Abstract

We establish a relation between the metric of the Furstenberg boundary of a symmetric space $\mathcal{X} = G/K$ and the geometry of maximal flats in $\mathcal{X}$. As an application, we prove that asymptotic cones of symmetric spaces are non-discrete Tits buildings.

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We establish here a relation, an inequality, between the distance function in the Furstenberg boundary $\mathcal{F}(\mathcal{X})$ of a symmetric space of non-compact type $\mathcal{X} = G/K$ and the intrinsic geometry of the space $\mathcal{X}$. Given $B, B' \in \mathcal{F}(\mathcal{X})$, we denote by $F(B, B')$ the maximal flat of $\mathcal{X}$ closest to a base point $x_0 \in \mathcal{X}$ and asymptotic simultaneously to both B and B'. We find there are constants $D \geq 1$ and $\delta > 0$, depending only on the symmetric space $\mathcal{X}$, such that

$$\theta(B, B') \leq De^{-\delta d\left(x_0, F(B,B')\right)}$$

for B and B' close enough, where $\theta(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are the Riemannian distance functions in $\mathcal{F}(\mathcal{X})$ and $\mathcal{X}$, respectively (Theorem 2).

As an application of this result, we will conclude that the asymptotic cone of a symmetric space is a non-discrete Tits building. A non-discrete Tits building is a pair consisting of a set and a family of injections which has as domain a fixed Euclidean space and satisfy a collection of five axioms. The two last (out of five) axioms are the most difficult to verify. We prove the fifth using the Baker-Campbell-Hausdorf formula. The hardest one to prove is the fourth axiom and the proof given here is the one that follows from Theorem 2.

In the introduction we fix the notation and introduce the concepts needed to prove Theorem 2. Any of the text books [He], [SM], [Va] or [Wa] may be a good reference for this section. The second section is devoted to the proof of this theorem. The last section is devoted to the application, which is preceded by some explanations about ultra-limits, asymptotic cones and also Tits buildings (Theorem 3.3).

1 Introduction

Let $\mathcal{X}$ be a symmetric space of non-compact type. We let $G = \mathrm{Isom}^0(\mathcal{X})$ be the identity component of the isometry group of $\mathcal{X}$ and K the stabilizer (in G) of a point in $\mathcal{X}$. Then $\mathcal{X} = G/K$, G is a *real semi-simple Lie group* and K a *maximal compact* subgroup of G. We shall denote by x_0 the point of $\mathcal{X}$ fixed by K.

Since G is semi-simple the *Cartan-Killing form*

$$B(X,Y) = \mathrm{Tr}\,(\mathrm{adX} \circ \mathrm{adY})$$

is a non-degenerate bilinear form on $g \times g$, where g is the Lie algebra of G. If we denote by k the Lie algebra of K and by p its orthogonal complement we get a *Cartan decomposition* $g = k \oplus p$ (direct sum) , with $[k,k] \subseteq k$, $[p,p] \subseteq k$ and $[k,p] \subseteq p$.

A Cartan involution of g is an automorphism $\nu : g \longrightarrow g$ such that

$$\nu(X_k + X_p) = X_k - X_p,$$

where $X_k + X_p$ is the decomposition of X relative to a given Cartan decomposition of g. The quadratic form

$$\langle X, Y \rangle = -B(X, \nu(Y))$$

is a positive definite quadratic form on g invariant under the action of $\mathrm{Ad}(K)$.

The Hadamard-Cartan Theorem assures that $\exp : T_{x_0}\mathcal{X} \to \mathcal{X}$ is a diffeomorphism. The subspace p can be identified with the tangent spaces of $\mathcal{X}$ by the map $\exp^{-1} \circ \pi \circ \exp : p \to \mathcal{X}$ (where $\exp^{-1} : \mathcal{X} \to T_{x_0}\mathcal{X}$ is the inverse of the usual Riemannian exponential, $\pi : G \to \mathcal{X} = G/K$ is the projection and $\exp : g \to G$ is the usual exponential map). Up to rescaling it by a constant, the restriction of the quadratic form $\langle \cdot, \cdot \rangle$ to p coincides with any G-invariant Riemannian metric of $\mathcal{X}$.

We have also an *Iwasawa decomposition* $G = KAN$, where A is a maximal abelian subgroup and N a maximal nilpotent subgroup. We denote by a and n the Lie algebras of A and N respectively. A *flat* in $\mathcal{X}$ is an isometrically embedded Euclidean space. It can be easily proved that flats in $\mathcal{X}$ containing the point x_0 are associated (by the exponential map) with commutative subalgebras of g. So, $F = Ax_0$ is a maximal flat in $\mathcal{X}$. Since commutative subalgebras in g are all conjugated, every maximal flat in $\mathcal{X}$

has the form $F' = gF = gAx_0$, with $g \in G$. The *rank* of a symmetric space is the dimension of a maximal flat and, by the preceding argument, it equals the dimension of A.

The *root space decomposition* of g is given by

$$g = g_0 \oplus \sum_{\lambda \in \Lambda} g_\lambda$$

where $\lambda \in \mathrm{Hom}\,(\mathrm{a}, \mathrm{R})\,, \mathrm{g}_\lambda = \{\mathrm{Y} \in \mathrm{g} |\, [\mathrm{H}, \mathrm{Y}] = \lambda\,(\mathrm{H})\,\mathrm{Y}, \mathrm{forallH} \in \mathrm{g}_0\}$ and $\Lambda = \{\lambda \in \mathrm{Hom}\,(\mathrm{a}, \mathrm{R})\,|\mathrm{g}_\lambda \neq \{0\}\}$. The λ's in Λ are called *roots* of g and each g_λ a *root subspace.*

The hyperplanes $\{H \in a | \lambda\,(H) = \{0\}\}$ for $\lambda \in \Lambda$, divide a into finitely many open convex cones, and the closure of each one is called a *Weyl chamber.* This same division of a leads (by the exponential map) to a similar division of $A = \exp a$ and hence of $F = Ax_0$ and of every flat $F' = gF$ in $\mathcal{X}$. If we denote by a^+ a Weyl chamber of a and by $A^+ = \exp\, a^+$ its image in G, we shall call gA^+x_0 a *Weyl sector,* to any $g \in G$. The point $gx_0 \in gA^+x_0$ is called the *base point* of the sector. The choice of a Weyl chamber a^+ corresponds to the choice of a set of *positive roots* $\Lambda^+ = \{\lambda \in \Lambda \mid \lambda\,(H) > 0, foreveryH \in a^+\}$ and a set of *negative roots* $\Lambda^- = \{\lambda \in \Lambda \mid \lambda\,(H) < 0, foreveryH \in a^+\}$. To a set Λ^+ of positive roots we associate a (unique) *system of simple roots* $\widetilde{\Lambda} = \{\lambda_1, \lambda_2, ...\lambda_r\}$ where the λ_i's are roots and every positive (negative) root λ may be expressed uniquely in the form

$$\lambda = n_1\lambda_1 + n_2\lambda_2 + ...n_r\lambda_r$$

with the n_i's positive (negative) integers. Given a sector A^+x_0 and its system of simple roots $\widetilde{\Lambda}$, each $k-1$ flat $\exp\,(F_{\lambda_i})\,x_0$, with

$$F_{\lambda_i} = \{Y \in a \mid \lambda_i\,(Y) = 0\}\,,\ \lambda_i \in \widetilde{\Lambda},$$

intersects A^+ in an $k-1$ dimensional Euclidean set whose interior (in F_{λ_i}) we call a *wall* of A^+x_0. Similarly, we define a $(k-s)$-wall $(0 \leq (k-s) \leq k-1)$ to be the interior (in $\exp\left(F_{\lambda_{i_1}} \cap ...F_{\lambda i_s}\right)$) of the intersection of the closure of A^+x_0 with the $k-s$ flat $\exp\left(F_{\lambda_{i_1}} \cap ...F_{\lambda i_s}\right)$.

If we let M' be the subgroup of K that leaves a maximal flat $F = Ax_0$ invariant and M the (normal) subgroup of M' that fixes F pointwise, we get the *Weyl group* $W = M'/M$, that acts simply transitively on the set of Weyl sectors of F.

The *Furstenberg Boundary* $\mathcal{F}(\mathcal{X})$ of $\mathcal{X}$ is the homogeneous space G/P, where $P = MAN$, is a minimal parabolic subgroup of G. Since K also acts transitively on $\mathcal{F}(\mathcal{X})$, we have that $\mathcal{F}(\mathcal{X})$ is diffeomorphic to K/M.

We should also note that an Iwasawa decomposition of G is defined by the choice of a Weyl sector A^+, K being the stabilizer of the base point, A as the maximal abelian subgroup containing the sector A^+ and N as $\exp\left(\sum_{\lambda\in\Lambda^+} g_\lambda\right)$.

Let us consider the set of all geodesic rays $\alpha : R^+ \to \mathcal{X}$. We say two such rays α and β are asymptotic if there is an $a \geq 0$ such that $d(\alpha(t), \beta(t)) \leq a$ for every $t \geq 0$. This is an equivalence relation on the set of all geodesic rays. We denote the equivalence class determined by a geodesic ray α by $\alpha(\infty)$ and the set of all equivalence classes is called the *ideal boundary* of $\mathcal{X}$, denoted by $\partial_\infty\mathcal{X}$.

A Weyl sector $gA^+x_0 \subset \mathcal{X}$ gives rise to a subset

$$gA^+x_0(\infty) = \left\{\gamma(\infty) \mid \gamma(t) \subset gA^+x_0, fort \geq 0\right\} \subset \partial_\infty\mathcal{X},$$

which we call an *ideal Weyl chamber*. In a similar way we define an *ideal wall*.

Let $\alpha_1(t)$ and $\alpha_2(t)$ be two geodesic rays contained in the interior of Weyl sectors $g_1A^+x_0$ and $g_2A^+x_0$ respectively. It is a remarkable known fact that if $\alpha_1(\infty) = \alpha_2(\infty)$, then, for every geodesic ray $\beta_1(t) \subset g_1A^+x_0$ there is a geodesic ray $\beta_2(t) \subset g_2A^+x_0$ such that $\beta_1(\infty) = \beta_2(\infty)$. In other words, in the ideal boundary, two Weyl sectors give rise to sets that are either coincident or with disjoint interior. A similar statement may be applied to walls at infinity, just taking the necessary care to consider walls of the same dimension. The asymptotic image of a flat

$$F(\infty) = \{\gamma(\infty) \mid \gamma(t) \subset F fort \geq 0\} \subset \partial_\infty\mathcal{X},$$

is called an *apartment* of $\partial_\infty(\mathcal{X})$.

Since G acts transitively on the set of all Weyl sectors of $\mathcal{X}$ and the asymptocity relation is preserved by isometries, we find that G also acts transitively on the set of all ideal Weyl sectors. Moreover, given a Weyl chamber A^+ and a corresponding Iwasawa decomposition $G = KAN$, the parabolic subgroup $P = MAN$ is precisely the stabilizer of the ideal Weyl sector $A^+x_0(\infty)$. So, the Furstenberg Boundary $\mathcal{F}(\mathcal{X})$ may be identified with the set of all ideal Weyl sectors. We will use this identification in all following sections.

Given a Weyl sector a^+ and a corresponding set of positive roots Λ^+, the sector $a^- = \{H \in a | \lambda (H) \leq 0, \lambda \in \Lambda^+\}$ is said to be *opposed* to a^+. Two Weyl chambers with same base point are *opposed* if they are the image of opposed Weyl sectors. It is not difficult to see that Weyl chambers are opposed if and only if they nilpotent subgroups determined by the corresponding Iwasawa decompositions have trivial intersection. We denote by n and n^- the nilpotent subalgebras determined by a^+ and a^-, respectively. Two ideal Weyl chambers $gA^+x_0(\infty)$ and $g'A^+x_0(\infty)$ are *opposed* if there is a flat $F = hA_{x_0}$ such that $hA^+x_0(\infty) = gA^+x_0(\infty)$ and $hA^-x_0(\infty) = g'A^+x_0(\infty)$.

2 Metric relation between $\mathcal{X}$ and $\mathcal{F}(\mathcal{X})$

From here on, we denote by $\theta(\cdot,\cdot)$ a K-invariant metric in the homogeneous space $\mathcal{F}(\mathcal{X})$, to which we will refer as the *angle* between two ideal Weyl chambers.

Let $A_i(\infty) = A_i^+x_0(\infty)$ and $B_i(\infty) = B_i^+x_0(\infty)$ be two sequences of ideal Weyl chambers. We will be concerned here with sequences of ideal Weyl chambers that have the same limit. Since $\mathcal{F}(\mathcal{X})$ is an homogeneous space, we loose no generality by assuming that sequence $A_i(\infty)$ is constant, defined by a Weyl sector $A^+(\infty) = A^+x_0(\infty)$.

We put $\theta_i = \theta(A^+(\infty), B_i(\infty))$ and assume that $\lim\limits_{i\to\infty} B_i = A^+$or equivalently, that $\lim\limits_{i\to\infty} \theta_i = 0$. We will prove here that the rate at which the angle θ_i between B_i and A^+ decreases, depends on the distance between the base point x_0 and F_i, the flat closest to x_0 and asymptotic simultaneously to both A^+ and B_i. To be more precise, we are going to prove there are constants $D \geq 1$ and $\delta > 0$, depending just on the symmetric space $\mathcal{X}$, such that

$$\theta_i \leq De^{-\delta d(x_0,F_i)}$$

We start now with some preliminaries to show the angle θ_i decrease exponentially.

We consider the adjoint representation $\mathrm{Ad}(g)$ of G in his Lie algebra g. The adjoint representation extends naturally to the j-fold exterior product $\bigwedge^j g$ for every $j > 0$, by defining it on indecomposable elements

$$\mathrm{Ad}(g)(X_1 \wedge X_2 \wedge \cdots \wedge X_j) = \mathrm{Ad}(g)X_1 \wedge \mathrm{Ad}(g)X_2 \wedge \cdots \wedge \mathrm{Ad}(g)X_j$$

which, for sake of simplicity, we will denote simply by $g(X_1 \wedge X_2 \wedge \cdots \wedge X_j)$ when no doubt about the action may rise.

The quadratic form $\langle X, Y\rangle = -B(X, \nu(Y))$ (as every inner product) extends also to the $j-$fold product $\bigwedge^j g$ by the formula

$$\langle X_1 \wedge \cdots \wedge X_j, Y_1 \wedge \cdots \wedge Y_j\rangle = \det\left(\langle X_i, Y_k\rangle\right)_{i,k=1}^{j}$$

for indecomposable elements and extending it by linearity to decomposable ones.

We let n be, as usual, a maximal nilpotent sub-algebra, $d = \dim n$ and look at the image of n^d at $\bigwedge^d g$. To be more precise, we choose any base $\{X_1 X_2, \ldots, X_d\}$ of n and look at the (1-dimensional) subspace

$$\bigwedge^d n = \langle X_1 \wedge X_2 \wedge \cdots \wedge X_d\rangle$$

of $\bigwedge^d g$ generated by $X_1 \wedge X_2 \wedge \cdots \wedge X_d$ and consider its orbit

$$\{\mathrm{Ad}(g)(X_1 \wedge X_2 \wedge \cdots \wedge X_j) \mid g \in G\}$$

by the action of G.

A maximal nilpotent subgroup determines a Weyl chamber and a corresponding Iwasawa decomposition. The stabilizer of n is thus the minimal parabolic subgroup $P = M'AN$. If we consider only the projective space $\mathcal{P}\left(\bigwedge^d g\right)$ and look at the action of G on it (induced by the action on $\bigwedge^d g$), we may identify the orbit of the subspace $\bigwedge^d n$ with G/P, that is with the set of Weyl chambers at infinity, or alternatively with the set of Weyl sectors of X based at x_0. This orbit is diffeomorphic to the space G/P, and since it is compact, this is a bi-Lipschitz diffeomorphism.

In this representation the chambers represented by $C^+ = \bigwedge^d n$ and $C^- = \bigwedge^d n^-$are opposite chambers.

Let B^+ be the Weyl sector contained in Ax_0, based at x_0 and asymptotically fixed by P. For given sectors B' and B'' we find elements $g', g'' \in G$ such that $g'B^+(\infty) = B'(\infty)$, $g''B^+(\infty) = B''(\infty)$. We identify the sectors B' and B'' with

$$g'B^+ = g'\left(\bigwedge^d n\right) = g'C^+, \quad g''B^+ = g''\left(\bigwedge^d n\right) = g''C^+,$$

and then, the angle $\theta(B', B'')$ may be approximated by the projective distance

$$d_{\mathcal{P}}\left(g'C^+, g''C^+\right),$$

where $d_{\mathcal{P}}(\cdot,\cdot)$ is the distance in the projective space $\mathcal{P}\left(\bigwedge^{d} g\right)$.

But for B' and B'' sufficiently close, we may consider the two-sheet spherical covering , and we find that

$$\theta(B',B'') \approx d_{\mathcal{P}}\left(g'C^{+},g''C^{+}\right) = \left\| \frac{g'C^{+}}{\|g'C^{+}\|} - \frac{g''C^{+}}{\|g''C^{+}\|} \right\|$$

where the norm is the one induced on $\bigwedge^{d} g$ by the Cartan-Killing form.

In the situation we are studying, we have that each B_i is a sector opposed to the fixed sector A^{+}. We use the following lemma that assures there are uniquely determined $n_i \in N$ such that $n_i A^{-} = B_i$ where A^{-} is the chamber opposite to A^{+}contained in the same flat as A^{+}.

A chamber $B^{-}(\infty)$ is opposite to $A^{+}(\infty) = \bigwedge^{d} n$, if and only if there is an $n \in N = \exp n$ such that $B^{-}(\infty) = nA^{+}(\infty)$

The group G acts as isomorphism of the Tits building of $\partial_{\infty}\mathcal{X}$.Given a chamber $B^{-}(\infty)$ opposite to $A^{+}(\infty)$ there is an (unique) apartment containing both of them. But the apartments containing the chamber $A^{+}(\infty)$ are all of the kind $nF(\infty)$, where $n \in N = \exp n$. Since the action of G preserves the relation of opposition in apartments and $nA^{+}(\infty) = A^{+}(\infty)$, N acts simply transitively on the set of all chambers opposite to $A^{+}(\infty)$ and the lemma is proven.

So, if we put $C^{-} = \bigwedge^{d} n^{-}$, with n^{-} being the subspace of the Lie algebra g determined by the negative roots of the sector A^{+}, and denote by C^{+} the image of B^{+} in $\bigwedge^{d} g$, we have that

$$\theta_i = \theta\left(B_i, B^{+}\right) \approx \left\| \frac{C^{+}}{\|C^{+}\|} - \frac{n_i C^{-}}{\|n_i C^{-}\|} \right\|.$$

Since we loose no generality by assuming $\|C^{+}\| = 1$, we may also assume that

$$\theta_i \approx \left\| C^{+} - \frac{n_i C^{-}}{\|n_i C^{-}\|} \right\|.$$

We fix now a base for $\bigwedge^{d} g$. For any given point $y \in \bigwedge^{d} g$, we denote by $(y_1,...,y_l)$ its coordinates in this given base, and by $[y_1,...,y_l]$ we denote its equivalence class in $\mathcal{P}\left(\bigwedge^{d} g\right)$. We also fix a base for n and denote by

$(x_1,...,x_d)$ the coordinates in n. Since n is nilpotent, there is an $M>0$ such that

$$\mathrm{ad}^m(x_1,...,x_d)=0, \qquad \forall m\geq M, \forall (x_1,...,x_d)\in n$$

and for the adjoint action of $N=\exp n$ on n we get that

$$\mathrm{Ad}(\exp(x_1,...,x_d))=\sum_{j=1}^{M}\frac{\mathrm{ad}^j(x_1,...,x_d)}{j!}$$

and find that the linear action of G on $\overset{d}{\bigwedge} g$ and the induced action on $\mathcal{P}\left(\overset{d}{\bigwedge} g\right)$ also depends polynomially on the coordinates $(x_1,...,x_d)$. It follows that given $n=\exp(x_1,...,x_d)\in N$, we have that

$$n\left[C^-\right]=[P_1(x_1,...,x_d),...,P_l(x_1,...,x_d)]$$

where the P_i are polynomials in $(x_1,...,x_d)$ with real coefficients.

Consider $C^-=\overset{d}{\bigwedge} n^-$ and $N=\exp n^+$, as defined above. Let $n=\exp X,\ X\in n^+$ and assume that $\lim\limits_{\|X\|\to\infty} nC^-(\infty)=C^+(\infty)$. Then, there are positive constants $\alpha\in N$ and $c\in R$ such that

$$\left\|\left[C^+\right]-\left[nC^-\right]\right\|_{\mathcal{P}\left(\overset{d}{\bigwedge} g\right)}\leq c\|X\|_n^{-\alpha}$$

There is no loss of generality in assuming that C^+ is the last vector of the chosen base to $\overset{d}{\bigwedge} g$, so that its coordinates in the projective space are $[0,...,0,1]$. Then, if we look at the coordinates of nC^-, they are given by polynomials $[P_1(X),...,P_l(X)]$, and we get that

$$\lim_{\|X\|\to\infty}[P_1(X),...,P_l(X)]=[0,...,0,1]$$

Remember we are looking at the convergence on the projective space and this means exactly that, for $\|X\|\gg 1$, $P_l(X)\neq 0$ and that

$$\lim_{\|X\|\to\infty}\frac{P_i(X)}{P_l(X)}=0$$

for every $i\in\{1,...,l-1\}$.

If we put $Z = X\|X\|^2$ we get that $X = Z\|Z\|^2$ and $\|X\| = 1\|Z\|$, so that $\|X\| \to \infty$ if and only if $\|Z\| \to 0$. So, we have that

$$\lim_{\|Z\|\to 0} \frac{P_i\left(\frac{Z}{\|Z\|^2}\right)}{P_l\left(\frac{Z}{\|Z\|^2}\right)} = 0$$

for every $i \in \{1, ..., l-1\}$.

We note that $\frac{P_i\left(\frac{Z}{\|Z\|^2}\right)}{P_l\left(\frac{Z}{\|Z\|^2}\right)}$ is a rational function on Z such that the denominator is non zero in a neighborhood of 0 (excluding eventually the point itself) and converging to 0 when $Z \to 0$. So, in order ro conclude the proof of the proposition, it is enough to prove the following lemma:

Let $F(Z)$ be a rational function on $Z = (z_1, ..., z_d)$ such that the denominator is non-zero in a neighborhood of 0 (excluding eventually the point itself). If $\lim\limits_{\|Z\|\to 0} F(Z) = 0$, there are positive constants $C \in R$ and $1 \leq \alpha \in N$ such that

$$|F(Z)| \leq C\|Z\|^{\alpha}$$

for $\|Z\| \ll 1$.

We write $F(Z) = P(Z)Q(Z)$ and we look at two possible situations: $Q(0) = 0$ or $Q(0) \neq 0$. If $Q(0) \neq 0$, then $1Q(Z)$is bounded on a neighborhood of 0, so that $|F(Z)| \leq \text{const} \cdot |P(Z)|$ and since $P(0) = 0$, looking at the first non-zero term of the Taylor expansion, we approximate $P(Z)$ by an homogeneous $P_1(Z)$ polynomial of degree $\beta \geq 1$. We choose $r_0 > 0$ such that

$$A = \sup\{|P_1(Z)| \mid \|Z\| = r_0\} > 0$$

Since $P_1(Z)$ is homogeneous, we get that

$$|P_1(Z)| = \left\|\frac{Z}{r_o}\right\|^{\beta} P_1\left(\frac{r_0 Z}{\|Z\|}\right) \leq A\frac{\|Z\|^{\beta}}{r_0^{\beta}} = B\|Z\|^{\beta}.$$

So, it is left to prove the case when $Q(0) = 0$. First of all, we should notice that 0 must be either a point of local maximum or a minimum of $Q(Z)$. Indeed, if it was not so, we would find open sets $\mathcal{A}^+$ and $\mathcal{A}^-$, as close as wanted to $Z = 0$, such that $Q(Z) > 0$ for $Z \in \mathcal{A}^+$ and $Q(Z) < 0$ for $Z \in \mathcal{A}^-$. Given $Z_i^+ \in \mathcal{A}^+$ and $Z_i^- \in \mathcal{A}^-$, the segment $(1-s)Z_i^+ + sZ_i^-$ must contain a point $Z_i = (1-s_i)Z_i^+ + s_iZ_i^-$ such that $Q(Z_i) = 0$. By taking a small perturbation of Z_i^+ (if needed), we find that $Z_i \neq 0$. So, when Z_i^+ and Z_i^- converges to 0, we would find a sequence of points Z_i, with $Q(Z_i) = 0$ and $\lim_{i\to\infty} Z_i = 0$, contradicting the hypothesis that $Z = 0$ is an isolated zero of $Q(Z)$.

We loose no generality by assuming that $Q(Z) \geq 0$ in a neighborhood of $Z = 0$. Then, the minimal degree monomials have all coefficients of the same sign (positive) and each variable appear with even power. Let $Q^+(Z)$ be the monomials of $Q(Z)$ where all variable appear with even power and $Q^-(Z) = Q(Z) - Q^+(Z)$ the monomials where some variable appears with odd power. We can express

$$Q^+(Z) = Q_1^+(Z) + Q_2^+(Z) + ... + Q_r^+(Z)$$

where $Q_i^+(Z)$ is homogeneous of degree $2\alpha_i$, with $0 < \alpha_1 < \alpha_2 < ... < \alpha_r$. Since all the coefficients of $Q_1^+(Z)$ are positive we find that, in an eventually smaller neighborhood of $Z = 0$,

$$Q^+(Z) \geq dQ_1^+(Z), \quad 0 < d < 1$$

since $(1-d)\,Q_1^+(Z)$, the homogeneous part of lower degree of

$$Q^+(Z) - dQ_1^+(Z) = (1-d)\,Q_1^+(Z) + Q_2^+(Z) + ... + Q_r^+(Z),$$

has only positive coefficients.. In the same way we can see that, in an eventually smaller neighborhood of $Z = 0$,

$$Q(Z) \geq d'Q^+(Z), \quad 0 < d' < 1.$$

Then, taking $D = d \cdot d'$, and restricting Z to the intersection of those two neighborhoods, we find that

$$|Q(Z)| \geq |d'Q(Z)| \geq \left|d \cdot d'Q_1^+(Z)\right| \geq D \cdot D' \|Z\|^\alpha$$

where $\alpha = 2\alpha_1$ and D' is the minimal coefficient of $Q_1^+(Z)$. Writing $B' = D \cdot D'$ we have that

$$\left|\frac{P_1(Z)}{Q_1^+(Z)}\right| \leq \frac{B}{B'} \|Z\|^{\beta-\gamma} = C \|Z\|^{\beta-\gamma}$$

and it is left to prove only that $\beta - \gamma > 0$. To do so, we choose a point Z_0 for which $P_1(Z_0) \neq 0$ and $Q_1^+(Z_0) \neq 0$. Since those are homogeneous polynomials, we have that

$$\left|\frac{P_1(tZ_0)}{Q_1^+(tZ_0)}\right| = t^{\beta-\gamma} \left|\frac{P_1(Z_0)}{Q_1^+(Z_0)}\right|$$

hence

$$\lim_{t\to 0} t^{\beta-\gamma}\left|\frac{P_1(Z_0)}{Q_1^+(Z_0)}\right| = \lim_{t\to 0}\left|\frac{P_1(tZ_0)}{Q_1^+(tZ_0)}\right| = 0$$

so that indeed, $\beta-\gamma>0$

Before we continue, we make an important remark:

The orbit of a maximal nilpotent subgroup $N\subset G$ is exponentially distorted in $\mathcal{X}=G/K$ ([B-G-S]). Indeed, let $(n_i)_{i=1}^{\infty}$ be a sequence in N with $n_i=\exp X_i$, $X_i\in n$ and assume $\|X_i\|$ grows less then exponential, i.e.,

$$\lim_{i\to\infty}\frac{\|X_i\|}{e^{ci}}=0, \qquad \forall c>0.$$

Let $H\in a$ belong to the Weyl sector determined by n be a unit vector and consider the sequence of geodesics

$$\gamma_i(t)=n_i\exp(tH)\,x_0$$

in our symmetric space $\mathcal{X}$. We have that , for every $c>0$

$$\begin{aligned} d\left(n_i\exp(icH)\,x_0,\exp(icH)\,x_0\right) &= d\left(\exp(-icH)\,n_i\exp(icH)\,x_0,x_0\right)\\ &= d\left(\exp\left(e^{-ad(icH)}X_i\right)x_0,x_0\right)\end{aligned}$$

If we consider the decomposition

$$X_i=\sum_{\lambda\in\Lambda_i^+}X_{i,\lambda}$$

we get that

$$d\left(\exp\left(e^{-ad(c\cdot iH)}X_i\right)x_0,x_0\right)=d\left(\exp\left(\sum_{\lambda\in\Lambda_i^+}e^{-c\cdot i\lambda(H)}X_{i,\lambda}\right)x_0,x_0\right)$$

and since $\lim_{i\to\infty}\|X_i\|e^{ci}=0$ and for every component $\|X_{i,\lambda}\|\le\|X_i\|$, we find that

$$\lim_{i\to\infty}\left\|e^{-c\cdot i\lambda(H)}X_{i,\lambda}\right\|=0$$

for every root λ, so that

$$\lim_{i\to\infty}\frac{1}{i}d\left(\exp\left(\sum_{\lambda\in\Lambda_i^+}e^{-c\cdot i\lambda(H)}X_{i,\lambda}\right)x_0,x_0\right)=0$$

and by continuity follows that the orbit $n_i x_0$ is bounded.

We are able now to prove our main theorem:

Let $G = KAN$ be a semi-simple Lie group of non-compact type and consider the symmetric space $\mathcal{X} = G/K$. Let $n_i = \exp X_i$, $X_i \in n$ be a sequence with $\|X_i\| \to \infty$. We let B^- be a sector in $\mathcal{X}$ based at x_0, B_i the sector based at x_0 asymptotic to $n_i B^-$ and B^+ the sector based at x_0 such that

$$\theta_i = \theta\left(B_i, B^+\right)$$

converges to zero. Then, there are constants $D \geq 1$ and $\delta > 0$, depending only on the symmetric space $\mathcal{X}$, such that

$$\theta_i \leq De^{-\delta d(x_0, F_i)}$$

where F_i is the flat of X asymptotic to both B_i and B^+ closest to the base point x_0.

As we did before , we identify the sectors B_i and $B^+$with elements $n_i C^-$ and C^+ of the projective orbit of the linear action of G on $\bigwedge^d g$, so that we can approximate

$$\theta_i \approx \left\| C^+ - \frac{n_i C^-}{\|n_i C^-\|} \right\|_{\mathcal{P}\left(\bigwedge^d g\right)} \leq D \|X_i\|^{-\alpha}$$

where the inequality was proved in proposition 2.

But

$$d\left(x_0, F_i\right) \leq d\left(x_0, n_i x_0\right)$$

and it is known that the orbits of N are exponentially distorted in $\mathcal{X}$, i.e.,

$$\|X_i\| \leq e^{-\eta d(x_0, n_i x_0)}$$

for some constant $\eta > 0$.

Gathering those three inequalities we find that

$$\theta_i \leq D \|X_i\|^{-\alpha} \leq De^{-\alpha\eta d(x_0, n_i x_0)} \leq De^{-\delta d(x_0, F_i)}$$

where $\delta = \alpha\eta$.

3 Application

Before we state the application we made of our main theorem, we must introduce some definitions and notations.

3.1 Ultra-filters and Asymptotic Cones

A *non-principal ultra-filter* ([Si]) over the natural numbers is a finitely additive measure ϖ defined on all subsets $A \subseteq N$ enjoying the following properties:

1. ϖ assumes values in $\{0,1\}$.
2. $\varpi(A) = 0$ for every finite $A \subset N$.

Ultra-filters has the property that for every $A \subset N$, either $\varpi(A) = 1$ or $\varpi(A^c) = 1$, hence it may be viewed as a family of subsets of $\mathcal{P}(N)$ that contains either A or A^c, so that we may use indistinguishably both the notations $\varpi(A) = 1$ or $A \in \varpi$.

Given such an ultra-filter we define the concept of ultra-limit of a sequence:

Definition 3.1 *Let X be a topological space and $(x_i)_{i=1}^{\infty}$ a sequence in X. We say that $x_0 \in X$ is the* ultra-limit *of $(x_i)_{i=1}^{\infty}$, and denote it by $\varpi - \lim x_i = x_0$ if, for every neighborhood V of $x_{0,}$ the set $\{i \in N |\ \mathbf{x}_i \in V\}$ is in ϖ. We say a sequence $(x_i)_{i=1}^{\infty}$ is* ultra-convergent *if its ultra-limit exists.*

The ultra-limit of a sequence (whenever it exist) is unique and has the following essential properties:

1. Whenever $(x_i)_{i=1}^{\infty}$ is convergent it is also ultra-convergent and $\lim\limits_{i\to\infty} x_i = \varpi - \lim_{i\to\infty} x_i$.
2. If X is a compact Hausdorff space, then every sequence $(x_i)_{i=1}^{\infty}$ in X is ultra-convergent.
3. Ultra-limits are linear operators on sequences in vector spaces:

$$\varpi - \lim_{i\to\infty}(x_i + y_i) = \varpi - \lim_{i\to\infty} x_i + \varpi - \lim_{i\to\infty} y_i$$

The above cited properties of ultra-limits enable us to define the ultra-limit of a sequence of metric spaces and the asymptotic cone of a given one.

Consider a sequence (X_i, d_i, p_i) of pointed metric spaces and let

$$\overline{X} = \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i\in N} X_i \mid d_i(p_i, x_i) is bounded \right\}.$$

The function $d((x_i),(y_i)) = \varpi - \lim d_i(x_i, y_i)$ is a semi-metric, and by considering the equivalence relation defined by $(x_i)_{i=1}^{\infty} \sim (y_i)_{i=1}^{\infty}$ if and only if $d((x_i),(y_i)) = 0$, this semi-metric induces a metric $d_{\varpi}(\cdot,\cdot)$ in the quotient space $X = \overline{X}/\sim$. We shall say that this metric space is the ultra-limit of the sequence (X_i, d_i, p_i) of pointed metric spaces.

We should note that whenever the given sequence os spaces converges in the Hausdorff topology on subsets ([G-L-P]), this convergence coincides with the ultra-convergence defined above. Criteria for the late convergence, can be found in ([Pa]).

The asymptotic cone of a metric (X, d) space is a special instance of the ultra-limit of spaces when we put $X_i = X$, $d_i(x,y) = \frac{1}{i}d(x,y)$ and $x_i = x_0$. Since this definition is crucial in what follows, we will repeat it explicitly.

Let (X, d) be a metric space, x_0 a fixed point in it and consider the set of all sequences $x = (x_i)_{i=1}^{\infty}$ in X that distance away from x_0 at most linearly, i.e., sequences for which there is a constant $c_x \in R^+$ such that $d(x_0, x_i) \le c_x \cdot i$ for every $i \in N$. For any pair of such sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$, we have that $\frac{1}{i}d(x_i, y_i) \le c_x + c_y$ so that it has an ultra-limit. The ultra-limit defines a semi-metric on the space of all such sequences, and, by identifying those sequences for which $\varpi - \lim \frac{1}{i}d(x_i, y_i) = 0$, we get a metric space. Following M. Gromov ([Gr]), we call this metric space the *asymptotic cone* of X over ϖ and denote it by $\mathrm{cone}_{\infty}X$. We denote the equivalence class of $x = (x_i)_{i=1}^{\infty}$ by $[x_i]$ and the metric on $\mathrm{cone}_{\infty}X$ by $d_{\varpi}(\cdot,\cdot)$.

3.2 Non-discrete Buildings

The concept of a non discrete Tits building appears first at [Ti]. The definition we adopt here is the one used in [Ro] and differs from that of Tits only in the fifth axiom (that was changed later by Tits himself).

Let V be a (finite dimensional) vector space with an inner product and $\overline{W} \subset Gl(V)$ a finite group generated by reflections in hyperplanes (hence a Coxeter group) $\{W_1, W_2, ...W_r\}$. Let W be the maximal (affine) group of isometries of V which linear part is $\overline{W}$, i.e., $W = \overline{W}V$. For each $p \in V$ the coset $\overline{W} \times \{p\}$ divides V into finitely many open subsets (with p being the intersection of the closure of them all). Each connected component of $V \setminus \bigcup\limits_{i=1,\dots,r} (p + W_i)$ is called a *sector* (*based at* p). Given a sector A^+ based at p and a hyperplane $p + W_i$ such that $\overline{A}^+ \bigcap (p + W_i)$ has codimension 1 in V ($\overline{A}^+$ being the topological closure of A^+), we call the interior (in $p + W_i$)

of $\overline{A}^{+} \bigcap (p + W_i)$ a *wall* of the sector A^+. In a similar way we define lower dimensional walls of a sector. The hyperplanes $p + W_i$ divides V into two half spaces, each one we call an *half-apartment*.

Let now $(\Delta, \mathcal{F})$ be a pair where Δ is a set and $\mathcal{F}$ a collection of injections of V into Δ. We define an *apartment, half apartment, sector* and *wall* in Δ to be the image under an element of $\mathcal{F}$ of V itself, an half apartment, sector or wall in V respectively. Given two sectors A_1, A_2, we say A_2 is a *sub-sector* of A_1 if $A_2 \subseteq A_1$. We are now able to give the desired definition:

Definition 3.2 *A pair* $(\Delta, \mathcal{F})$ *defined as above is called a* non-discrete building *(or affine building) if it satisfies the following axioms:*

1. *If* $w \in W$ *and* $f \in \mathcal{F}$ *then* $f \circ w \in \mathcal{F}$.

2. *Given* $f, f' \in \mathcal{F}, Y = f^{-1}(f'(V))$ *is closed and convex in* V *and there is an* $w \in W$ *such that* $f\mid_Y = f' \circ w \mid_Y$.

3. *Given any two points in* Δ*, there is an apartment containing both of them.*

4. *Given any two sectors* A_1, A_2 *of* Δ*, there are sub-sectors* $A_i' \subseteq A_i, i = 1, 2$ *and an apartment* $F \subset \Delta$ *containing both* A_1' *and* A_2'.

5. *If* F, G *and* H *are three apartments such that the intersection of any two of them is an half-apartment, then* $F \bigcap G \bigcap H \neq \emptyset$.

3.3 Asymptotic Cones and Buildings

We use our main result (Theorem 2) to show that the asymptotic cone of a symmetric space $\mathcal{X} = G/K$ has a structure of a non-discrete Tits Building. This fact was first proved in [K-L], but the approach given here is quite different: We define explicitly the set of sectors and apartments of the building and show what are the apartments containing a given sector or pair of sectors.

The sectors and apartments of Δ are respectively the equivalence class $[A_i]$ and $[F_i]$ of sequences $(g_i A^+ x_0)_{i=1}^{\infty}$ and $(g_i A x_0)_{i=1}^{\infty}$ where $A^+ x_0$ and Ax_0 are respectively a sector and a flat of X, $g_i \in G$ and $\omega - \lim_{i \to \infty} 1id\,(x_0, g_i x_0)$ exist.

As we already noticed, the first three axioms are relatively simple to prove. The fifth is proved by mean of a detailed examination of the root decomposition, an application of the Hausdorff-Campbell formula, and a reduction of its application to a subalgebra isomorphic to $sl\,(2, R)$, where the convergence of the series is global. The fourth axiom is the one that makes use of Theorem 2.

We shall prove the axioms in the same order they where stated.

The pair $(cone_\infty \mathcal{X}, \mathcal{F})$ satisfies axiom (A1), i.e., given $w \in W$ and an apartment $[g_i F]$, $[g_i w F]$ is also an apartment.

This is obvious, we just need to take an representative of w in G.

The pair $(\mathrm{cone}_\infty \mathcal{X}, \mathcal{F})$ satisfies axiom (A2): given $f, f' \in \mathcal{F}, Y = f^{-1}\,(f'\,(V))$ is closed and convex in V and there is an $w \in W$ such that $f\mid_Y = f' \circ w \mid_Y$.

The closeness and convexity of the intersection of the apartments follows from the fact they are embedded isometrically in $\mathrm{cone}_\infty \mathcal{X}$ and each apartment is closed and convex and so is their intersection.

It is left to prove that if $Y = f^{-1}\,(f'\,(V))$ there is an $w \in W$ such that $f\mid_Y = f' \circ w \mid_Y$.

We remember that each $f \in \mathcal{F}$ is actually a family of isometric embedding $g_i : A \mapsto g_i A x_0$. Let f and f' be defined by the sequences $(g_i)_{i=1}^\infty$ and $(g_i')_{i=1}^\infty$ respectively. We denote by $F = [g_i A x_0]$ and $F' = [g_i' A x_0]$ the flats in $\mathrm{cone}_\infty X$ defined by those apartments. We loose no generality by assuming that $[x_0] \in [g_i A x_0] \cap [A x_0]$. Moreover, we may assume that $[g_i x_0] = [x_0]$. If it was not the case, there would be a sequence $b_i \in A$ such that $[g_i b_i x_0] = [x_0]$ and since $b_i A = A$, we could substitute each g_i by $b_i g_i$. We consider the polar decomposition $g_i = k_{i,1} h_i k_{i,2}$, with $k_{i,1}, k_{i,2} \in K$ and $h_i \in A^+$. We claim that $[k_{i,1} h_i k_{i,2} A x_0] = [k_{i,1} k_{i,2} A x_0]$. Indeed, since the metric on $\mathcal{X}$ is left invariant, it is sufficient to prove that $[h_i k_{i,2} A x_0] = [k_{i,2} A x_0]$.

Since we are assuming that $[g_i x_0] = [x_0]$, from the fact that

$$\begin{aligned} d\,(k_{i,1} h_i k_{i,2} x_0, x_0) &= d\,(k_{i,1} h_i x_0, x_0) \\ &= d\left(h_i x_0, k_{i,1}^{-1} x_0\right) \\ &= d\,(h_i x_0, x_0) \end{aligned}$$

we find that $\omega - \lim 1 i d\,(h_i x_0, x_0) = 0$. Given $x_i \in \mathcal{X}$, it is uniquely described as $a_i n_i x_0$, for some $a_i \in A$ and $n_i \in N$, the nilpotent subgroup of G determined by the Weyl sector A^+ and we have that

$$\begin{aligned} d\,(h_i a_i n_i x_0, a_i n_i x_0) &= d\left(a_i^{-1} h_i a_i n_i x_0, n_i x_0\right) \\ &= d\,(h_i n_i x_0, n_i x_0) \\ &\leq d\,(h_i n_i x_0, h_i x_0) + d\,(h_i x_0, x_0) + d\,(x_0, n_i x_0) \end{aligned}$$

$$
\begin{aligned}
&= d\left(n_i x_0, x_0\right) + d\left(h_i x_0, x_0\right) + d\left(x_0, n_i x_0\right) \\
&= 2d\left(n_i x_0, x_0\right) + d\left(h_i x_0, x_0\right).
\end{aligned}
$$

Since we have that $\omega{-}\lim 1id\left(h_i x_0, x_0\right) = 0$, we find that $\omega{-}\lim_{i\to\infty} 1id\left(h_i x_i, x_i\right) = \omega - \lim_{i\to\infty} 1id\left(h_i a_i n_i x_0, a_i n_i x_0\right) = 0$ whenever $\omega - \lim_{i\to\infty} 1id\left(n_i x_0, x_0\right) = 0$. But this is the case for $[x_i] \in [Ax_0]$, since we can represent x_i just as $x_i = a_i x_0$. So, for $x_i \in Ax_0$ we find that

$$
\begin{aligned}
[g_i x_i] &= [k_{i,1} h_i k_{i,2} x_i] \\
&= [k_{i,1} k_{i,2} x_i].
\end{aligned}
$$

Since $k_{i,1}, k_{i.2} \in K$, a compact and closed subgroup of G, there exist the ultra limit $\omega - \lim_{i\to\infty}\left(k_{i,1}k_{i,2}\right) = k \in K$. It follows that, for $x_i \in A$, $\omega - \lim_{i\to\infty} 1id\left(g_i x_i, kx_i\right) = \omega - \lim_{i\to\infty} 1id\left(k_{i,1}k_{i,2}x_i, kx_i\right) = 0$, or, in other words, for $x_i \in Ax_0$, we have that $[g_i x_i] = [kx_i]$.

Now, we finally reach our problem: If we assume that $[g_i x_i] \in [g_i Ax_0] \cap [Ax_0]$, we get that $[kx_i] \in [Ax_0]$. But, if $[kx_i]$ belongs to the interior of a sector, this happens if and only if $k \in M'$, the normalizer of A in K. We define $w = \overline{k}$, the class of k in $W = M'/M$ and we find that $[g_i x_i] = [wx_i]$, for every $[x_i] \in Y$. In case there is no $[kx_i] \in [g_i Ax_0] \cap [Ax_0]$ belonging to interior of a sector of $\mathrm{cone}_\infty \mathcal{X}$, we have that is contained in a wall of a sector. In this case we find that k may belongs to a subgroup $gM'g^{-1}$conjugated to M'. But this causes now problem, since, restricted to such a wall, the action of k will coincide with the action of $k' := g^{-1}kg \in M'$. For the particular case when $[g_i Ax_0] \cap [Ax_0]$ consist of the single point $[x_0]$, we can just define $k' = e$, the identity element of G.

The pair $(cone_\infty X, \mathcal{F})$ satisfies axiom (A3), i.e. given two points $[x_i]$,$[y_i] \in cone_\infty X$ there is an apartment containing both of them.

This is also obvious, just take F_i to be a flat in X containing both x_i and y_i and we get that $[F_i]$ is the desired apartment, because the boundedness of $1id\left(x_i, x_0\right)$ implies that of $1id\left(F_i, x_0\right)$.

In order to prove the fourth axiom, namely, that given to sectors in $\mathrm{cone}_\infty \mathcal{X}$, there is an apartment containing subsectors of both of them, we need a preliminary result:

Let B_i and B_i' be sequences of sectors in the symmetric space $\mathcal{X}$, based at x_0. Let $F_i := F(B_i, B_i')$ be the flat in $\mathcal{X}$ closest to the base point x_0 and asymptotic simultaneously to B_i and B_i'. If

$$\frac{1}{i} d\left(x_0, F_i\right)$$

is unbounded, then, in the asymptotic cone, we have that

$$[B_i] = [B_i']$$

Since $\mathcal{X}$ is an homogeneous space, it has bounded curvature (the homogeneity condition assures the curvature is defined on a Grassmannian, hence compact, space). We lose no generality by assuming it has curvature bounded from below by -1. This condition implies that geodesic rays in the hyperbolic plane H^2 distance one from the other faster then in $\mathcal{X}$. To be more precise, if $\gamma(t)$ and $\gamma'(t)$ are unit speed geodesic (the prime is just a superscript, not the derivative) in $\mathcal{X}$ with $\gamma(0) = \gamma'(0)$ and angle θ and $\widetilde{\gamma}(t)$ and $\widetilde{\gamma'}(t)$ are unit speed geodesic rays in H^2 having the same angle θ at their intersection point $\widetilde{\gamma}(0) = \widetilde{\gamma'}(0)$ then

$$d\left(\gamma(t), \gamma'(t)\right) \leq d_{H^2}\left(\widetilde{\gamma}(t), \widetilde{\gamma'}(t)\right), foreveryt \geq 0$$

For each $i \in N$, we choose a regular geodesic ray γ_i contained in B_i with $\gamma_i(0) = x_0$. If $B_i' = g_i B_i$ with $g_i \in K$, we put $\gamma_i' = g_i \gamma_i$. Then, the remark above assures that, for every $c \geq 0$.

$$\varpi - \lim \frac{1}{i} d\left(\gamma_i(c \cdot i), \gamma_i'(c \cdot i)\right) \leq \varpi - \lim \frac{1}{i} d_{H^2}\left(\widetilde{\gamma}_i(c \cdot i), \widetilde{\gamma'}_i(c \cdot i)\right)$$

By the hyperbolic cosines law we know that

$$\begin{aligned}
\cosh d_{H^2}\left(\widetilde{\gamma}_i(c \cdot i), \widetilde{\gamma'}_i(c \cdot i)\right) &= \cosh^2(c \cdot i) - \sinh^2(c \cdot i)\cos\theta_i \\
&= (1 - \cos\theta_i)\cosh^2(c \cdot i) - \cos\theta_i \\
&= \frac{e^{2ci} + e^{-2ci} + 2}{4}(1 - \cos\theta_i) - \cos\theta_i
\end{aligned}$$

and since $\operatorname{arccosh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ we can approximate

$$d_{H^2}\left(\widetilde{\gamma}_i(c \cdot i), \widetilde{\gamma'}_i(c \cdot i)\right) \sim \ln \frac{e^{2ci}(1 - \cos\theta_i)}{2}$$

so that

$$\begin{aligned}\varpi-\lim_{i\to\infty}\frac{1}{i}d\left(\gamma_i\left(c\cdot i\right),\gamma_i'\left(c\cdot i\right)\right) &\leq \varpi-\lim_{i\to\infty}\frac{1}{i}d_{H^2}\left(\widetilde{\gamma}_i\left(c\cdot i\right),\widetilde{\gamma'}_i\left(c\cdot i\right)\right)\\ &= \varpi-\lim_{i\to\infty}\frac{1}{i}\ln\frac{e^{2ci}\left(1-\cos\theta_i\right)}{2}\end{aligned}$$

and, by Theorem 2 we find that

$$\varpi-\lim_{i\to\infty}\frac{1}{i}\ln\frac{e^{2ci}\left(1-\cos\theta_i\right)}{2}=0.$$

Hence $[\gamma_i\left(c\cdot i\right)]=[\gamma_i'\left(c\cdot i\right)]$ for every $c\geq 0$ and this means $[\gamma_i]=[\gamma_i']$.

Given a point $[x_i]\in[B_i]$, we choose for each $i\in N$ a geodesic ray $\beta_i\subset B_i$ containing x_i and put $\beta_i'=g_i\beta_i'$. As we just saw, $[\beta_i]=[\beta_i']$. Since $[\beta_i']\subset[B_i']$, we find that $[B_i]\subseteq[B_i']$. The same argument, starting with a point $[x_i']\in[B_i']$ shows the inverse inclusion, so that $[B_i]=[B_i']$

The pair $(cone_\infty X,\mathcal{F})$ satisfies axiom (A4), i.e., given sectors $[A_i],[B_i]\subset$ cone$_\infty\mathcal{X}$, there is an apartment containing subsectors $[A_i']\subseteq[A_i],[B_i']\subseteq[B_i]$ of both $[A_i]$ and $[B_i]$.

Let $(g_iA^+x_0)_{i=1}^\infty$ and $(h_iA^+x_0)_{i=1}^\infty$ be representatives of sectors $[A_i]$ and $[B_i]$, respectively.

There is a natural candidate to be the apartment containing both of them: For each $i\in N$, we let $F_i=k_iAx_0$ be a flat asymptotic to both $g_iA^+x_0$ and $h_iA^+x_0$. Such a flat always exist and, whenever $\omega-\lim\limits_{i\to\infty}1id\left(k_ix_0,x_0\right)$ exist, it determines an apartment in cone$_\infty\mathcal{X}$. By construction, for each $i\in N$ there are sectors $A_i^*,B_i^*\subset F_i$ asymptotic respectively to $[A_i]$ and $[B_i]$. We define

$$[A_i']:=[A_i^*]\cap[A_i]\,,\;[B_i']:=[B_i^*]\cap[B_i]$$

and, in this case, it is left to prove that $[A_i']$ and $[B_i']$ are indeed subsectors of $[A_i]$ and $[B_i]$ respectively.

As we did in the proof of Proposition 3.3, we lose no generality if we assume that $A_ix_0=Ax_0$ for every $i\in N$. In this situation we have that for ω-all $i\in N$, $A_i^*x_0=n_iA^+x_0$ with A^+x_0 a Weyl sector in Ax_0 and $n_i\in N$, the positive nilpotent subgroup determined by A^+. Moreover, we are assuming that $\omega-\lim\limits_{i\to\infty}1id\left(n_ix_0,x_0\right)$ exists. If we write $n_i=\exp\left(\sum_{\lambda\in\Lambda^+}X_{i,\lambda}\right)$ with $X_{i,\lambda}\in g_\lambda$, and put $X_i:=\sum_{\lambda\in\Lambda^+}X_{i,\lambda}$, we find, as in Remark 2, there is a nonnegative constant c such that

$$\lim_{i\to\infty}\frac{\|X_i\|}{e^{ci}}=0.$$

Let us consider a geodesic ray $\gamma(t) = \exp(tH)x_0$ contained in the Weyl sector A^+x_0, with $H \in a^+$ and define $\gamma_a(i) = \gamma(ai)$. We find that

$$\begin{aligned} d(n_i\gamma_a(i), \gamma_a(i)) &= d(\exp\left(\sum_{\lambda\in\Lambda^+} X_{i,\lambda}\right)\exp(aiH)x_0, \exp(aiH)x_0) \\ &= d(\exp(-aiH)\exp\left(\sum_{\lambda\in\Lambda^+} X_{i,\lambda}\right)\exp(aiH)x_0, x_0) \\ &= d(\exp(\sum_{\lambda\in\Lambda^+} e^{-ai\lambda(H)}X_{i,\lambda})x_0, x_0). \qquad (1) \end{aligned}$$

By the remark above, since $\lambda(H) > 0$, we find that, for a sufficiently large, $[\gamma_a(i)] = [n_i\gamma_a(i)]$. We note that, a-priori, a depends on H. However, since we are considering H to be a unit vector in the Weyl chamber a^+ and this choice is continuous, there is an a_0 such that equality (1) above holds for every $H \in a^+$ and every $a \geq a_0$. So we proved that $[A_i'] = [A_i^*] \cap [A_i]$ is indeed a subsector of $[A_i]$, by construction contained in the apartment $[F_i]$. Obviously, the same demonstration works for $[B_i'] = [B_i^*] \cap [B_i]$, and the proposition is proved if $\omega - \lim_{i\to\infty} 1id\,(k_ix_0, x_0)$ does exist.

It is left to treat the case when $\omega - \lim_{i\to\infty} 1id\,(k_ix_0, x_0)$ does not exist. In this case, we let A_i^* and B_i^* be the sectors based at x_0 asymptotic to A_i and B_i respectively. Then, lemma 3.3 implies that $[A_i^*] = [B_i^*]$

We put

$$\begin{aligned} [A_i'] : &= [A_i^*] \cap [A_i] \cap [B_i] \\ &= [B_i^*] \cap [A_i] \cap [B_i] \\ &= [B_i']\,. \end{aligned}$$

This is a subsector of both $[A_i]$ and $[B_i]$ and hence, we may consider any sequence of flats F_i containing A_i' (or B_i') and this defines an apartment $[F_i]$ as required.

In order to prove Axiom 5 we need some intermediate results. We start proving that it is possible to choose special representatives for apartments intersecting in half apartments.

Let $\left[\widetilde{H}_i\right], [F_i] \subset \text{cone}_\infty\mathcal{X}$ be two apartments such that $\left[\widetilde{H}_i\right] \cap [F_i]$ is an half apartment. Then we can find a sequence of flats $(H_i)_{i=1}^\infty$such that $[H_i] = \left[\widetilde{H}_i\right]$ with the following property: If $\left[A_i^+\right] \subset [F_i], \left[B_i^+\right] \subset [H_i]$ are sectors (with

$A_i^+ \subset F_i, B_i^+ \subset H_i$) such that $\left[A_i^+\right] = \left[B_i^+\right]$, then, A_i^+ is ϖ-always asymptotic to B_i^+ i.e.,

$$\varpi\left(\left\{i \in N | A_i^+ isasymptotictoB_i^+\right\}\right) = 1$$

We loose no generality by assuming all the B_i^+ are based at x_0. We can also assume that the base point of $\left[A_i^+\right]$ (and hence the base point of $\left[B_i^+\right]$, since they are the same sector in the cone) belongs to the border of the half apartment $\left[\widetilde{H}_i\right] \cap [F_i]$. If it was not the case, we could take a convenient displacement A_i^* of A_i^+ along F_i and also displacement $B_i^*$of B_i^+ along $\widetilde{H}_i$ such that $[A_i^*] = [B_i^*]$ and this sector would be based at the border of the half apartment $\left[\widetilde{H}_i\right] \cap [F_i]$. We loose also no generality by assuming $\widetilde{H}_i \equiv H = \exp Ax$, and we consider the Iwasawa decomposition $G = KAN$, where $N = \exp\left(\sum_{\lambda\in\Lambda^+} g_\lambda\right)$ is the nilpotent group determined by the half apartment $\left[\widetilde{H}_i\right] \cap [F_i]$, i.e., if $X \in a = \mathrm{Lie}\,(A)$ is such that

$$[\exp\left(c \cdot i \cdot X\right) x_0] \in \left[\widetilde{H}_i\right] \cap [F_i]\,, \qquad \forall c \geq 0$$

then $\lambda\left(X\right) > 0$. Every flat in X is of the form nkH, where $n \in N$ and $k \in K$, in particular $F_i = n_i k_i H$. We take now $X \in a{=}Lie\,(A)$ such that

$$[\exp\left(c \cdot i \cdot X\right) x_0] \in \left[\widetilde{H}_i\right] \cap [F_i]\,, \qquad \forall c \geq 0$$

To shorten the notation, we denote $\gamma\left(ci\right) = \exp\left(c \cdot i \cdot X\right) x_0$. Then we have that

$$[\gamma\left(ci\right)] = [n_i k_i \gamma\left(ci\right)]\,, \qquad \forall c \geq 0$$

(if this is not the case, we can change the choice of k_i up to an element of the Weyl group). But this means, by definition, that

$$\varpi - \lim_{i\to\infty} \frac{1}{i} d\left(n_i k_i \gamma\left(ci\right), \gamma\left(ci\right)\right) = 0 \qquad \forall c \geq 0$$

and if we put $n_i = \exp X_i$, we get

$$\begin{aligned}
\frac{1}{i} d\left(n_i k_i \gamma\left(ci\right), \gamma\left(ci\right)\right) &= \frac{1}{i} d\left(\exp\left(-ciX\right) n_i k_i \exp\left(ciX\right) x_0, x_0\right) \\
&= \frac{1}{i} d\left(\exp(\sum_{\lambda\in\Lambda^+} e^{-ci\lambda(X)} X_{i,\lambda}) \exp(-ciX) k_i \gamma(ci), x_0\right)
\end{aligned}$$

where $X_{i=} \sum_{\lambda\in\Lambda} X_{i,\lambda}$ is the root space decomposition of X_i. But since $\lambda(X) > 0$, the term $\exp\left(\sum_{\lambda\in\Lambda^+} e^{-ci\lambda(X)} X_{i,\lambda}\right)$ converges exponentially to the identity and we get that

$$\begin{aligned} \varpi - \lim_{i\to\infty} \frac{1}{i} d\left(n_i k_i \gamma(ci), \gamma(ci)\right) &= \varpi - \lim_{i\to\infty} \frac{1}{i} d\left(\exp(-ciX) k_i \gamma(ci), x_0\right) \\ &= \varpi - \lim_{i\to\infty} \frac{1}{i} d\left(k_i \gamma(ci), \gamma(ci)\right) = 0 \end{aligned}$$

and this would imply also that $\varpi - \lim_{i\to\infty} \frac{1}{i} d\left(k_i\gamma(-ci), \gamma(-ci)\right) = 0$. Since X was chosen arbitrarily within the condition that $\lambda(X) > 0$, we get that in fact the flats $\tilde{H}_i \equiv H$ and $H_i = k_i \tilde{H}_i$ gives rise to the same apartment, that is, $[H_i] = \left[\widetilde{H}_i\right]$ and the sequence $(H_i)_{i=1}^{\infty}$ is the sequence we were looking for.

We need one more lemma before we prove that $\text{cone}_\infty X$ satisfies the fifth axiom of the definition of a non-discrete Tits building:

Let $F = Ax_0$ be an flat in $\mathcal{X}$, let λ be a root and V a unitary vector in the Lie algebra a of A such that $\lambda(V) > 0$ and V is orthogonal to the wall $\{U \in A | \lambda(U) = 0\}$ determined by the root λ. Consider the flats nF and mF where $n = \exp Z$ and $m = \exp Y$ with $Z \in g_\lambda$ and $Y \in g_{-\lambda}$. Suppose $W \in a$ is a (unitary) vector such that the geodesic ray $\alpha(t) = n\exp(-tW)x_0$ is asymptotic to the geodesic ray $\beta(t) = m\exp(tV)x_0$. Then we have that $V = W$.

We consider the apartments $F(\infty)$, $mF(\infty)$ and $nF(\infty)$ in $\partial_\infty X$. The intersection

$$F(\infty) \cap mF(\infty)$$

is a half apartment in $\partial_\infty X$ consisting of

$$\{\gamma(\infty) | \gamma(t) = \exp tU, with U \in a and - \lambda(U) \geq 0\}$$

whose boundary

$$\{\gamma(\infty) | \gamma(t) = \exp tU, with U \in a and - \lambda(U) = 0\}$$

we denote as usual by $\partial(F(\infty) \cap mF(\infty))$.

If we remember that we are denoting by $d_T(\cdot,\cdot)$ the Tits metric on $\partial_\infty X$, since V is orthogonal to the wall determined by λ we have that

$$d_T(\beta(\infty), \gamma(\infty)) = \frac{\pi}{2}$$

for all $\gamma(\infty) \in \partial(F(\infty) \cap mF(\infty))$. We note that the same holds for $\widetilde{\beta}(\infty)$, where $\widetilde{\beta}(t) = \exp(tV)x_0$, i.e.,

$$d_T\left(\widetilde{\beta}(\infty), \gamma(\infty)\right) = \pi 2$$

whenever $\gamma(\infty) \in \partial(F(\infty) \cap mF(\infty))$. Now if we define

$$F^+(\infty) = \{\gamma(\infty) \,|\gamma(t) = \exp tU, withU \in aand\lambda(U) \geq 0\}$$

we get that

$$\partial\left(F^+(\infty)\right) = \partial(F(\infty) \cap mF(\infty))$$

so that

$$F^+(\infty) \cup (F(\infty) \cap mF(\infty))$$

is an apartment in $\partial_\infty \mathcal{X}$ and we see in this apartment that

$$d_T\left(\beta(\infty), \widetilde{\beta}(\infty)\right) = \pi$$

If we define

$$mF^+(\infty) = \{m\gamma(\infty) \,|\gamma(t) = \exp tU, withU \in aand\lambda(U) \geq 0\}$$

we see immediately that $\mathrm{mF}^+(\infty) \cap (F(\infty) \cap mF(\infty)) = F^+(\infty) \cap (F(\infty) \cap mF(\infty))$ $= \{\gamma(\infty) \,|\gamma(t) = \exp tU, withU \in aand\lambda(U) = 0\}$ and this implies that the union of the two half apartments $mF^+(\infty) \cup F^+(\infty)$ is an apartment in $\partial_\infty \mathcal{X}$.

The hypothesis that $\alpha(\infty) = \beta(\infty)$ implies $\alpha(\infty) \in mF^+(\infty)$ while $\widetilde{\beta}(\infty) \in F^+(\infty)$ by definition. Since

$$d_T\left(\widetilde{\beta}(\infty), \gamma(\infty)\right) = d_T(\alpha(\infty), \gamma(\infty)) = \frac{\pi}{2}$$

for all $\gamma(\infty) \in mF^+(\infty) \cap F^+(\infty)$ we get

$$d_T\left(\widetilde{\beta}(\infty), \alpha(\infty)\right) = \pi$$

But we also have, by definition, that $d_T(\alpha(\infty), \alpha(-\infty)) = \pi$ and since $nF^+(\infty) = F^+(\infty)$ (because $n = \exp Z$ with $Z \in g_\lambda$ and $\lambda(V) > 0$) and by definition $\widetilde{\beta}(\infty) \in nF^+(\infty)$ we have two points in the apartment $mF^+(\infty) \cup F^+(\infty)$ (namely $\alpha(-\infty)$ and $\widetilde{\beta}(\infty)$) opposed to $\alpha(\infty)$. Because every apartment is isometric to a unit sphere we get $\alpha(-\infty) = \widetilde{\beta}(\infty)$ and consequently $V = W$.

The pair $(\mathrm{cone}_\infty \mathcal{X}, \mathcal{F})$ satisfies axiom (A5), i.e. if F, H and L are three apartments such that the intersection of any two of them is an half-apartment, then $F \cap L \cap H \neq \emptyset$.

First of all we take representatives for our apartments: $F = [F_i]\,, L = [L_i]$ and $H = [H_i]$. Again, with no loss of generality, we may assume $x_0 \in F_i$ and that $[x_0]$ is in the (topological) border $\partial\,(F \cap L)$ of $F \cap L$. We also loose no generality by assuming $F_i \equiv Ax_0$. The half-apartment $F \cap L$ is determined by a sequence $(\lambda_i)_{i=1}^{\infty}$ of roots of the corresponding root-space decomposition: if a is the Lie algebra of A and $0 \neq X_i \in a$ are such that $[\exp X_i] \in F \cap L$ then $\lambda_i\,(X_i) > 0$. In other words, for each $i \in N$ we may consider all the Weyl sectors contained in F that give rise to sectors in the intersection $F \cap L$. For each such sector there is a set of corresponding positive roots and then $\{\lambda_i\}$ is the intersection of all such sets of positive roots. Such a sequence is well defined up to a subset of indexes with zero ϖ-measure.

By lemma 3.3, we may assume that an half apartment of L_i is asymptotic to an half apartment of F_i for ϖ-almost every $i \in N$ so that we have $L_i = n_i F_i$, with $n_i = \exp Z_i$, $Z_i \in g_{\lambda_i}$. The same holds for the intersection $F \cap H$, hence $H_i = m_i F_i$ where $m_i = \exp Y_i$ with $Y_i \in g_{-\lambda_i'}$.

If $\lambda_i \neq \lambda_i'$ the half apartments in F determined by λ_i and any translation (in F) of the apartment determined by $-\lambda_i'$ intersect one the other and contains a Weyl sector, hence, if

$$\varpi\left(\left\{i \in N \mid \lambda_i \neq \lambda_i'\right\}\right) = 1$$

we will have a hole sector contained in $F \cap L \cap H$.

So, we may look now at the case when ϖ-always $\lambda_i = \lambda_i'$.

Let $V_i \in A$ be a sequence of unit vectors with $\lambda_i\,(V_i) > 0$. Then we have that

$$[\gamma_i\,(c \cdot i)] = \exp\,(c \cdot iV_i)\,x_0 \in F \cap L,$$

for all $c > 0$. From here on, we will assume V_i to be orthogonal to the wall

$$\{V \in a | \lambda_i\,(V) = 0\}$$

Since $F \cap H$ is determined by sequence of parallel walls (up to a subset on indexes not contained in ϖ), there is a sequence $s_i \in R$ such that, if we put

$$\widetilde{\gamma}_i\,(t) = \gamma_i\,(-\,(s_i + t))$$

we get that $[\widetilde{\gamma}_i(0)] \in \partial(F \cap H)$ and $[\widetilde{\gamma}_i(c \cdot i)] \in F \cap H$ for all $c > 0$. If we put

$$\alpha_i(t) = m_i \widetilde{\gamma}_i(t)$$

we find that

$$[\alpha_i(c \cdot i)] = [\widetilde{\gamma}_i(c \cdot i)] \, for every c > 0$$

because $m_i \in g_{-\lambda_i}$, $-\lambda_i(-V_i) > 0$ and

$$\begin{aligned} d(\alpha_i(t), \widetilde{\gamma}_i(t)) &= d(m_i \exp((-s_i - t) V_i) x_0, \exp((-s_i - t) V_i) x_0) \\ &= d(\exp(Y_i) \exp((-s_i - t) V_i) x_0, \exp((-s_i - t) V_i) x_0) \\ &= d(\exp((s_i + t) V_i) \exp(Y_i) \exp((-s_i - t) V_i) x_0, x_0) \\ &= d\left(\exp\left(\left(e^{-(s_i+t)\lambda_i(V_i)}\right) Y_i\right) x_0, x_0\right) \end{aligned}$$

and the only instance in which $e^{-(s_i+t)\lambda_i(V_i)}$ would not converge to zero is when $V_i \to 0$ but, in this case, we have that $[m_i x_0] = [x_0] \in F \cap L \cap H$ and there is nothing left to prove.

By the same reasoning we did before, for a suitable choice of a sequence $t_i \in R$ and by putting

$$\widetilde{\alpha}_i(t) = \alpha_i(-(t_i + t))$$

we get that $[\widetilde{\alpha}_i(0)] \in \partial(L \cap H)$ while $[\widetilde{\alpha}_i(c \cdot i)] \in L \cap H$ for every $c > 0$.

Now, for a suitable choice of unitary $W_i \in a$ we will have the geodesic ray $n_i \exp(tW_i)x_0$ asymptotic to the ray $\widetilde{\alpha}_i(t)$, for $t > 0$. Since we have chosen V_i to be orthogonal to the wall, lemma 3.3 assures that actually $W_i = V_i$.

Independently on the fact of V_i being orthogonal to the wall, for a suitable translation along the wall determined by λ_i, i.e., for a suitable choice of $a_i = \exp A_i$, with $A_i \in a$ and $\lambda_i(A_i) = 0$ and a shift determined by r_i, if we put

$$\beta_i(t) = n_i \exp((r_i + t) V_i) a_i x_0$$

we have that $[\beta_i(c \cdot i)] = [\widetilde{\alpha}_i(c \cdot i)]$.

We note here that we have only decomposed the displacements along the flats $n_i A x_0$ into components, one belonging to the wall λ_i and the other in a transversal direction determined by the geodesic ray $n_i \exp(tV_i)$. Since

$$a = RV_i \oplus \{V \in a \mid \lambda_i(V) = 0\}$$

this decomposition is possible and unique.

The distances between the initial points of our geodesic rays are given by

$$\begin{cases} d\left(\left[\gamma_i(0)\right],\left[\widetilde{\gamma}_i(0)\right]\right)=d\left(\left[\gamma_i(0)\right],\left[\alpha_i(0)\right]\right)=\varpi-\lim s_i i \\ d\left(\left[\widetilde{\gamma}_i(0)\right],\left[\widetilde{\alpha}_i(0)\right]\right)=d\left(\left[\widetilde{\gamma}_i(0)\right],\left[\beta_i(0)\right]\right)=d\left(\left[\alpha_i(0)\right],\left[\widetilde{\alpha}_i(0)\right]\right)=\varpi-\lim t_i i \\ d\left(\left[\widetilde{\alpha}_i(0)\right],\left[a_i x_0\right]\right)=d\left(\left[\beta_i(0)\right],\left[a_i x_0\right]\right)=\varpi-\lim r_i i \\ d\left(\left[\gamma_i(0)\right],\left[a_i x_0\right]\right)=\varpi-\lim \|A_i\| i \end{cases}$$

and to make more clear the place we are standing on we stress that

$$\begin{gathered} \left[\gamma_i(0)\right],\left[a_i x_0\right]\in F\cap L \\ \left[\alpha_i(0)\right]=\left[\widetilde{\gamma}_i(0)\right]\in F\cap H \\ \left[\widetilde{\alpha}_i(0)\right]=\left[\beta_i(0)\right]\in L\cap H \end{gathered}$$

and we will prove one of those point belongs also to the third apartment. Let us note that if we prove that either $\varpi-\lim s_i i=0$ or $\varpi-\lim t_i i=0$ or $\varpi-\lim r_i i=0$ the proposition will follow at once.

Since $W_i=V_i$ we have that

$$\begin{aligned} & d\left(\beta_i(0),\widetilde{\alpha}_i(0)\right)=d\left(n_i\exp\left(r_iV_i\right)a_ix_0, m_i\exp\left(\left(t_i-s_i\right)V_i\right)x_0\right) \\ & =d\left(\exp Z_i\exp\left(r_iV_i\right)a_ix_0,\exp Y_i\exp\left(\left(t_i-s_i\right)V_i\right)x_0\right) \\ & =d\left(\exp\left(-e^{(t_i-s_i)\lambda_i(V_i)}Y_i\right)\exp\left(\left(t_i-s_i+r_i\right)V_i\right)\exp\left(e^{-r_i\lambda_i(V_i)}Zi\right)a_ix_0,x_0\right) \end{aligned}$$

If the sequence $(r_i)_{i=1}^{\infty}$ is bounded, we have that

$$d\left(\left[\widetilde{\alpha}_i(0)\right],\left[a_ix_0\right]\right)=\varpi-\lim r_i i=0$$

and hence $\left[\widetilde{\alpha}_i(0)\right]\in F\cap L\cap H$ and we proved the intersection is not empty. Hence we may suppose the sequence $(r_i)_{i=1}^{\infty}$ is unbounded, so that $\exp\left(e^{-r_i\lambda_i(V_i)}Zi\right)$ converges exponentially to the identity isometry and since we are interested only in linear factors, we can approximate

$$d\left(\beta_i(0),\widetilde{\alpha}_i(0)\right)\approx d\left(\exp\left(-e^{(t_i-s_i)\lambda_i(V_i)}Y_i\right)\exp\left(\left(t_i-s_i+r_i\right)V_i\right)a_ix_0,x_0\right).$$

If we substitute $a_i=\exp A_i$ and consider the fact that A_i commute with V_i we get that

$$\begin{aligned} d\left(\beta_i(0),\widetilde{\alpha}_i(0)\right) &= d(\exp(-e^{(t_i-s_i)\lambda_i(V_i)}Y_i)\exp\left(\left(t_i-s_i+r_i\right)V_i\right)\exp A_ix_0,x_0) \\ &= d(\exp(-e^{(t_i-s_i)\lambda_i(V_i)}Y_i)\exp((t_i-s_i+r_i)V_i+A_i)x_0,x_0) \end{aligned}$$

At this point we would like to use any result to relate $\exp X\exp Y=\exp Z$, or to be more precise, we want to find U_i such that

$$\exp U_i=\exp\left(-e^{-(s_i-t_i)\lambda_i(V_i)}Y_i\right)\exp\left(\left(t_i-s_i+r_i\right)V_i+A_i\right)$$

We could use for instance the **Baker-Campbell-Hausdorf** formula:

$$\exp X \exp Y = \exp\left(\sum_{n=1}^{\infty} C_n\left(X : Y\right)\right)$$

where $C_1\left(X : Y\right) = X + Y$ and

$$\begin{aligned} (n+1)\,C_{n+1}\left(X : Y\right) &= \frac{1}{2}\left[X - Y, C_n\left(X : Y\right)\right] \\ &\quad + \sum_{p\geq 1} \\ 2p \leq n K_{2p} &\sum_{k_1,\cdots,k_{2p}>0} \\ k_1 + \ldots + k_{2p} = n &\Big[C_{k_1}, \Big[\ldots \Big[C_{k_{2p}}, X + Y\Big]\Big]\Big] \end{aligned}$$

where the K_{2p} are rational numbers and C_n stands for $C_n\left(X : Y\right)$.

The problem in the use of this formula is that the convergence of the series is generally only local and we need global convergence. However, we note that

$$\begin{gathered} \left[A_i, V_i\right] = 0 \\ \left[Y_i, A_i\right], \left[Y_i, V_i\right] \in g_{-\lambda_i} \end{gathered}$$

and this implies that Y_i and $\left(t_i - s_i + t_i\right) V_i + A_i$ generates, for each $i \in N$, a two dimensional semi-simple subalgebra of g, hence we are in fact in a subalgebra isomorphic to $\mathbf{sl}\left(2\right)$ and the convergence of the Baker-Hausdorf-Campbell serie is in fact global and we may apply it to our case globally.

Without calculating it explicitly, we know that $C_n\left(X : Y\right)$ is a sum of brackets of order n involving X and Y. In our specific case, this brackets involves Y_i and $\left(t_i - s_i + r_i\right) V_i + A_i$ and hence, for every $n \geq 2$ and every $i \in N$

$$C_n\left(Y_i : \left(t_i - s_i + r_i\right) V_i + A_i\right) \in g_{-\lambda_i}$$

and we get that

$$\begin{gathered} \exp\left(-e^{-(s_i - t_i)\lambda_i(V_i)} Y_i\right) \exp\left(\left(t_i - s_i + r_i\right) V_i + A_i\right) \\ = \exp\left(\left(\left(t_i - s_i + r_i\right) V_i + A_i\right) + f_i Y_i\right) \end{gathered}$$

where f_i is defined by

$$f_i Y_i = -e^{-(s_i - t_i)\lambda_i(V_i)} Y_i + \sum_{n=2}^{\infty} C_n\left(\left(t_i - s_i + r_i\right) V_i + A_i : -e^{-(s_i - t_i)\lambda_i(V_i)} Y_i\right)$$

By definition of the vectors, since A_i and V_i are contained in the subalgebra a and $Y_i \in g_{-\lambda_i}$, we have that $(t_i - s_i + r_i) V_i + A_i$ and $f_i Y_i$ are linearly independent.

But,

$$\begin{aligned} & \varpi - \lim_{i\to\infty} \frac{1}{i} d\left(\beta_i\left(0\right), \widetilde{\alpha}_i(0)\right) \\ = \quad & \varpi - \lim_{i\to\infty} \frac{1}{i} d\left(\exp\left(\left(t_i - s_i + r_i\right) V_i + A_i + f_i Y_i\right) x_0, x_0\right) \end{aligned}$$

so that we must have

$$\varpi - \lim_{i\to\infty} \frac{1}{i}\left(\left(t_i - s_i + r_i\right) V_i + A_i + f_i Y_i\right) = 0$$

and from the linear independence we cited above we get that

$$\varpi - \lim_{i\to\infty} \frac{1}{i}\left(\left(t_i - s_i + r_i\right) V_i + A_i\right) = 0.$$

But V_i and A_i are also linearly independent so, we must have

$$\varpi - \lim_{i\to\infty} \frac{1}{i}\left(t_i - s_i + r_i\right) = 0.$$

From the linearity of the ultra-limit (property 3, page 13), we find that

$$\begin{aligned} d\left(\left[x_0\right], \left[\widetilde{\alpha}_i\left(0\right)\right]\right) + d\left(\left[\widetilde{\alpha}_i\left(0\right)\right], \left[\alpha_i\left(0\right)\right]\right) &= \varpi - \lim_{i\to\infty} \frac{1}{i} r_i + \varpi - \lim_{i\to\infty} \frac{1}{i} t_i \\ &= \varpi - \lim_{i\to\infty} \frac{1}{i} s_i = d\left(\left[x_0\right], \left[\alpha_i\left(0\right)\right]\right) \end{aligned}$$

so that the triangle $\triangle\left(\left[x_0\right], \left[\alpha_i\left(0\right)\right], \left[\widetilde{\alpha}_i\left(0\right)\right]\right)$ is a degenerated triangle, and $\left[\alpha_i\left(0\right)\right]$ is actually contained in the side determined by $[x_0]$ and $\left[\widetilde{\alpha}_i\left(0\right)\right]$. But this side of the triangle is contained in the flat $L = [L_i]$ and $\left[\alpha_i\left(0\right)\right]$ is contained in the intersection of $F = [F_i]$ and $H = [H_i]$ so that

$$\left[\alpha_i\left(0\right)\right] \in F \cap L \cap H$$

as desired.

The pair $(\mathrm{cone}_\infty \mathcal{X}, \mathcal{F})$ is a non-discrete Tits building.

We proved the axioms are satisfied in propositions 3.3, 3.3, 3.3, 3.3 and 3.3.

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