GL\left(n,\mathbb{R}\right)/\mathrm{Z} = \mathrm{Z} \oplus SL\left(n,\mathbb{R}\right)/\mathrm{Z} \simeq SL\left(n,\mathbb{R}\right)$ orbit, that is, to lattices generated by vectors $\{Be_1, Be_2, ..., Be_n\}$ where $\{e_1, e_2, ..., e_n\}$ is a given and fixed base of $\mathbb{R}^n$ and $B \in SL\left(n\right)$.

Moreover, given $B \in O\left(n\right)$, it acts in $\mathbb{R}^n$ as isometry, thus both $\mu\left(\mathcal{D}_0\right)$ and $\mu\left(B\left(0, R/2\right)\right)$ are preserved by the action of B, and in particular, $\rho\left(B\left(\mathcal{A}\right)\right) = \rho\left(\mathcal{A}\right)$. So, we may reduce our universe of search of optimal (respective to the maximization of the density function) solution to the symmetric space $SL\left(n\right)/SO\left(n\right)$, the space of positive definite symmetric forms.

Considering the polar decomposition $G = KA^+K$ (valid for any semi-simple Lie group of non-compact type), in our particular case we get that $K = SO\left(n\right)$ and

$$A^+ = \left\{ \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix} | \lambda_1 \geq \cdots \geq \lambda_n, \lambda_1 + \cdots + \lambda_n = 0 \right\}.$$

We will adopt the notation $\mathrm{diag}\left(\lambda_1, .., \lambda_n\right) = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}$.

What we are going to prove is the following:

Theorem 1 *Let $E_n \in KA^+K$ be a sequence such that $\lim_{n\to\infty} \|E_n\| = \infty$. Then, $\lim_{n\to\infty} \rho\left(E_n\left(\mathcal{A}\right)\right) = 0$.*

Here we may consider any usual norm for a matrix, for example , if $E = \left(e_{ij}\right)_{i,j=1}^n$, we may take $\|E\| = \max\left\{|e_{ij}| \, | i,j = 1, ..., n\right\}$.

To prove the theorem, we need the following:

Lemma 1 *Let $\{v_1, ..., v_n\}$ be an orthonormal base of $\mathbb{Q}^n$. Then, given $B = \mathrm{diag}\left(\lambda_1, .., \lambda_n\right) \in A^+$, there are sequences $m_1\left(k\right), ..., m_n\left(k\right) \in \mathbb{Z}$ such that*

$$\lim_{k\to\infty} \left\| B^k \left(m_1\left(k\right) v_1 + m_2\left(k\right) v_2 + ... + m_n\left(k\right) v_n\right)\right\| = 0.$$

In other words, there is a sequence of vectors

$$w_k = m_1\left(k\right) v_1 + m_2\left(k\right) v_2 + ... + m_n\left(k\right) v_n \in \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus ... \oplus \mathbb{Z}v_n$$

such that

$$\lim_{k\to\infty}\left\|B^{k}\left(w_{k}\right)\right\|=0.$$

Proof: Let $\{e_1,e_2,...,e_n\}$ be the canonical base of $\mathbb{R}^n$ and consider $(\varepsilon_k)_{k=1}^{\infty}$ a sequence converging to 0.

Writing each $v_i=\sum_{j=1}^{n}a_{ij}e_j$ as a linear combination of the base, we have that each $a_{ij}\in\mathbb{Q}$, because each $v_i\in\mathbb{Q}^n$. Also, we find that

$$\begin{aligned} w_k &= \sum_{i=1}^{n} m_i(k)\, v_i \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} m_i(k)\, a_{ij}e_j\right) \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e_j \end{aligned}$$

and since each e_j is an eigenvector of B^k corresponding to the eigenvalue $e^{k\lambda_j}$ we find that

$$\begin{aligned} B^k(w_k) &= B^k\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e_j\right) \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e^{k\lambda_j}e_j. \end{aligned}$$

But $\lim_{k\to\infty}\left\|B^k(w_k)\right\|=0$ if and only if $\lim_{k\to\infty}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e^{k\lambda_j}=0$, for every $j=1,2,...,n$. Let us observe that we need not to care about those j for which $\lambda_j<0$: in those cases we only need to choose sequences $(m_i(k))_{k=1}^{\infty}$ that grows less then exponentially with k. Taking the square of the norm we find that

$$\begin{aligned} \left\|\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e^{k\lambda_j}\right\|^2 &= \left(\sum_{i=1}^{n} m_i^2(k)\, a_{ij}^2 + 2\sum_{1=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij}a_{lj}\right)e^{2k\lambda_j} \\ &= \begin{pmatrix} m_1^2(k)\, a_{1j}^2 + m_1(k)\left(2\sum_{i=2}^{n} m_i(k)\, a_{1j}a_{ij}\right) \\ +2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij}a_{lj} \end{pmatrix} e^{2k\lambda_j} \end{aligned}$$

and so,

$$\left\|\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right)e^{k\lambda_j}\right\|<\varepsilon_k \tag{2}$$

if and only if

$$\begin{aligned} & m_1^2(k)\left(a_{1j}^2\right) + m_1(k)\left(2\sum_{i=2}^{n} m_i(k)\, a_{1j}a_{ij}\right) \\ & +\left(\left(2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij}a_{lj}\right)-\frac{\varepsilon_k}{e^{2k\lambda_j}}\right)<0. \end{aligned} \tag{3}$$

This is a quadratic polynomial in $m_1(k)$ with (possibly real) solutions

$$m_1(k)=\frac{-2\sum_{i=2}^{n} m_i(k)\, a_{1j}a_{ij}\pm\sqrt{\Delta}}{2a_{1j}^2} \tag{4}$$

where

$$\begin{aligned}
\Delta &= \left(2\sum_{i=2}^{n} m_i(k)\, a_{1j} a_{ij}\right)^2 - 4a_{1j}^2\left(\left(2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij} a_{lj}\right) - \frac{\varepsilon_k}{e^{2k\lambda_j}}\right) \\
&= 4a_{1j}^2\left(\sum_{1=2}^{n} m_i(k)\, a_{ij}\right)^2 - 4a_{1j}^2\left(\left(2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij} a_{lj}\right) - \frac{\varepsilon_k}{e^{2k\lambda_j}}\right) \\
&= 4a_{1j}^2\left(\left(2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij} a_{lj}\right) + \sum_{i=2}^{n} m_i^2(k)\, a_{ij}^2\right) \\
&\quad -4a_{1j}^2\left(\left(2\sum_{2=i<l}^{n} m_i(k)\, m_l(k)\, a_{ij} a_{lj}\right) - \frac{\varepsilon_k}{e^{2k\lambda_j}}\right) \\
&= 4a_{1j}^2\left(\left(\sum_{i=2}^{n} m_i^2(k)\, a_{ij}^2\right) - \frac{\varepsilon_k}{e^{2k\lambda_j}}\right).
\end{aligned}$$

and since we are assuming $\lambda_j > 0$, for almost every (apart from a finite number of possibilities) choice of $m_2(k), m_3(k), ..., m_n(k)$ we find that $\Delta > 0$. Hence, for almost every choice, the equation 2 is satisfied in the interval

$$\left[\frac{-2\sum_{i=2}^{n} m_i(k)\, a_{1j} a_{ij} - \sqrt{\Delta}}{2a_{1j}^2}, \frac{-2\sum_{i=2}^{n} m_i(k)\, a_{1j} a_{ij} + \sqrt{\Delta}}{2a_{1j}^2}\right].$$

But those intervals are centered at the points $\frac{-\sum_{i=2}^{n} m_i(k) a_{ij}}{a_{1j}}$ and, since a_{ij} is rational for every $i, j = 1, 2, ..., n$, for suitable choices of the integers $m_2(k), m_3(k), ..., m_n(k)$, we find that $\frac{-\sum_{i=2}^{n} m_i(k) a_{ij}}{a_{1j}}$ is integer and we may take any such choice of integers $m_2(k), m_3(k), ..., m_n(k)$ and get another integer $m_1(k) = \frac{-\sum_{i=2}^{n} m_i(k) a_{ij}}{a_{1j}}$. Let us notice that since all the a_{1j} are not 0, and we are assuming $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, we get that $\lim_{k\to\infty} \left\| \left(\sum_{i=1}^{n} m_i(k)\, a_{i1}\right) e^{k\lambda_1} \right\| = 0$ implies that

$$\lim_{k\to\infty} \left\| \left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right) e^{k\lambda_j} \right\| = 0$$

for every $j = 1, 2, ..., n$, so that

$$\begin{aligned}
\lim_{k\to\infty} B^k(w_k) &= \lim_{k\to\infty}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right) e^{k\lambda_j} e_j\right) \\
&= \sum_{j=1}^{n}\left(\lim_{k\to\infty}\left(\sum_{i=1}^{n} m_i(k)\, a_{ij}\right) e^{k\lambda_j}\right) e_j = 0.
\end{aligned}$$

□

Remark 1 *If we start with a given ordered orthonormal base of* $\mathbb{R}^n$*, say* $e_1 = (1,0,...,0), ..., e_n = (0,...,0,1)$*, we may find an orthogonal matrix* C *such that* $v_i = C(e_i)$ *for* $i = 1,2,...,n$*. For such an orthonormal base, the vectors of minimal length are just the vectors* $\pm e_i, i = 1,2,...,n$*. This is not the case in general, and that is the reason why, when looking for vectors of decreasing norm, we may need to take vectors with increasing integer coefficients. This happens for example when we take* $v_1 = \left(1/\sqrt{2}, 1/\sqrt{2}\right), v_2 = \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ *and* $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ *with* $\lambda \neq \pm 1, 0$*. In such a situation, for*

$$v_{m,n} = mv_1 + mv_2 = \frac{1}{\sqrt{2}}(m-n, m+n) \in \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$$

we have that

$$\left\|A^k(v_{m,n})\right\|^2 = \frac{1}{2}\left(\lambda^k (m-n)^2 + \frac{1}{\lambda^k}(m+n)^2\right)$$

so that $\|v_1\|^2 = \|v_1\|^2 = \left(\lambda^k + \frac{1}{\lambda^k}\right)/2$ *and we find that*

$$\|v_1 + v_2\|^2 = \frac{2}{\lambda^k} \leq \frac{\left(\lambda^k + \frac{1}{\lambda^k}\right)}{2}$$

whenever $k \geq \log_\lambda \sqrt{5}$*.*

Corollary 1 *Let* $\{v_1, ..., v_n\}$ *be an orthonormal base of* $\mathbb{R}^n$*. Then, given* $B = \operatorname{diag}(\lambda_1, .., \lambda_n) \in A^+$*, there are sequences* $m_1(k), ..., m_n(k) \in \mathbb{Z}$ *such that*

$$\lim_{k\to\infty} \left\|B^k (m_1(k) v_1 + m_2(k) v_2 + ... + m_n(k) v_n)\right\| = 0.$$

Proof: Let $(\varepsilon_k)_{k=1}^{\infty}$ be a sequence of positive real numbers converging to 0. Then, for each k there is an orthonormal base $v_1(k), v_2(k), ..., v_n(k)$ with rational coefficients such that $\|v_i(k) - v_i\| < \frac{\varepsilon_k}{2ne^{k\lambda_1}}$.

We choose $m_1(k), ..., m_n(k) \in \mathbb{Z}$ (as in lemma 1) such that

$$\left\|B^k (m_1(k) v_1(k) + m_2(k) v_2(k) + ... + m_n(k) v_n(k))\right\| < \frac{\varepsilon_k}{2}$$

and denote

$$\begin{aligned} u_k &= B^k (m_1(k) v_1 + ... + m_n(k) v_n), \\ w_k &= B^k (m_1(k) v_1(k) + ... + m_n(k) v_n(k)). \end{aligned}$$

With this notation we find that

$$\begin{aligned} \|u_k\| &\leq \|u_k - w_k\| + \|w_k\| \\ &\leq \sum_{i=1}^{k} e^{\lambda_i} \|v_i(k) - v_i\| + \|w_k\| \\ &\leq e^{k\lambda_1} \sum_{i=1}^{k} \|v_i(k) - v_i\| + \|w_k\| \\ &\leq e^{k\lambda_1} n \frac{\varepsilon_k}{2ne^{k\lambda 1}} + \frac{\varepsilon_k}{2} = \varepsilon_k \end{aligned}$$

□

Theorem 2 *Let $A = CBC^{-1}$ be a invertible $n \times n$ matrix, with $C \in O(n, \mathbb{R})$ and $B \in A^{+}$. Then, $\lim_{k\to\infty} \rho^{*}\left(A^{k}\right) = 0$.*

Proof: Let $\alpha = \{e_1, e_2, ..., e_n\}$ be an orthonormal base of $\mathbb{R}^n$, $\mathcal{A}$ the packing determined by the lattice $\mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_n$ and $\mathcal{A}_k$ the packing determined by the lattice $\mathbb{Z}A^k(e_1) \oplus ... \oplus \mathbb{Z}A^k(e_n)$. We want to prove that $\lim_{k\to\infty} \rho(\mathcal{A}_k) = 0$.

First of all we notice that $A^k = CB^kC^{-1}$. Since $C \in O(n, \mathbb{R})$, the density of the packings determined by A^k and B^kC^{-1} are the same. If $\mathcal{D}$ is a Voronoi domain of the original lattice, we find that $\mu(\mathcal{D}) = 1$, since α is an orthonormal base. But $\det(A) = \det\left(A^k\right) = 1$, and hence the Voronoi domain $\mathcal{D}_k$ of the lattice $\mathbb{Z}A^k(e_1) \oplus ... \oplus \mathbb{Z}A^k(e_n)$ also has volume 1. But the preceding corollary assures that $\lim_{k\to\infty} \left\|B^kC^{-1}(e_i)\right\| = 0$ for every $i = 1, 2, ..., n$ and we find that the minimal distance R_k and the inscribed radius $R_k/2$ of the lattice goes 0, so that

$$\lim_{k\to\infty} \rho(\mathcal{A}_k) = \lim_{k\to\infty} \frac{\mu(B(0, R_k/2))}{\mu(\mathcal{D}_k)} = 0$$

□

What we just proved for a special kind of sequence, we are able to prove in some more generality, just enough to prove our main result.

Lemma 2 *Given $D, C \in O(n)$ and*

$$B(t) = \operatorname{diag}\left(e^{t\lambda_1}, e^{t\lambda_2} ..., e^{t\lambda_n}\right) = \exp\left(\operatorname{diag}(t\lambda_1, t\lambda_2, ..., t\lambda_n)\right) \in A^{+},$$

we have that

$$\lim_{t\to\infty} \rho^{*}(DB(t)C) = 0,$$

where ρ^{*} is the induced density function defined in 1.

Proof: First of all we notice that, being D an orthogonal matrix,

$$\rho^{*}(DB(t)C) = \rho^{*}(B(t)C).$$

Given $t \in \mathbb{R}^{+}$ there are $n \in \mathbb{N}$ and $s \in [0, 1]$ such that $t = n \cdot s$, so that $B(t)C = B^n(s)C$. In exactly the same way we did in the preceding theorem, we find that $\lim_{n\to\infty} \rho^{*}(B^n(s)C) = 0$ and hence, $\lim_{t\to\infty} \rho^{*}(B(t)C) = 0$. □

We finally get to our main theorem:

Theorem 3 *Let $E_k \in SL(n, \mathbb{R})$ be a sequence such that $\lim_{k\to\infty} \|E_k\| = \infty$. Then, $\lim_{k\to\infty} \rho^{*}(E_k) = 0$.*

Proof: Let us consider a ray $DB(t)C \in SL(n, \mathbb{R})$. As we saw in the preceding lemma, given $\varepsilon > 0$ there is a M (depending on D, C and $B(1)$) such that $\rho^*(DB(t)C) < \varepsilon$ for $t > M$, where $B(t) = \operatorname{diag}\left(e^{t\lambda_1}, e^{t\lambda_2} ..., e^{t\lambda_n}\right)$.

Such ray is determined by the choice of elements $D, C \in O(n, \mathbb{R})$ and an element of

$$\mathcal{C} = \{(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n | \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \text{and} \lambda_1 + \lambda_2 + ... + \lambda_n = 0\}.$$

The action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n$ is obviously analytic. Also, the density function depends continuously on the variation of the lattices of $\mathbb{R}^n$, since the density function is determined by the minimal length of a vector in the lattice (remember we are restricting ourselves to lattices determined by elements $E_k \in SL(n, \mathbb{R})$and this minimum is indeed attained. So, we find that $\rho^* : SL(n, \mathbb{R}) \to (0, 1)$ is continuous. But both $O(n, \mathbb{R})$ and $\mathcal{C}$ are compact, and the continuity of ρ^* assures there is an M (now independent on D, C and $B(1)$) such that $\rho^*(DB(t)C) < \varepsilon$ for $t > M$.

Given sequence $(E_k)_{k=1}^{\infty}$ of matrices, we consider its decomposition $E_k = D_k B_k C_k$. For each $k \in \mathbb{N}$ we can find $\left(\lambda_1^k, \lambda_2^k, ..., \lambda_n^k\right) \in \mathcal{C}$ such that $B_k = \exp\left(t_k \operatorname{diag}\left(\lambda_1^k, \lambda_2^k, ..., \lambda_n^k\right)\right)$ and $\lim_{k\to\infty} \|B_k\| = \infty$ if and only if $\lim_{k\to\infty} t_k = \infty$. So, given $\varepsilon > 0$ there is an K such that $t_k > M$ for $k > K$, and hence $\rho^*(D_k B_k C_k) < \varepsilon$, that is, $\lim_{k\to\infty} \rho^*(E_k) = 0$. □

As a consequence of this theorem, we can easily prove that the lattice-packing problem has a solution:

Corollary 2 *The lattice packing problem has a solution, that is, there is a lattice packing $E_0(\mathcal{A})$ such that $\rho(E_0(\mathcal{A})) \geq \rho(E(\mathcal{A}))$, for every $E \in GL(n)$.*

Proof: First of all, from the discussion above, it is enough to consider packings associated to elements $B \in A^+K$. The density function of a lattice packing is continuous and strictly positive on irreducible lattices, exactly those attained as a $GL(n)$ orbit. The theorem assures us that it is arbitrarily small outside of closed, hence compact, balls. In other words, given $\varepsilon > 0$, there is an $R > 0$ such that $\rho(B(\mathcal{A})) < \varepsilon$ if $\|B\| > R$. Hence, the supremum is reachable by a sequence of elements B_n with $\|B\| \leq R$. The continuity of the function and the compacity of closed balls in $\mathbb{R}^n$ assures us that the supremum is indeed a maximum. □

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