Control Sets on Flag Manifolds and Ideal Boundaries of Symmetric Spaces*

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Abstract

Given a semigroup S with non-empty interior, contained in a semi-simple real Lie group of non-compact type G, the effective control sets of S in the flag manifolds are well known. In this work we consider the orbits of S in a symmetric space and its images by the Weyl group and describe the effective control sets in the flag manifolds as images of those orbits.

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1 Introduction

Let G be a semisimple real Lie group of non compact type and finite center and let $S \subset G$ be a semigroup of non empty interior. Let us consider the left action of S in the flag manifold G/P, where P is a parabolic subgroup

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of G. In a previous work ([4]) we considered the ideal boundary $\partial_\infty(X)$ of the symmetric space $X = G/K$, where K is a maximal compact subgroup of G, and described the relation between the invariant control set of S on G/P and the set of points in the ideal boundary of $\partial_\infty(X)$ that belong to the closure of any orbit Sx, where x is an arbitrary point of the symmetric space X.

In this work we **complete** the previous one ([4]) by describing the relation of (non invariant) control sets in the flag manifold G/P and the ideal boundary $\partial_\infty(X)$. This will be done using the characterization of control sets established by San Martin and Tonelli in [9], where the control sets are indexed by elements of the Weyl group W.

2 Basic constructions

In this section we merely establish the vocabulary and notations that will

be needed to explain and prove the results.All definitions and constructions needed in this work are presented with some explanations in [4] and we refer the reader to that work for further details.

Let $\mathcal{X}$ be a symmetric space of non-compact type, $G = \mathrm{Isom}^0(\mathcal{X})$ the identity component of the isometry group of $\mathcal{X}$ and K the stabilizer (in G) of a point $x_0 \in \mathcal{X}$. Then $\mathcal{X} = G/K$, G is a *real semi-simple Lie group* and K a *maximal compact* subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{x}$ the *Cartan decomposition*, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{x}$ its orthogonal complement (relatively to the Cartan-Killing form).

The *root space decomposition* of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda$$

where $\lambda \in \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$,

$$\mathfrak{g}_\lambda = \{Y \in \mathfrak{g} |\, [H, Y] = \lambda(H)\, Y, \text{ for all } H \in \mathfrak{a}\}$$

and

$$\Lambda = \{\lambda \in \mathrm{Hom}(\mathfrak{a}, \mathbb{R}) \,|\mathfrak{g}_\lambda \neq \{0\}\}.$$

The λ's in Λ are called *roots* of $\mathfrak{g}$ and each $\mathfrak{g}_\lambda$ a *root subspace.* Each root $\lambda \in \Lambda$ determines a hyperplane $\mathcal{H}_\lambda = \{H \in \mathfrak{a} | \lambda(H) = \{0\}\}$. Each component of

$$\mathfrak{a} \setminus \bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda$$

is said to be an *open Weyl chamber.* A *Weyl chamber* is the closure of an open Weyl Chamber. A Weyl chamber $\overline{\mathfrak{a}^+}$ uniquely determines a set of positive roots

$$\Pi^+ = \left\{\lambda \in \Lambda | \lambda(H) \geq 0 \text{ for every } H \in \overline{\mathfrak{a}}^+\right\}$$

a set of negative roots $\Pi^- = -\Pi^+$, and a set of *simple roots* Σ.

A Weyl chamber $\mathfrak{a}^+$ (alternatively, a set of positive roots Π^+ or a set of associated simple roots Σ) determines maximal nilpotent subalgebras

$$\mathfrak{n}^\pm = \sum_{\lambda \in \Pi^\pm} \mathfrak{g}_\lambda.$$

We denote by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A *minimal parabolic subalgebra* is any algebra conjugate in $\mathfrak{g}$ to

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

The subalgebras $\mathfrak{a}$, $\mathfrak{n}^+$ and $\mathfrak{m}$ are determined by the choice of a Weyl chamber $\mathfrak{a}^+$. For a subset $\Theta \subseteq \Sigma$, let $\mathfrak{n}^-(\Theta)$ stands for the subalgebra spanned by the root spaces $\mathfrak{g}_{-\lambda}$, for $\lambda \in \langle\Theta\rangle$, where $\langle\Theta\rangle$ is the set of (positive) roots generated by Θ and let we denote by $\mathfrak{p}_\Theta$ the *parabolic subalgebra*

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}.$$

We note that, $\mathfrak{p}_\emptyset = \mathfrak{p}$ and $\mathfrak{p}_\Sigma = \mathfrak{g}$.

An *Iwasawa decomposition* of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$.

For all objects introduced so far in the Lie algebra $\mathfrak{g}$ we find corresponding objects in the Lie group G and the symmetric space $\mathcal{X} = G/K$. We denote by x_0 the base point in $\mathcal{X}$. The subspace $\mathfrak{x} \subset \mathfrak{g}$ is identified with the tangent space of $\mathcal{X}$ at x_0 so that geodesics in $\mathcal{X}$ with initial point x_0 are defined as $\eta(t) = \exp(tY)\, x_0$, for some vector $Y \in \mathfrak{x}$, with $\|Y\| = 1$.

By defining $A = \exp \mathfrak{a}$, $K = \exp \mathfrak{k}$ and $N^+ = \exp \mathfrak{n}^+$, we get an *Iwasawa decomposition* $G = KAN^+$.

A *flat* in $\mathcal{X}$ is an isometrically embedded Euclidean space and $F = Ax_0$ is a maximal flat in $\mathcal{X}$. Every maximal flat in $\mathcal{X}$ is of the form $F' = gF = gAx_0$, with $g \in G$. The *rank* of a symmetric space is the dimension of a maximal flat and it equals the dimension of A.

The structure of Weyl chambers in $\mathfrak{a}$ carries over to the subgroup A and to the flat $F = Ax_0 \subset \mathcal{X}$: if we denote by $\mathfrak{a}^+$ an open Weyl chamber in $\mathfrak{a}$ and by $A^+ = \exp \mathfrak{a}^+$ its image in G, we shall call gA^+x_0 a *Weyl chamber*, to any $g \in G$ (whenever we mean a chamber in the Lie Algebra, Lie Group or Simmetric Space it should be clear by the notation and the context). The point $gx_0 \in gA^+x_0$ is called the *base point* of the chamber. A subalgebra $\mathcal{H}_\Theta$ gives rise to Θ-*flats* $gF_\Theta := g \exp(\mathcal{H}_\Theta) x_0$. In a similar way, we say that $g\overline{A}_\Theta^+ x_0 := g \exp\left(\overline{\mathfrak{a}}_\Theta^+\right) x_0$ is the Θ-*wall* of the chamber gA^+x_0.

A parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}$ determines a *parabolic subgroup*

$$P_\Theta = \{g \in G | \mathrm{Ad}(g)\, \mathfrak{p}_\Theta = \mathfrak{p}_\Theta\}.$$

The Weyl group of G is the quotient $W = M/M'$, where $M' := Z_K(A)$ is the centralizer of A in K. The Weyl group may naturally be seen as the group of symmetries of $\mathfrak{a}$ generated by the reflections $\sigma_\lambda$in the hyperplanes $\mathcal{H}_\lambda = \{H \in \mathfrak{a} | \lambda(H) = \{0\}\}$, for $\lambda \in \Lambda$ a root. It is actually generated by reflections in hyperplanes determined by any simple system of roots Σ. Given a subset $\Theta \subset \Sigma$, we denote by W_Θ the subgroup of W generated by the corresponding reflections.

Each parabolic subgroup determines a (compact) *flag manifold* G/P_Θ. If λ is a root determined by $\mathfrak{a}$, then $g\lambda$ is the root of $\mathrm{Ad}(g)\,\mathfrak{a}$ defined by the formula

$$g\lambda(H) = \mathrm{Ad}(g) \circ \lambda \circ \mathrm{Ad}\left(g^{-1}\right)(H), \text{ for all } H \in \mathrm{Ad}(g)\,\mathfrak{a}.$$

A parabolic subgroup is said to be of *type* Θ if it is determined by a set of roots of the form $g(\Theta)$ and the flag manifold $\mathbb{B}_\Theta$ may be viewed as the set of all type Θ parabolic subgroups. In particular, $P_\emptyset = P$ and $G/P_\emptyset$ is the set of all minimal parabolic subgroups.

Parabolic subgroups are partially ordered by inclusion, with $P_{\Theta_1} \subset P_{\Theta_2}$ iff $\Theta_1 \subset \Theta_2$. Hence there is a natural fibration

$$\begin{aligned} \rho_{\Theta_2}^{\Theta_1} : G/P_{\Theta_1} &\to G/P_{\Theta_2} \\ gP_{\Theta_1} &\mapsto gP_{\Theta_2} \end{aligned}$$

In the special case when $G/P_{\Theta_1} = G/P$, we denote $\rho_{\Theta_2}^{\Theta_1}$ by ρ_{Θ_2}.

A *Hadamard manifold* is a simply connected manifold of non positive curvature.

Let $\mathcal{X}$ be a Hadamard manifold. Two geodesic rays $\gamma, \beta : \mathbb{R}^+ \longrightarrow \mathcal{X}$ are said to be *asymptotic* if there is a constant $a \geq 0$ such that $d(\gamma(t), \beta(t)) \leq a$, for every $t \geq 0$. This defines an equivalence relation on the set of all geodesic rays in $\mathcal{X}$. We call the set of equivalence classes of asymptotic geodesic rays the *ideal boundary* of $\mathcal{X}$. We denote this space by $\partial_\infty \mathcal{X}$ and the equivalence class determined by γ we denote by $\gamma(\infty)$. Since $\mathcal{X}$ is simply connected, given any class $\gamma(\infty)$ and any point $x_0 \in \mathcal{X}$ there is a unique geodesic ray $\beta : \mathbb{R}^+ \longrightarrow \mathcal{X}$ with $\beta(0) = x_0$ and $\beta(\infty) = \gamma(\infty)$, so that we can identify $\partial_\infty \mathcal{X}$ with the unit tangent sphere and give it the usual metric of a unit sphere. We shall denote this metric by $d_S(\cdot, \cdot)$.

Let $\mathcal{X}$ be a Hadamard manifold. We fix a point $x_0 \in \mathcal{X}$ and for a given $\eta \in \partial_\infty \mathcal{X}$ we choose the unique geodesic ray $\eta(s)$ such that $\eta(0) = x_0$ and $\eta(\infty) = \eta$. Given a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $\mathcal{X}$, consider the sequence of geodesic rays $(\eta_n(s))_{n=1}^\infty$ such that $\eta_n(0) = x_0$ and $\eta_n(d(x_0, x_n)) = x_n$. We say that x_n converges to η if $\lim_{n\to\infty} d(x_0, x_n) = \infty$ and $\lim_{n\to\infty} \eta_n'(0) = \eta'(0)$, this last condition being equivalent to

$$\lim_{n\to\infty} \eta_n(\infty) = \eta$$

in $(\partial_\infty \mathcal{X}, d_S)$. This defines a topology on $\overline{\mathcal{X}} := \mathcal{X} \cup \partial_\infty \mathcal{X}$ that coincides with the metric topology on $\mathcal{X}$ and with the sphere metric d_S in $\partial_\infty \mathcal{X}$, and such that $\partial_\infty \mathcal{X}$ is closed and $\mathcal{X}$ open and dense in $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}}$ is a compact topological space.

For a subset $\mathcal{C} \subset \mathcal{X}$, we define its ideal boundary $\partial_\infty \mathcal{C} := \partial \mathcal{C} \cap \partial_\infty \mathcal{X}$, where $\partial \mathcal{C}$ stands for the usual boundary in $\overline{\mathcal{X}}$. If $\mathcal{C}$ is convex, then

$$\partial_\infty \mathcal{C} = \{\eta(\infty) \,|\, \eta(s) \text{ is a geodesic ray contained in } \mathcal{C}\}.$$

Let us assume from here on that $\mathcal{X} = G/K$ is a symmetric space of non-compact type and real rank at least 2. Since every Weyl chamber $B_g = gA^+x_0$ is convex, we have that

$$\begin{aligned} g\overline{A}^+(\infty) &= \partial_\infty \left(gA^+x_0\right) \\ &= \left\{\eta(\infty) \,|\, \eta(s) = g(\exp sX)\, x_0,\ X \in \mathfrak{a}^+\right\}. \end{aligned}$$

This is called an *open Weyl chamber at infinity*. Open Weyl chambers at infinity are either equal or disjoint.

Similar definitions hold also for *walls at infinity* and *flats at infinity*, denoted by $gA_{\Theta}^{+}(\infty)$ and $gA(\infty)$ respectively. We will consider only *closed* chambers $g\overline{A}^{+}(\infty)$ and walls at infinity:

$$\begin{aligned} g\overline{A}_{\Theta}^{+}(\infty) &= \partial_{\infty}\left(g\overline{A}_{\Theta}^{+}x_0\right) \\ &= \left\{\eta(\infty)\,|\eta(s) = g(\exp sX)\,x_0,\ X \in \mathfrak{a}_{\Theta}^{+}\right\}. \end{aligned}$$

We define a map

$$\pi : \partial_{\infty}\mathcal{X} \to \bigcup_{\Theta\subset\Sigma} G/P_{\Theta},$$

as follows: each $\eta \in \partial_{\infty}\mathcal{X}$ is of the form $\eta(\infty)$ with $\eta(s) = (g\exp sX)\,x_0$ with $|X| = 1$ and $X \in \mathring{\bigcup}_{\Theta\subseteq\Sigma}\mathfrak{a}_{\Theta}^{+}$, where we are considering the open chambers and open walls, so that the union is disjoint. So, we associate to η the parabolic subgroup $\pi(\eta) := gP_{\Theta}g^{-1}$ (where $X \in \mathfrak{a}_{\Theta}^{+}$). This association is independent of the choice of g.

We denote by $\partial_{\infty}^{\Theta}\mathcal{X}$ the inverse image $\pi^{-1}(G/P_{\Theta})$, the set of all Θ-*singular* geodesic rays and note that $\pi^{-1}(G/P_{\emptyset}) = \partial_{\infty}^{\emptyset}\mathcal{X}$ is an open and dense subset of $\partial_{\infty}\mathcal{X}$. Also, $\mathring{\bigcup}_{\lambda\in\Sigma}\partial_{\infty}^{\{\lambda\}}\mathcal{X}$ is open and dense in $\partial_{\infty}\mathcal{X}\backslash\partial_{\infty}^{\emptyset}\mathcal{X}$. In the same way, we find that $\mathring{\bigcup}_{\substack{\Theta\subseteq\Sigma \\ |\Theta|=k}}\partial_{\infty}^{\Theta}\mathcal{X}$ is open and dense in $\partial_{\infty}\mathcal{X}\backslash\left(\mathring{\bigcup}_{\substack{\Phi\subseteq\Sigma \\ |\Phi|<k}}\partial_{\infty}^{\Phi}\mathcal{X}\right)$, where $|\Theta|$ is just the cardinality of Θ and $k \leq \mathrm{r}(\mathfrak{g})$. The projection $\pi : \partial_{\infty}\mathcal{X} \to \bigcup_{\Theta\subseteq\Sigma} G/P_{\Theta}$ splits as a set of projections

$$\pi^{\Theta} : \partial_{\infty}^{\Theta}\mathcal{X} \to G/P_{\Theta}, \quad \Theta \subseteq \Sigma.$$

3 Semigroups and control sets

Let $X = G/L$ be an homogeneous manifold. We denote respectively by $\mathrm{cl}D$ and $\mathrm{int}D$ the closure and the interior of a subset D (of X or G, to be clearly understood from the context). A set S of diffeomorphisms of M is a *semigroup* if the composition of elements of S (with possible restrictions of domains) is still in S. A *control set* for S (an S-c.s.) is a subset $\mathrm{C} \subseteq X$ satisfying the conditions:

(i) $\text{int}(\mathrm{C}) \neq \emptyset$;

(ii) $\mathrm{C} \subseteq \text{cl}\,(Sx)$ for all $x \in \mathrm{C}$, and

(iii) C is maximal with properties *(i)* and *(ii)*.

We say that C is an *invariant control set* for S (an S-i.c.s.) if condition *(ii)* is substituted by the following:

(ii') $\text{cl}\,(Sx) = \text{cl}\,(\mathrm{C})$, for all $x \in \mathrm{C}$.

For the simplicity of the presentation, we assume that G is a semisimple Lie group of non-compact type and S a subsemigroup of G, even if some of the results do not depend on the semisimplicity of G. Regarding the control sets in a compact homogeneous space $X = G/L$ we have the following:

[9, Proposition 2.1] Let $X = G/L$ be a compact homogeneous space and S a subsemigroup of G with $\text{int}S \neq \emptyset$. Let $\mathrm{C} \subset X$ be an S-c.s and let $\mathrm{C}_0 = \{x \in \mathrm{C} | x \in (\text{int } S)\, x\}$. Then:

(i) $\mathrm{C}_0 = \text{int } (S)\,\mathrm{C} \cap \mathrm{C}$.

(ii) $\mathrm{C} \subset (\text{int } (S))^{-1}\, x$ for all $x \in \mathrm{C}_0$ (if $\mathrm{C}_0 \neq \emptyset$).

(iii) $\mathrm{C}_0 = \{x \in C | \exists g \in \text{int } S \text{ with } gx = x\}$.

(iv) $\mathrm{C}_0 = \{x \in C | \exists g \in \text{int } S \text{ with } g^{-1}x \in \mathrm{C}\}$.

(v) $\text{cl } \mathrm{C}_0 = \mathrm{C}$.

Because of property *(ii)* in the proposition above, C_0 is called the *set of transitivity* of C.

From here on we assume as standart hypothesis that G is a semisimple real Lie group of non compact type and finite center, $\mathcal{X} = G/K$ the associated symmetric space and $S \subset G$ is a semigroup of non empty interior.

The product MA is a closed subgroup of G. The homogeneous space G/MA may be seen as the set of Weyl chambers in $\mathfrak{g}$ or the set of Weyl chambers in G with base point at the identity. Alternatively, it may be seen as the choice of a Weyl chamber decomposition in each of the flats gAx_0 of $\mathcal{X}$. Each Weyl chamber $b = gMA$ is conjugate to the base chamber A^+: $b = gA^+g^{-1}$.

We assume throughout the rest of the paper that S has non-empty interior. Then, it has a unique S invariant control set C ([8, Theorem 3.1]). If we put

$$\Delta := \left\{b = gA^{+}g^{-1} \in G/MA | b \cap \operatorname{int} S \neq \emptyset\right\}, \tag{1}$$

we have the following:

Theorem 3.1 *[9, Theorem 3.1] Let* C *be the unique* S*-i.c.s. in* G/P *and* C_0 *be its set of transitivity. Let*

$$p : G/MA \to G/MAN^{+}$$

be the canonical projection. Then

$$\mathrm{C}_0 = p(\Delta).$$

Using this theorem we were able to describe the S invariant control set in the ideal boundary $\partial_\infty(\mathcal{X})$, as follows:

Theorem 3.2 *[4, Theorem 4.3] Let* S *be a sub-semigroup with non-empty interior of a semisimple Lie group* G*. Consider the boundary of an orbit* Sx_0 *in* G/K *and let* D *be the ideal boundary* $\partial_\infty(Sx_0)$*. Then* D *is the invariant control set of* S*. Moreover, if* C_Θ *is the unique* S*-i.c.s. in* G/P *and* $\mathrm{D}^\Theta = \mathrm{D} \cap \partial_\infty^\Theta(\mathcal{X})$*. Then,*

$$\pi^\Theta\left(\mathrm{D}^\Theta\right) = \mathrm{C}_\Theta.$$

Theorem 3.1 describes the invariant control sets as sets of fixed points of some elements in the interior of the semigroup S. The other control sets (the non-invariant ones) are given by other classes of fixed points, classes indexed by the elements of the Weyl group $W = M/M'$. In fact, there is a natural right action of W on G/MA that is given by

$$(gMA)\,w = g\widetilde{w}MA$$

where $\widetilde{w} \in M$ is any representative of the element w, that is, $w = \widetilde{w}M$. The control sets in the maximal flag G/MAN^{+} are determined by the action of the Weyl group in G/MA, in the following sense:

Theorem 3.3 *[9, Theorem 3.2] Let* $p : G/MA \to G/MAN^{+}$ *be be the canonical projection. Given* $w \in W$*, there is a unique control set* $\mathrm{C} \subset G/MAN^{+}$ *such that* $p(\Delta w) \subset \mathrm{C}$.

Not every control set may be described in such a way, but exactly the class of *effective control sets* (control sets with non-empty sets of transitivity) may be obtained in this way and we denote by E_w the (effective) control set containing $p(\Delta w)$. We note that E_e is the invariant control set ($e \in W$ is the identity element).

The control sets in other flag manifolds (not necessarily maximal) also admit a similar description:

Theorem 3.4 *[9, Proposition 5.1] Let G/P be the maximal flag manifold determined by G, P_Θ a parabolic subgroup and $\rho_\Theta : G/P \to G/P_\Theta$ the canonical fibration. Let $\mathrm{E} \subset G/P_\Theta$ be an effective control set for S. Then there exists $w' \in W$ such that $\rho_\Theta\left((\mathrm{D}w)_0\right) = \mathrm{E}_0$ for every representative $w \in w'W_\Theta$. Moreover, $\pi_\Theta(\mathrm{D}_0) = \mathrm{E}_0$ if D is an effective control set satisfying $D_0 \cap \rho_\Theta^{-1}(\mathrm{D}_0) \neq \emptyset$.*

It is important for the development of this work to note that the projection $p : G/MA \to G/MAN^+$ is invariant by the action of the group W. Indeed, consider $\alpha = gMA \in G/MA$ and $w = \widetilde{w}M \in W$. Then we have that

$$\begin{aligned} p(\alpha w) &= p(g\widetilde{w}MA) \\ &= g\widetilde{w}MAN^+ \\ &= \left(gMAN^+\right) w \\ &= p(\alpha)\, w. \end{aligned}$$

3.1 The Action of W in $\partial_\infty \mathcal{X}$

Our goal in this section is to show that the action of the Weyl group W on the ideal boundary $\partial_\infty \mathcal{X}$ is invariant by the projection $\pi : \partial_\infty \mathcal{X} \to \bigcup_{\Theta \subset \Sigma} G/P_\Theta$.

We start describing the action of W on $\partial_\infty \mathcal{X}$. Let us consider a point $\eta \in \partial_\infty \mathcal{X}$ and $w \in W$. Then, η is the equivalence class $\eta(\infty)$ of a geodesic ray

$$\eta(s) = \exp\left(\mathrm{Ad}(g)(sX)\right) x_0$$

where $\|X\| = 1$ (the norm determined by the Cartan-Killing form), $X \in \bigcup_{\Theta \subset \Sigma} \mathfrak{a}_\Theta^+$ and $g \in G$. If $\widetilde{w} \in M$ is a representative of $w \in W$, we define the action of w on $\partial_\infty \mathcal{X}$ by

$$\eta w := \alpha(\infty)$$

where

$$\alpha(s) := \exp\left(\mathrm{Ad}\left(g\widetilde{w}\right)(sX)\right)x_0.$$

Since $Ad(\widetilde{w})(X) = X$ for every $\widetilde{w} \in M$ and every $X \in \bigcup_{\Theta\subset\Sigma}\mathfrak{a}_\Theta^+$, this action is well defined.

In order to prove that the action of W on the ideal boundary is invariant by the projection π, let $X \in \bigcup_{\Theta\subset\Sigma}\mathfrak{a}_\Theta^+$ and assume that $X \in \mathfrak{a}_\Theta^+$. Then, by definition of η and π we have that

$$\pi(\eta) = gP_\Theta g^{-1}.$$

It follows that

$$\begin{aligned}\pi(\eta w) = \pi(\alpha(\infty)) &= (g\widetilde{w})\,P_\Theta\,(g\widetilde{w})^{-1}\\ &= g\left(\widetilde{w}P_\Theta\widetilde{w}^{-1}\right)g^{-1}\\ &= (gP_\Theta)\,w = \pi(\eta)\,w.\end{aligned}$$

3.2 Ideal Boundaries and Control Sets

Let E be a control set for the action on G/P of a sub-semigroup $S \subset G$. As we quoted in Theorem 3.3, E is an effective c.s if and a only if there is (a unique) $w \in W$ such that

$$p(\Delta w) \subset \mathrm{E}, \qquad (2)$$

where Δ is defined in (1) and $p : G/MA \to G/MAN^+$ is the canonical projection. In this situation, we denote the effective control set by E_w.

If we denote by C the (unique) invariant control set, we have that $p(\Delta) = \mathrm{C}_0$, the set of transitivity of C and, as shown in [4, Theorem 4.4] we have that $\pi^\emptyset\left(\mathrm{D}^\emptyset\right) = \mathrm{C}$, where $\mathrm{D} = \partial_\infty(Sx_0)$ and $\mathrm{D}^\emptyset = \mathrm{D}\cap\partial_\infty^\emptyset(\mathcal{X})$. So, by the invariance of the action of W in $\partial_\infty(\mathcal{X})$ we find that

$$\begin{aligned}\pi^\emptyset\left(\mathrm{D}^\emptyset w\right) &= \left(\pi^\emptyset\left(\mathrm{D}^\emptyset\right)\right)w\\ &= \mathrm{C}w = p(\Delta)\,w\end{aligned}$$

and by (2) it follows that

$$p(\Delta)\,w \subset \mathrm{E} = \mathrm{E}_w.$$

In other words we have proved the following:

Theorem 3.5 *Let S be a sub-semigroup with non-empty interior of a semi-simple Lie group G. Let E be an effective control set for the action of S on the flag manifold G/P. Consider the orbit Sx of any point x in the symmetric space $\mathcal{X} = G/K$ and its ideal boundary $\mathrm{D} := \partial_\infty (Sx)$ and $\mathrm{D}^\emptyset := \mathrm{D} \cap \partial_\infty^\emptyset (\mathcal{X})$. Then, there is $w \in W$ such that*

$$\pi^\emptyset \left(\mathrm{D}^\emptyset w\right) \subset \mathrm{E}.$$

We consider now the general case of an effective control set E in a (not necessarily maximal) flag manifold G/P_Θ. Theorem 3.4 assures the existence of $w \in W$ such that $\rho_\Theta \left((\mathrm{E}w)_0\right) = \mathrm{E}_0$, where $\rho_\Theta : G/P_\emptyset \to G/P_\Theta$ is the canonical projection and E_w is the unique effective control set in $G/P_\emptyset$ such that $p\,(\Delta w) \subset \mathrm{E}_w$. By Theorem 3.5 we have that

$$\pi^\emptyset \left(\mathrm{D}^\emptyset w\right) \subset \mathrm{E}_w,$$

hence the diagram

$$\begin{array}{ccc}
\partial_\infty^\emptyset(\mathcal{X}) & \xrightarrow{\quad \pi^\emptyset \quad} & G/P_\emptyset \\
\Big\downarrow p_\Theta & & \Big\downarrow \rho_\Theta \\
\partial_\infty^\Theta(\mathcal{X}) & \xrightarrow{\quad \pi^\Theta \quad} & G/P_\Theta
\end{array}$$

commutes and we find that

$$\begin{aligned}
\pi^\Theta \left(\mathrm{D}^\Theta w\right) &= \left(\pi^\Theta \left(\mathrm{D}^\Theta\right)\right) w \\
&= \left(\pi^\Theta \left(\rho_\Theta \left(\mathrm{D}^\emptyset\right)\right)\right) w \\
&= \left(\rho_\Theta \left(\pi^\emptyset \left(\mathrm{D}^\emptyset\right)\right)\right) w \\
&= \rho_\Theta \left(\pi^\emptyset \left(\mathrm{D}^\emptyset w\right)\right) \\
&\subset \rho_\Theta \left(\mathrm{E}_w\right)
\end{aligned}$$

so that

$$\pi^\Theta \left(\mathrm{D}^\Theta w\right)_0 \subset \rho_\Theta \left(\mathrm{E}_w\right)_0 = \mathrm{E}_0$$

and we have proved our final result:

Theorem 3.6 *Let S be a sub-semigroup with non-empty interior of a semi-simple Lie group G. Consider the orbit Sx of any point x in the symmetric space $\mathcal{X} = G/K$ and its ideal boundary $\mathrm{D} := \partial_\infty (Sx)$. If E is an effective control set for S in the flag manifold G/P_Θ and $\mathrm{D}^\Theta = \mathrm{D} \cap \partial_\infty^\Theta (\mathcal{X})$, then there is an element $w \in W$ such that $\pi^\Theta \left(\mathrm{D}^\Theta w\right)_0 \subset \rho_\Theta (\mathrm{E}_w)_0 = \mathrm{E}_0$.*

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