# A globally convergent Inexact-Newton method for solving reducible nonlinear systems of equations

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December 30, 1998

**Abstract**

A nonlinear system of equations is said to be reducible if it can be divided into $m$ blocks ($m > 1$) in such a way that the $i-$th block depends only on the first $i$ blocks of unknowns. Different ways of handling the different blocks with the aim of solving the system have been proposed in the literature. When the dimension of the blocks is very large, it can be difficult to solve the linear Newtonian equations associated to them using direct solvers based on factorizations. In this case, the idea of using iterative linear solvers to deal with the blocks of the system separately is appealing. In this paper, a local convergence theory that justifies this procedure is presented. The theory also explains the behavior of a Block–Newton method under different types of perturbations. Moreover, a globally convergent modification of the basic Block Inexact-Newton algorithm is introduced so that, under suitable assumptions, convergence can be ensured, independently of the initial point considered.

**Keywords.** Nonlinear systems, Inexact-Newton methods, reducible systems, decomposition. AMS: 65H10

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# 1 Introduction

We consider the problem of solving the nonlinear system of equations

$$F(x) = 0, \tag{1}$$

where $F : I\!R^n \rightarrow I\!R^n$ has lower block triangular structure. In other words we suppose that the components of $F$ can be partitioned into blocks of components such that (1) becomes

$$\begin{array}{l} F_1(x_1) = 0, \\ F_2(x_1, x_2) = 0, \\ \vdots \\ F_m(x_1, x_2, \ldots, x_m) = 0, \end{array} \tag{2}$$

with $x = (x_1, x_2, \ldots, x_m)$, $x_i \in I\!R^{n_i}$, $F_i : I\!R^{n_1} \times I\!R^{n_2} \times \ldots \times I\!R^{n_i} \rightarrow I\!R^{n_i}, i = 1, 2, \ldots, m$, $n_1 + n_2 + \ldots + n_m = n$. Such systems were considered in [4], where a large class of locally convergent $p$-step (quasi) Newton methods was proposed. Of course, in practice one is interested in the case $m >> 1$.

Systems with the structure (2) appear naturally in many practical applications. For example, the index $i \in \{1, \ldots, m\}$ could represent time and $x_i$ could be the vector of state variables of a physical, social or economic system. In this case, (2) says that the state of the system at a given time is the solution of a nonlinear system defined by the previous states. When one solves hyperbolic or parabolic partial differential equations using implicit or semi-implicit systems, a system with the structure (2) is generated, with $x_i$ representing the solution for the $i-$th time level. See [1, 12]. In many of these applications, the system that must be solved at the $i-$th level is large, so that Newton-like methods, based on the use of direct methods for solving linear systems, cannot be applied. The use of iterative linear solvers at each level $i$ can thus be appealing. This situation gives rise to the development of Block Inexact-Newton methods for solving (2), which is the subject of the present research.

The usefulness of research on Inexact-Newton methods does not rely only on its applicability to large-scale systems. On one hand, practical Newtonian iterations can be regarded as Inexact-Newton ones in the presence of rounding errors. On the other hand, the Inexact-Newton approach provides a useful representation of Newtonian iterations when the domain space is high– (perhaps infinite–) dimensional and the Jacobian employed comes from the consideration of coarser grids. See, for example [2]. In this sense, it can be claimed that Inexact-Newton theories *complete* the theory of the exact Newton method. Practical implementations of Newton's method would not be efficient if the classical Inexact-Newton convergence (and order-of-convergence) theorems were not true. Finally, all popular globalizations of Newton's method require a reduction in the norm of the local linear model of $F$ and, so, can be considered Inexact-Newton methods.

The same observations apply to globalization schemes. While global modifications of Newton's method (based on line searches or trust regions) are known for many years, globalization schemes for Inexact–Newton methods appeared only recently in the literature (see [6] and references therein). However, as we claimed in the case of local convergence properties, the reason why Newton globalization schemes are efficient in real life is that theoretically justified Inexact-Newton counterparts exist.

The relation between the results presented in this paper with the Block-Newton method introduced in [4] is the same that exists between the classical Inexact-Newton and the exact Newton method. We are going to prove that the Inexact-Newton version of the Block-Newton method of [4] is locally convergent, which means that, very likely, local convergence of the Block-Newton method will be reflected in practical computations. Even more important is the fact that the Inexact-Newton method can be globalized, since a global modification of the Block-Newton method was not known. So, the same globalization scheme can be applied to the Block-Newton method and, in practice, this procedure will be robust in the presence of rounding errors or other perturbations.

In Section 2 of this paper we introduce the "local version" of the Block Inexact-Newton method, which is based on the Dembo-Eisenstat-Steihaug [3] criterion for accepting an approximate solution of the linear Newtonian system at each block. In Section 3 we prove local linear and superlinear convergence. In Section 4 we introduce a modification of the local algorithm by means of which it turns out to be globally convergent. We also prove that, under suitable assumptions, the local and the global algorithm coincide near a solution and, so, superlinear convergence holds for the global method too. Possible applications and conclusions are drawn in Section 5.

## 2 Local Block Inexact-Newton method

The Block Inexact-Newton (BIN) method introduced in this paper is, as most methods for solving systems of equations, iterative. Iterates will be denoted $x^k$. Each $x^k \in I\!R^n$ can be partitioned into $m$ components belonging to $I\!R^{n_1}, \ldots, I\!R^{n_m}$, which will be denoted $x_i^k$. So, we write

$$x^k = (x_1^k, \ldots, x_m^k)$$

for all $k = 0, 1, 2, \ldots$, where $x_i^k \in I\!R^{n_i}, i = 1, \ldots, m$.

Each iteration of BIN consists of a sequence of $m$ steps. Roughly speaking, at the $i-$th step, $x_i^{k+1}$ (the $i-$th component of $x^{k+1}$) is computed, using $x_i^k$ (the $i-$th component of $x^k$) and the already obtained first $i-1$ components of $x^{k+1}$. Therefore, the vector whose components are the first $i-1$ components of $x^{k+1}$ followed by the $i-$th component of $x^k$ plays a very special role in the algorithm and deserves a special notation. In fact, we define $x^{k,1} = x_1^k$ and

$$x^{k,i} = (x_1^{k+1}, \ldots, x_{i-1}^{k+1},\ x_i^k) \quad \text{for} \quad i = 2, 3, \ldots, m.$$

Many times, given $x \in I\!R^n$, we will need to refer to the vector whose components are the first $i$ components of $x$. This vector, which belongs to $I\!R^{n_1} \times \ldots \times I\!R^{n_i}$, will be denoted $\bar{x}_i$. So, if $x = (x_1, \ldots, x_m)$, we have

$$\bar{x}_i = (x_1, \ldots, x_i), \quad i = 1, 2, \ldots, m.$$

With this notation, we can write

$$x^{k,i} = (\bar{x}_{i-1}^{k+1},\ x_i^k) \quad \text{for} \quad i = 2, 3, \ldots, m.$$

As we saw in the Introduction, the arguments of the function $F_i$ are $x_1, \ldots, x_i$ and its range space is $I\!R^{n_i}$. We assume that the derivatives of the components of $F_i$ with respect to the components of $x_i$ are well defined for all $\bar{x}_i$. Therefore, the corresponding Jacobian matrix of $F_i$ with respect to $x_i$ is a well defined square matrix $J_i(x_1, \ldots, x_i) \in I\!R^{n_i \times n_i}$. Derivatives of $F_i$ with respect to $x_1, \ldots, x_{i-1}$ are not assumed to exist at all.

Let us denote now $|\cdot|$ an arbitrary norm on $I\!R^{n_i}$ as well as its subordinate matrix norm. Assume that $x^0 \in I\!R^n$ is an arbitrary initial approximation to the solution of the nonlinear system (1). Given the $k-$th approximation $x^k = (x_1^k, \ldots, x_m^k)$ and the forcing parameter $\eta_k \in [0, 1)$, the (local) Block Inexact- Newton algorithm obtains $x^{k+1} = (x_1^{k+1}, \ldots, x_m^{k+1})$ by means of

$$J_i(x^{k,i})(x_i^{k+1} - x_i^k) = -F_i(x^{k,i}) + r_i^k, \tag{3}$$

where

$$|r_i^k| \leq \eta_k |F_i(x^{k,i})| \tag{4}$$

for $i = 1, 2, \ldots, m$.

As we mentioned in the Introduction, the computation of an increment $x_i^{k+1} - x_i^k$ that satisfies (3)-(4) involves the approximate solution of a linear system of equations, and the parameter $\eta_k$ represents the relative precision that is required for that solution. In most practical situations, an iterative linear solver will be used to compute the increment (3).

Convergence proofs make it necessary to define appropriate norms on the spaces $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_i}$. Here we define a norm $\|\cdot\|$ on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_i}$ by

$$\|\bar{x}_i\| = \sum_{j=1}^{i} |x_j|.$$

Let us now state the assumptions that will be used for proving local convergence of the method defined by (3) and (4). As usually in local convergence theories, we assume that there exists $x^* \in \mathbb{R}^n$ that solves the problem, that is:

$$F(x^*) = 0.$$

Assumptions **A1** and **A2** stated below say that the diagonal Jacobian blocks $J_i(x)$ are well behaved at $x^*$.

**(A1)** $J_i(\bar{x}_i)$ is continuous at $\bar{x}_i^*$, $\quad i = 1, 2, \ldots, m$;

**(A2)** $J_i(\bar{x}_i^*)$ is nonsingular for $i = 1, 2, \ldots, m$.

Since derivatives of $F_i$ with respect to variables other than $x_i$ are not assumed to exist, only assumptions relative to diagonal Jacobians are present in our theory. However, as in [4], an assumption is needed relative to the variation of $F_i$ with respect to the remaining variables.

**(A3)** There exist $\varepsilon_1 > 0, \beta \geq 0$ such that, whenever $\|\bar{x}_i - \bar{x}_i^*\| \leq \varepsilon_1$,

$$|F_i(\bar{x}_{i-1}, x_i) - F_i(\bar{x}_{i-1}^*, x_i)| \leq \beta \|\bar{x}_{i-1} - \bar{x}_{i-1}^*\|, \quad i = 2, 3, \ldots, m.$$

One should notice that these assumptions, together with (**A4**), which will be stated in Section 3, are the same as the ones used in [4] for proving local convergence of Newton-like methods.

The next two lemmas, the proofs of which can be found in [13], are going to be used for proving local results. Essentially, Lemma 1 says that the inverse of $J_i$ is continuous at $x^*$ and Lemma 2 states the differentiability of $F_i$ with respect to $x_i$.

**Lemma 1** *For all $\gamma > 0$ there exists $\varepsilon > 0$ such that $J_i(\bar{x}_i)$ is nonsingular and*

$$|J_i(\bar{x}_i)^{-1} - J_i(\bar{x}_i^*)^{-1}| < \gamma, \quad \textit{whenever } \|\bar{x}_i - \bar{x}_i^*\| < \varepsilon.$$

**Lemma 2** *For all $\gamma > 0$ there exists $\varepsilon > 0$ such that*

$$|F_i(\bar{x}_{i-1}^*, x_i) - F_i(\bar{x}_{i-1}^*, x_i^*) - J_i(\bar{x}_i^*)(x_i - x_i^*)| \le \gamma |x_i - x_i^*|,$$

*whenever $|x_i - x_i^*| < \varepsilon$.*

We finish this section proving a technical lemma which is crucial for the local convergence proofs. Roughly speaking, the quantities $e_i^k$ mentioned in this technical result will represent componentwise errors in the local convergence theorem which will be proved in the next section.

**Lemma 3** *Let $e_i^k$, $i = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots$, $\rho \in [0, 1)$ and $C > 0$ be real numbers such that $e_i^k \ge 0$ for all $i = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots$ and*

$$e_i^{k+1} \le \rho e_i^k + C \sum_{j=1}^{i-1} e_j^{k+1} \tag{5}$$

*for all $i, k$. Then the sequence $\left\{e^k\right\}$, with $e^k = \left(e_1^k, \ldots, e_m^k\right)$, converges to $0$ and the convergence is q-linear in a suitable norm.*

*Proof.* Define the matrix $A \in I\!R^{n \times n}$, $A = [a_{ij}]$, by

$$a_{ij} = \begin{cases} 1, & i = j, \\ -C, & i > j, \\ 0, & i < j \end{cases}.$$

Clearly, (5) is equivalent to

$$Ae^{k+1} \le \rho e^k.$$

It is easy to see that the entries of $A^{-1}$ are nonnegative. Therefore, we can multiply both sides of the previous inequality by $A^{-1}$, obtaining

$$e^{k+1} \leq \rho A^{-1} e^k \tag{6}$$

for all $k$. Since the spectral radius of $A^{-1}$ is equal to 1, there exists a norm $\| \cdot \|_C$ on $\mathbb{R}^n$ such that, given $\rho_0 \in (\rho, 1)$, we have

$$\|\rho A^{-1}\|_C \leq \rho_0.$$

The construction of this norm in the case that the matrix is lower-triangular imposes that $\|x\|_C = \|Dx\|_2$, where $D$ is a positive diagonal matrix. Therefore, by (6),

$$De^{k+1} \leq \rho D A^{-1} e^k.$$

Thus, by the monotonicity of $\| \cdot \|_2$,

$$\|e^{k+1}\|_C = \|De^{k+1}\|_2 \leq \rho \|DA^{-1}e^k\|_2$$

$$= \rho \|A^{-1}e^k\|_C \leq \rho \|A^{-1}\|_C \|e^k\|_C \leq \rho_0 \|e^k\|_C.$$

So, the desired result is proved. $\square$

# 3 Local convergence

In this section we prove local convergence results for the algorithm BIN, defined by (3)-(4). We are going to prove that, if the forcing parameters $\eta_k$ are bounded away from 1 and the initial estimate is close enough to the solution, then $\{x^k\}$ converges to $x^*$ with a $q$-linear rate determined by an upper bound of $\eta_k$ in a specific norm. This implies $r$-linear convergence in any other norm.

As so happens to occur with the classical Inexact–Newton method introduced in [3], the linear convergence rate is associated to a special norm defined by the Jacobian at $x^*$. Similarly, here we will use some auxiliary norms. Let $|\cdot|$ be the norm used in Lemma 3. The induced norms are

$$|x_i|_* = |J_i(\bar{x}_i^*) x_i| \quad \text{and} \quad \|\bar{x}_i\|_* = \sum_{j=1}^{i} |x_j|_*.$$

for each $i = 1, \ldots, m$ and every $\bar{x}_i \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_i}$.

**Theorem 1** *Let $0 \le \eta_k \le \eta < 1$. There exists $\varepsilon > 0$ such that if $\|x^0 - x^*\| \le \varepsilon$, then the sequence $\{x^k\}$ generated by BIN is well defined and converges q-linearly (in a suitable norm) to $x^*$.*

*Proof.* Let us denote

$$e_i^k = |x_i^k - x_i^*|_*, \quad i = 1, 2, \ldots, m, \quad k = 0, 1, 2, \ldots.$$

Since $J_i(\bar{x}_i^*)$ is nonsingular for all $i = 1, 2, \ldots, m$, there exists $\mu > 0$ such that

$$\mu = \max_{1 \le i \le m} \{|J_i(\bar{x}_i^*)|, |J_i(\bar{x}_i^*)^{-1}|\} \tag{7}$$

and

$$\frac{1}{\mu}|x_i| \le |x_i|_* \le \mu |x_i|, \quad \text{for} \quad x_i \in I\!R^{n_i},\ i = 1, 2, \ldots, m. \tag{8}$$

Choosing $\gamma > 0$ small enough, we have that

$$(1 + \gamma\mu)[\eta(1 + \gamma\mu) + 2\gamma\mu] \le \rho < 1.$$

By Lemma 1, Lemma 2 and Assumption **A2** there exists $\varepsilon \in (0, \varepsilon_1)$ such that

$$|J_i(\bar{x}_i) - J_i(\bar{x}_i^*)| \le \gamma, \tag{9}$$

$$|J_i(\bar{x}_i)^{-1} - J_i(\bar{x}_i^*)^{-1}| \le \gamma, \tag{10}$$

$$|F_i(\bar{x}_{i-1}^*, x_i) - F_i(\bar{x}_{i-1}^*, x_i^*) - J_i(\bar{x}_i^*)(x_i - x_i^*)| \le \gamma |x_i - x_i^*| \tag{11}$$

for $\|\bar{x}_i - \bar{x}_i^*\| \le \mu^2 \varepsilon$ and all $i = 1, 2, \ldots, m$.

We are going to prove, by induction on $i$, that the sequence $\{e_i^k\}$ satisfies (5).

Suppose that $\|x^k - x^*\| \le \mu^2 \varepsilon$. Then

$$\begin{aligned} J_1(x_1^*)(x_1^{k+1} - x_1^*) &= J_1(x_1^*)(x_1^k - x_1^* - J_1(x_1^k)^{-1} F_1(x_1^k) + J_1(x_1^k)^{-1} r_1^k) \\ &= (I + J_1(x_1^*)(J_1(x_1^k)^{-1} - J_1(x_1^*)^{-1})) \end{aligned}$$
$$\times [r_1^k - (F_1(x_1^k) - F_1(x_1^*) - J_1(x_1^*)(x_1^k - x_1^*)) - (J_1(x_1^*) - J_1(x_1^k))(x_1^k - x_1^*)],$$

where $I$ is the identity matrix. So by (7), (9)-(11) and condition (4) we have:

$$e_1^{k+1} \le (1 + \mu\gamma)(\eta_k |F_1(x_1^k)| + 2\gamma |x_1^k - x_1^*|).$$

Since

$$F_1(x_1^k) = J_1(x_1^*)(x_1^k - x_1^*) + F_1(x_1^k) - F_1(x_1^*) - J_1(x_1^*)(x_1^k - x_1^*),$$

taking norms and using (11) we obtain

$$|F_1(x_1^k)| \le e_1^k + \gamma |x_1^k - x_1^*|.$$

Therefore

$$e_1^{k+1} \le (1+\mu\gamma)[(1+\mu\gamma)\eta_k + 2\gamma\mu]e_1^k \le \rho e_1^k$$

and $|x_1^{k+1} - x_1^*| \le \mu^2\varepsilon$. Assume as inductive hypothesis that

$$e_j^{k+1} \le \rho e_j^k + \beta\mu(2+\mu\gamma)\sum_{l=1}^{j-1} e_l^{k+1} \quad \text{and} \quad \|(\bar{x}_{i-1}^{k+1}, x_i^k) - \bar{x}_i^*\| \le \mu^2\varepsilon,$$

for $j = 1, 2, \ldots, i-1$, $i \ge 2$. Then

$$\begin{aligned} J_i(\bar{x}_i^*)(x_i^{k+1} - x_i^*) = (I + J_i(\bar{x}_i^*)(J_i(x^{k,i})^{-1} - J_i(\bar{x}_i^*)^{-1})) \\ \times[r_i^k - (F_i(x^{k,i}) - F_i(\bar{x}_{i-1}^*, x_i^k)) \\ -(F_i(\bar{x}_{i-1}^*, x_i^k) - F_i(\bar{x}_i^*) - J_i(\bar{x}_i^*)(x_i^k - x_i^*)) - (J_i(\bar{x}_i^*) - J_i(x^{k,i}))(x_i^k - x_i^*)] \end{aligned}$$

and

$$\begin{aligned} F_i(x^{k,i}) &= J_i(\bar{x}_i^*)(x_i^k - x_i^*) + (F_i(x^{k,i}) - F_i(\bar{x}_{i-1}^*, x_i^k)) \\ &\quad + (F_i(\bar{x}_{i-1}^*, x_i^k) - F_i(\bar{x}_i^*) - J_i(\bar{x}_i^*)(x_i^k - x_i^*)). \end{aligned}$$

Taking norms, using (11) and Assumption **A3** we have that

$$\begin{aligned} e_i^{k+1} \le (1+\mu\gamma)[\eta_k(e_i^k + \gamma|x_i^k - x_i^*| + \beta\|\bar{x}_{i-1}^{k+1} - \bar{x}_{i-1}^*\|) \\ +\beta\|\bar{x}_{i-1}^{k+1} - \bar{x}_{i-1}^*\| + 2\mu\gamma e_i^k], \end{aligned}$$

so

$$e_i^{k+1} \le \rho e_i^k + \beta\mu(2+\mu\gamma)\sum_{j=1}^{i-1} e_j^{k+1}.$$

Since $e_i^{k+1}$ satisfies inequality (5) with $C = \beta\mu(2+\mu\gamma)$ it follows from Lemma 3 that

$$\lim_{k\to\infty} e^k = 0$$

and the convergence is $q$-linear.□

In the next theorem we will prove superlinear convergence of the sequence $\{x^k\}$ generated by BIN when $\lim \eta_k = 0$. To the end of this section we suppose that $F$ satisfies the assumption **A4** in addition to assumptions (**A1**–**A3**).

**(A4)** There exists $L > 0$ such that

$$|J_i(\bar{x}_i) - J_i(\bar{x}_i^*)| \le L\|\bar{x}_i - \bar{x}_i^*\|$$

for $i = 1, 2, \ldots, m$ and $\|\bar{x}_i - \bar{x}_i^*\| < \varepsilon_1$.

As a consequence of this assumption we have that

$$|F_i(\bar{x}_{i-1}^*, x_i) - F_i(\bar{x}_{i-1}^*, x_i^*) - J_i(\bar{x}_i^*)(x_i - x_i^*)| \le \frac{L}{2}|x_i - x_i^*|^2 \qquad (12)$$

for $i = 1, 2, \ldots, m$ and $\|\bar{x}_i - \bar{x}_i^*\| < \varepsilon_1$.

**Theorem 2** *Assume that the BIN iterates $\{x^k\}$ converge to $x^*$ and $\lim_{k\to\infty} \eta_k = 0$. Then the convergence is superlinear.*

*Proof.* We will prove by induction on $i$ that

$$|x_i^{k+1} - x_i^*| = o(\|\bar{x}_i^k - \bar{x}_i^*\|).$$

By the convergence of the sequence and Lemma 1, there exists $C > 0$ such that

$$C \ge (|J_i(\bar{x}_i^*)^{-1}| + |J_i(x^{k,i})^{-1} - J_i(\bar{x}_i^*)^{-1}|),$$

$i = 1, 2 \ldots, m$. For $i = 1$ we have, as in the proof of Theorem 1,

$$|x_1^{k+1} - x_1^*| \le (|J_1(x_1^*)^{-1}| + |J_1(x_1^k)^{-1} - J_1(x_1^*)^{-1}|)\cdot$$

$$(|r_1^k| + |F_1(x_1^k) - F_1(x_1^*) - J_1(x_1^*)(x_1^k - x_1^*)| + |J_1(x_1^*) - J_1(x_1^k)||x_1^k - x_1^*|)$$

$$\le C(\eta_k|F_1(x_1^k)| + \frac{L}{2}|x_1^k - x_1^*|^2 + L|x_1^k - x_1^*|^2),$$

and since $\eta_k \to 0$ and

$$|F_1(x_1^k)| \le e_1^k + |F_1(x_1^k) - F_1(x_1^*) - J_1(x_1^*)(x_1^k - x_1^*)|,$$

we have

$$|x_1^{k+1} - x_1^*| = o(|x_1^k - x_1^*|) = o(|\bar{x}_1^k - \bar{x}_1^*|).$$

Assume, as inductive hypothesis,

$$|x_j^{k+1} - x_j^*| = o(\|\bar{x}_j^k - \bar{x}_j^*\|),\ j = 0, 1, \ldots, i-1.$$

As in the proof of Theorem 1, we have that

$$|x_i^{k+1} - x_i^*| \le (|J_i(\bar{x}_i^*)^{-1}| + |J_i(x^{k,i})^{-1} - J_i(\bar{x}_i^*)^{-1}|)$$
$$\times[|r_i^k| + |F_i(x^{k,i}) - F_i(\bar{x}_{i-1}^*, x_i^k)| + |F_i(\bar{x}_{i-1}^*, x_i^k) - F_i(\bar{x}_i^*) - J_i(\bar{x}_i^*)(x_i^k - x_i^*)|$$
$$+|(J_i(\bar{x}_i^*) - J_i(x^{k,i}))(x_i^k - x_i^*)|]$$

and

$$F_i(x^{k,i}) = J_i(\bar{x}_i^*)(x_i^k - x_i^*) + (F_i(x^{k,i}) - F_i(\bar{x}_{i-1}^*, x_i^k))$$
$$+(F_i(\bar{x}_{i-1}^*, x_i^k) - F_i(\bar{x}_i^*) - J_i(\bar{x}_i^*)(x_i^k - x_i^*)).$$

So, using (12) we have

$$|x_i^{k+1} - x_i^*| \le C[\eta_k(|x_i^k - x_i^*||J_i(\bar{x}_i^*)| + \beta \sum_{j=1}^{i-1} |x_j^{k+1} - x_j^*| + \frac{L}{2}|x_i^k - x_i^*|^2)$$

$$+\beta \sum_{j=1}^{i-1} |x_j^{k+1} - x_j^*| + \frac{L}{2}|x_i^k - x_i^*|^2 + L(\sum_{j=1}^{i-1} |x_j^{k+1} - x_j^*| + |x_i^k - x_i^*|)|x_i^k - x_i^*|]$$

$$\le Co(|x_i^k - x_i^*|) + M \sum_{j=1}^{i-1} |x_j^{k+1} - x_j^*|,$$

for some constant $M$. From this inequality and the inductive hypothesis we obtain that

$$|x_i^{k+1} - x_i^*| = o(\|\bar{x}_i^k - \bar{x}_i^*\|),$$

so

$$\|x^{k+1} - x^*\| = \sum_{i=1}^{m} |x_i^{k+1} - x_i^*| = o(\|x^k - x^*\|)$$

and the sequence $\{x^k\}$ converges superlinearly to $x^*$. $\square$

# 4 Global convergence

Up to now, we have proved some local convergence properties of the Algorithm BIN, introduced in Section 2. Obviously, since Newton's method is a particular case of BIN (when $m = 1$ and $\eta_k \equiv 0$), this algorithm is not globally convergent. (Examples of nonconvergence of Newton's method in one-dimensional cases are easy to construct.) By this we mean that convergence to solutions or other special points depends on the initial approximation. In this section we introduce a modification of BIN that makes it globally convergent in the sense that will be specified later. Moreover, we will prove that, near a solution and under suitable assumptions, the modification coincides with the original BIN, so that local properties also hold.

Unlike the local convergence results, based on the assumptions A1-A4, our global convergence theorems assume that $F(x)$ has continuous partial derivatives with respect to all the variables. We will denote $F'(x)$ the Jacobian matrix of $F(x)$. Moreover, for all $i = 2, \dots, m$, $j = 1, \dots, i-1$ we denote $\frac{\partial F_i(x)}{\partial \bar{x}_j}$ the matrix of partial derivatives of $F_i$ with respect to $x_1, \dots, x_j$. Therefore, $\frac{\partial F_i(x)}{\partial \bar{x}_j} \in \mathbb{R}^{n_i \times (n_1 + \dots + n_j)}$. For proving that the globalized algorithm converges to a solution independently of the initial approximation we will need the assumption stated below.

**(A5)** For all $x \in \mathbb{R}^n$, $F'(x)$ is continuous and nonsingular.

Assumption **A5** implies that all the entries of $F'(x)$ and $F'(x)^{-1}$ are bounded on bounded sets, as well as the norms of all the submatrices of $F'(x)$ and $F'(x)^{-1}$.

From now on we denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^{n_i}$, $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$ (so $\|x\|^2 = \sum_{i=1}^m |x_i|^2$) and we denote $\langle \cdot, \cdot \rangle$ the standard scalar product.

The global modification of BIN is stated below. Essentially, given an iterate $x^k$, a "pure" BIN iteration may be tried first, expecting sufficient decrease of $\|F(x)\|$. If this decrease is not obtained, closer points to $x^k$ will be tried, following rules that are similar to classical backtracking (see [5]), but introducing a suitable change of direction each time the step is reduced. The idea is that, for small steps, the direction obtained approximates a Newton direction for the whole system.

**Algorithm GBIN**
Assume that $\sigma \in (0,1)$ and $0 < \theta_1 < \theta_2 < 1, \eta \in [0,1)$ are constants independent of $k$. For given $x^k$ $(k \geq 0)$ such that $F\left(x^k\right) \neq 0$ and $0 \leq \eta_k \leq \eta \in [0,1)$, $x^{k+1}$ is obtained by means of the following steps.
**Step 1.**
Set
$$\alpha \leftarrow 1.$$

**Step 2.** (BIN iteration)

Choose $\delta \in (0, \alpha]$.
For $i = 1, 2, \ldots, m$, define

$$y_i = \overline{x}_i^k + \alpha \left(\overline{d}_{i-1}, 0\right) = (x_1^k + \alpha d_1, \ldots, x_{i-1}^k + \alpha d_{i-1}, x_i^k),$$

$$z_i = \overline{x}_i^k + \delta \left(\overline{d}_{i-1}, 0\right) = (x_1^k + \delta d_1, \ldots, x_{i-1}^k + \delta d_{i-1}, x_i^k),$$

$$w_i \in \{\bar{x}_i^k, y_i\},$$

($y_1 = z_1 = x_1^k$) and find $d_i \in I\!R^{n_i}$ such that

$$|J_i(w_i)d_i + F_i(\bar{x}_i^k) + \frac{1}{\delta}(F_i(z_i) - F_i(\bar{x}_i^k))| \leq \eta_k |F_i(w_i)|. \tag{13}$$

**Step 3.** (Test sufficient decrease)
Define $\gamma_k = 1 - \eta_k^2$ and $d = (d_1, \ldots, d_m)$.
If

$$\|F(x^k + \alpha d)\| \leq (1 - \frac{\sigma\gamma_k\alpha}{2})\|F(x^k)\| \tag{14}$$

set

$$\alpha_k = \alpha, \quad d^k = d \quad \text{and} \quad x^{k+1} = x^k + \alpha_k d^k.$$

Otherwise choose

$$\alpha_{new} \in [\theta_1\alpha, \theta_2\alpha], \quad \alpha \leftarrow \alpha_{new}$$

and repeat Step 2.

**Remarks.** The step $d_i$ computed in (13) is an approximate solution of the linear system

$$\frac{1}{\delta}(F_i(z_i) - F_i(\bar{x}_i^k)) + J_i(w_i)d_i + F_i(\bar{x}_i^k) = 0.$$

When $\delta$ is small this system is an approximation of the system considered at the $i-$th stage of a block lower-triangular solver for a step of Newton's method applied to $F(x) = 0$. This is obviously true for $i = 1$. For $i > 1$, the statement above follows from

$$F_i(z_i) - F_i(\bar{x}_i^k) = F_i(x_1^k + \delta d_1, \ldots, x_{i-1}^k + \delta d_{i-1}, x_i^k) - F_i(x_1^k, \ldots, x_{i-1}^k, x_i^k)$$

$$\approx \delta[\frac{\partial F_i}{\partial x_1}(\bar{x}_i^k)d_1 + \ldots + \frac{\partial F_i}{\partial x_{i-1}}(\bar{x}_i^k)d_{i-1}].$$

Since

$$\frac{\sigma \gamma_k \alpha}{2} = \frac{\sigma(1-\eta_k^2)\alpha}{2} = \frac{\sigma(1+\eta_k)}{2}(1-\eta_k)\alpha$$

and $\frac{\sigma(1+\eta_k)}{2} < 1$, the algorithm GBIN can be formulated setting $\sigma' \in (0,1)$ and replacing (14) by

$$\|F(x^k + \alpha d)\| \leq (1 - \sigma'(1-\eta_k)\alpha)\|F(x^k)\|. \tag{15}$$

This is the condition for acceptability of a step in the backtracking algorithm of Eisenstat and Walker [6]. In our formulation of GBIN we prefer the Euclidean norm to measure progress towards the solution because the differentiability of $\|F(x)\|^2$ allows one to justify the use of safeguarded interpolation schemes for choosing $\alpha_{new} \in [\theta_1\alpha, \theta_2\alpha]$. Global convergence proofs probably hold for arbitrary norms.

The choices of $\delta$ and $w_i$ at Step 2 of GBIN deserve some discussion. If we fix $\delta = \alpha$ and $w_i = y_i$, the solution of (13) is an approximate solution of the linear system

$$J_i(y_i)d_i + \frac{1}{\alpha}F_i(y_i) + (1 - \frac{1}{\alpha})F_i(\bar{x}_i^k) = 0. \tag{16}$$

For $\alpha = 1$ this corresponds exactly to an iteration of the local algorithm defined in Section 2. For this reason we say that GBIN is a generalization of BIN. When $\alpha$ is reduced, the scheme based on the inexact solution of (16) preserves the spirit of decomposition in the sense that the Jacobian $J_i$ is computed at the displaced point $y_i$, as an attempt to use fresh information on the local variation of the current block. However, the independent term of the linear system (16) must change in such a way that, in the limit, $d_i$ approximates a Newton direction. This scheme might not be economic because each time $\alpha$ is reduced, $J_i(y_i)$ and the approximate solution of (16) must be computed again. A modified algorithm could consist on choosing $\delta$ "very small" and $w_i = \bar{x}_i^k$. In this case, the same $\delta$ can be used through successive reductions of $\alpha$ and, so, the solution of (13) does not change. The resulting algorithm turns out to be an Inexact-Newton backtracking method in the sense of [6] and the convergence proof can be obtained using the acceptance criterion (15) and the arguments of [6]. In this case, no intermediate iterates are computed, but the decomposition structure of the method disappears and, in fact, GBIN ceases to be a generalization of BIN.

The discussion above reveals that, in some sense, full preservation of the decomposition philosophy conflicts with low computational cost requirements, at least if the first trial point is not accepted. However, the formulation of GBIN given above allows us to combine decomposition and pure backtracking in several ways. For example, one can choose $\delta = \alpha$ and $w_i = y_i$ when $\alpha = 1$ but $\delta$ "very small" and $w_i = \bar{x}_i^k$ when $\alpha < 1$. Since we expect that, at most iterations, $\alpha = 1$ will be accepted near a solution, (see Theorem 4 below), the resulting method will be, in many cases, very similar to the decomposition algorithm BIN. The first time $\alpha$ is reduced, a new linear system must be solved but at the remaining reductions the same $d_i$ can be used.

The question of how to choose parameters $\eta_k$ in ordinary Inexact-Newton methods has been object of several studies. See [7], [9] and references therein. Many of the strategies used for the (one-block) Inexact-Newton method can be easily adapted to the Block Inexact-Newton case. As so happens to be with $\eta_k$, the parameters $\sigma$, $\theta_1$ and $\theta_2$ are adimensional and so, it is possible to recommend specific choices independently of the problem. See [5, 7].

The global algorithm can be stated using an approximation $\widetilde{J}_i$ for the Jacobian $J_i$ in the equation (13). Convergence statements stay valid with minor technical changes in the proofs if the approximation $\widetilde{J}_i$ is nonsingular, $\|\widetilde{J}_i^{-1}\|$ is bounded and $\widetilde{J}_i$ can be made close to $J_i\left(\overline{x}_i^k\right)$.

In the rest of this section, we prove the global convergence results. The first lemma we need to prove says that, independently of the current point $x^k$ there exists a common bound for the directions $d$ computed at Step 2 of GBIN.

**Lemma 4** *Let $\{x^k\}$lie in a compact set. Then, the directions $d_i$ computed at Step 2 of GBIN are well defined and uniformly bounded independently of $x^k$.*

*Proof.* By Assumption **A5**, $F'(x)$ is nonsingular. So, by the structure of this matrix, $J_i(x)$ is nonsingular. This implies that directions $d_i$ satisfying (13) always exist. Now, by (13),

$$|J_1(x_1^k)d_1 + F_1(x_1^k)| \leq \eta |F(x_1^k)|.$$

This implies that

$$|J_1(x_1^k)d_1| \leq (1+\eta)|F(x_1^k)|.$$

So,

$$|d_1| \leq |J_1(x_1^k)^{-1}||J_1(x_1^k)d_1| \leq 2|J_1(x_1^k)^{-1}||F(x_1^k)|.$$

Therefore, the directions $d_1$ are bounded.

Suppose that $i \geq 2$ and that the directions $d_1, \ldots, d_{i-1}$ are bounded. Then, by (13),

$$|d_i| \leq |J_i(w_i)^{-1}|[|F_i(\bar{x}_i^k) + \frac{1}{\delta}(F_i(z_i) - F_i(\bar{x}_i^k))| + \eta|F_i(w_i)|].$$

By the inductive hypothesis and the boundedness of $\{x^k\}$, we have that $\{|F_i(w_i)|\}$ is bounded. Therefore, we only need to prove that $\frac{1}{\delta}(F_i(z_i) - F_i(\bar{x}_i^k))$ is bounded independently of $\alpha$.

But

$$\frac{1}{\delta}(F_i(z_i) - F_i(\bar{x}_i^k)) = \frac{1}{\delta}(F_i(x_1^k + \delta d_1, \ldots, x_{i-1}^k + \delta d_{i-1}, x_i^k) - F_i(x_1^k, \ldots, x_{i-1}^k, x_i^k))$$

$$= \sum_{j=1}^{i-1} [\int_0^1 \frac{\partial F_i}{\partial x_j}(x_1^k + t\delta d_1, \ldots, x_{i-1}^k + t\delta d_{i-1}, x_i^k)dt] d_j. \quad (17)$$

So, the desired result follows from the inductive hypothesis , the boundedness of $\{x^k\}$ and the continuity and boundedness of the partial derivatives. $\square$

In the following lemma, we prove that, independently of the current point $x^k$, after a finite number of reductions of $\alpha$ a trial point is found that satisfies the sufficient decrease condition (14). Therefore, a single iteration of GBIN finishes in finite time.

**Lemma 5** *Algorithm GBIN is well defined. That is, $\alpha$ cannot be decreased infinitely many times within a single iteration.*

*Proof.* If $F(x^k) = 0$ the algorithm terminates, so let us assume $F(x^k) \neq 0$. Suppose that $\alpha$ is decreased infinitely many times i.e. GBIN generates infinite sequences $\{\alpha_j\}$, $\{\delta_j\}$ and $\{d^j\}$ at iteration $k$ such that

$$\|F(x^k + \alpha_j d^j)\|^2 > (1 - \frac{\sigma\gamma_k\alpha_j}{2})^2\|F(x^k)\|^2. \quad (18)$$

Let us denote $z_1^j = y_1^j = x_1^k$,

$$y_i^j = (x_1^k + \alpha_j d_1^j, \ldots, x_{i-1}^k + \alpha_j d_{i-1}^j,\ x_i^k), \quad i = 2, \ldots, m,$$

$$z_i^j = (x_1^k + \delta_j d_1^j, \ldots, x_{i-1}^k + \delta_j d_{i-1}^j,\ x_i^k), \quad i = 2, \ldots, m,$$

and $w_i^j \in \{\bar{x}_i^k, y_i^j\}$ as in Step 2 of GBIN.

Since, as in Lemma 4, all the directions $d^j$ generated at Step 2 of GBIN are bounded, there exists an infinite set of indices $K_1$, such that

$$\lim_{j \in K_1} d_i^j = d_i^*, \ i = 1, 2, \ldots, m, \tag{19}$$

and, since $\alpha_j \to 0$, we have

$$\lim_{j \in K_1} \delta_j d_i^j = \lim_{j \in K_1} \alpha_j d_i^j = 0.$$

Consequently,

$$\lim_{j \in K_1} w_i^j = \lim_{j \in K_1} y_i^j = \lim_{j \in K_1} z_i^j = \bar{x}_i^k. \tag{20}$$

Define

$$A_i = J_i(w_i^j) d_i^j + \frac{1}{\delta_j}(F_i(z_i^j) - F_i(\bar{x}_i^k)), \ i = 1, 2, \ldots, m.$$

Then, by (13), we have that

$$\langle A_i + F_i(\bar{x}_i^k), A_i + F_i(\bar{x}_i^k) \rangle \le \eta_k^2 \langle F_i(w_i^j), F_i(w_i^j) \rangle.$$

This implies, using $\langle A_i, A_i \rangle \ge 0$, that

$$2\langle F_i(\bar{x}_i^k), J_i(w_i^j) d_i^j + \frac{1}{\delta_j}(F_i(z_i^j) - F_i(\bar{x}_i^k)) \rangle \le \eta_k^2 |F_i(w_i^j)|^2 - |F_i(\bar{x}_i^k)|^2, \tag{21}$$

for $i = 1, 2, \ldots, m$.

Define $\phi(x) = \frac{1}{2}\|F(x)\|^2$, then $\nabla \phi(x) = F'(x)^T F(x)$. Using (19), (20), the identity

$$\frac{1}{\delta_j}(F_i(z_i^j) - F_i(\bar{x}_i^k)) = \frac{1}{\delta}(F_i(x_1^k + \delta d_1^j, \ldots, x_{i-1}^k + \delta d_{i-1}^j, x_i^k) - F_i(x_1^k, \ldots, x_{i-1}^k, x_i^k))$$

and taking limits on both sides of (21) we obtain:

$$\langle \nabla \phi(x^k), d^* \rangle \le -\frac{\gamma_k}{2} \|F(x^k)\|^2. \tag{22}$$

On the other hand, our assumption (18) gives

$$\frac{\phi(x^k + \alpha_j d^j) - \phi(x^k)}{\alpha_j} > -\frac{\sigma \gamma_k}{2} \|F(x^k)\|^2 + \frac{\sigma^2 \gamma_k^2 \alpha_j}{8} \|F(x^k)\|^2$$

and, by the Mean - Value Theorem there exists $\xi_j \in [0,1]$ such that

$$\langle \nabla\phi(x^k + \xi_j \alpha_j d^j), d_j\rangle > -\frac{\sigma\gamma_k}{2}\|F(x^k)\|^2 + \frac{\sigma^2\gamma_k^2\alpha_j}{8}\|F(x^k)\|^2.$$

Taking limits in the last inequality we obtain

$$\langle \nabla\phi(x^k), d^*\rangle \geq -\frac{\sigma\gamma_k}{2}\|F(x^k)\|^2.$$

This contradicts (22). Therefore, $\alpha$ cannot be decreased infinitely many times within one iteration. □

The main convergence result for GBIN is given in the following theorem. We are going to prove that if $\{x^k\}$ is bounded, every cluster point of the sequence is a solution of (1). Boundedness of $\{x^k\}$ can be guaranteed if $\|F(x)\|$ has bounded level sets. Again, Lemma 4 is crucial for proving this result since it guarantees that search directions are bounded.

**Theorem 3** *Let $\left\{x^k\right\}$ be a bounded infinite sequence generated by GBIN. Then all its limit points are solutions of (1).*

*Proof.* Assume that $K_2$ is an infinite set of indices such that

$$\lim_{k\in K_2} x^k = x^*.$$

Let us first suppose that $\alpha_k \geq \bar{\alpha} > 0$ for $k \geq k_1$. Then,

$$1 - \frac{\sigma\gamma_k\alpha_k}{2} \leq 1 - \frac{\sigma\gamma\bar{\alpha}}{2} = r < 1, \quad \text{for} \quad k \geq k_1.$$

so $\|F(x^{k+1})\| \leq r\|F(x^k)\|$ for all $k \geq k_1$. This implies that $\|F(x^k)\| \to 0$ and $F(x^*) = 0$.

Assume now that $\liminf_{k\to\infty} \alpha_k = 0$. Let $K_3$ be an infinite subset of $K_2$ such that $\alpha$ is decreased at every iteration $k \in K_3$ and $\lim_{k\in K_3} \alpha_k = 0$. So, for $k \in K_3$ we have a steplength $\alpha_k'$ which immediately precedes $\alpha_k$ for which the sufficient descent condition (14) does not hold. Let $\delta_k' \in (0, \alpha_k']$ as in Step 2 of GBIN ($\alpha_k$ is chosen in $[\theta_1\alpha_k', \theta_2\alpha_k']$.) Let $g^k$ be the direction corresponding to $\alpha_k'$, generated at Step 2 of GBIN. By Lemma 4 and the hypothesis of this theorem, $\{|g_i^k|\}$ is a bounded sequence for $i =$

$1, 2, \ldots, m,\ k \in K_3$. So, there exists an infinite set of indices $K_4 \subset K_3$ such that

$$\lim_{k \in K_4} g_i^k = g_i^*,\ \lim_{k \in K_4} \eta_k = \tilde{\eta},\ \lim_{k \in K_4} \gamma_k = \tilde{\gamma} = 1 - \tilde{\eta}^2 > 0$$

and

$$\lim_{k \in K_4} \alpha_k' g^k = \lim_{k \in K_4} \delta_k' g^k = 0.$$

Define, for $k \in K_4$,

$$\begin{aligned} y_1^k &= z_1^k = x_1^k, \\ y_i^k &= (x_1^k + \alpha_k' g_1^k, \ldots, x_{i-1}^k + \alpha_k' g_{i-1}^k,\ x_i^k), \\ z_i^k &= (x_1^k + \delta_k' g_1^k, \ldots, x_{i-1}^k + \delta_k' g_{i-1}^k,\ x_i^k), \end{aligned}$$

and $w_i^k \in \{\bar{x}_i^k, y_i^k\}$ as in Step 2 of GBIN, $i = 2, \ldots, m$. Defining $\phi$ as in Lemma 5, using the arguments of that Lemma and uniform continuity on compact sets, we deduce that

$$\langle \nabla \phi(x^*), g^* \rangle \leq -\frac{\tilde{\gamma}}{2} \|F(x^*)\|^2. \tag{23}$$

Now, by the definition of $\alpha_k'$, we have that

$$\|F(x^k + \alpha_k' g^k)\|^2 > (1 - \frac{\sigma \gamma_k \alpha_k'}{2})^2 \|F(x^k)\|^2 \quad \text{for all} \quad k \in K_4.$$

So, as in Lemma 5,

$$\frac{\phi(x^k + \alpha_k' g^k) - \phi(x^k)}{\alpha_k'} > -\frac{\sigma \gamma_k}{2} \|F(x^k)\|^2 + \frac{\sigma^2 \gamma_k^2 \alpha_k'}{8} \|F(x^k)\|^2.$$

The Mean–Value Theorem implies that there exists $\xi_k \in [0, 1]$ such that

$$\langle \nabla \phi(x^k + \xi_k \alpha_k' g^k), g^k \rangle > -\frac{\sigma \gamma_k}{2} \|F(x^k)\|^2 + \frac{\sigma^2 \gamma_k^2 \alpha_k'}{8} \|F(x^k)\|^2.$$

Taking limits in the last inequality we obtain

$$\langle \nabla \phi(x^*), g^* \rangle \geq -\frac{\sigma \tilde{\gamma}}{2} \|F(x^*)\|^2. \tag{24}$$

By (23) and (24), it follows that $F(x^*) = 0$. □

To complete the analysis of global convergence we will prove that GBIN matches well with local method analyzed in the previous section. The following classical lemma will be used.

**Lemma 6** *Let $F'(x^*)$ be nonsingular and*

$$P = \max\{\|F'(x^*)\| + \frac{1}{2Q}, 2Q\}$$

*where $Q = \|F'(x^*)^{-1}\|$. Then*

$$\frac{1}{P}\|y - x^*\| \le \|F(y)\| \le P\|y - x^*\|$$

*for $\|y - x^*\|$ sufficiently small.*

In the next theorem we assume that algorithm GBIN runs in such a way that, at each iteration, the $m$ Newtonian linear equations are solved with increasing accuracy $\eta_k \to 0$. In this case, it will be proved that, if a solution $x^*$ is a limit point of $\{x^k\}$ where the basic assumptions hold, then the first trial direction $d_k$ will be asymptotically accepted. As a consequence, GBIN coincides asymptotically with BIN in this case and $\eta_k \to 0$ implies that the convergence is superlinear.

**Theorem 4** *Assume that the sequence $\{x^k\}$ is generated by GBIN and that for all $k = 0, 1, 2, \ldots$, whenever $\alpha = 1$ we choose $\delta = \alpha$, $w_i = y_i$ and we have that (13) holds with $\eta_k \to 0$. Suppose that $x^*$ is a limit point of $\{x^k\}$ and that the basic assumptions (A1)-(A5) are satisfied in some neighborhood of $x^*$. Then $x^*$ is a solution of (1) and $x^k \to x^*$ superlinearly.*

*Proof.* Let $\varepsilon_1 > 0$ be such that the basic assumptions are satisfied for $\|x - x^*\| < \varepsilon_1$. Let $k$ be large enough such that $\|x^k - x^*\| \le \varepsilon_1$. Since $\lim_{k\to\infty} \eta_k = 0$, using the same argument as in the proof of Theorem 2 we can prove that

$$\lim_{k\to\infty} \frac{\|x^k + d^k - x^*\|}{\|x^k - x^*\|} = 0$$

where $d^k$ is the increment obtained with $\alpha = 1$. Thus, since $\sigma\gamma_k \in (0, 1)$ from Lemma 6 it follows that, for $k$ large enough,

$$\|F(x^k + d^k)\| \le (1 - \frac{\sigma\gamma_k}{2})\|F(x^k)\|.$$

So $x^k + d^k$ satisfies the sufficient descent condition for $k$ large enough. This means that, for $k$ large enough, GBIN coincides with BIN. Therefore, Theorem 2 applies and the convergence is superlinear. $\square$

## 5 Conclusions

In the Introduction we emphasized the relevance of the Block Inexact-Newton theory to complete the convergence theory of the Block-Newton method introduced in [4]. We also mentioned the fact that, when $n_i$ is large and the structure of $J_i$ is not good for matrix factorizations, using Newton's method is not possible and the employment of iterative linear solvers on each block is unavoidable. The practical question is: Is it better to perform just *one* Inexact-Newton iteration on the $i$-th block (as suggested by this paper) or to execute *several* Inexact-Newton iterations on block $i$ before dealing with block $i+1$? The common practice is, in many cases, to "solve" block $i$ before considering block $i+1$. ("Solving" is an ambiguous word in this case since we deal with infinite processes.) Perhaps, this is the most effective alternative in a lot of practical situations. Numerical experiments in [4] seem to indicate that using "several" iterations on each block is, frequently, more efficient than performing only one. However, some arguments can be invoked in defense of the "one-iteration" strategy:

1. It is easy to construct small-dimensional (even $2 \times 2$) examples (say $f_1(x_1) = 0, f_2(x_1, x_2) = 0$) where "to solve" $f_1(x_1) = 0$ before dealing with $f_2$ leads to disastrous results. This is what happens when the behavior of $f_1(x_1^*, x_2)$ is good only for $x_2$ close to the solution. In these cases it is better to approach slowly to $x_1^*$, mixing these attempts with approximations to the solution of the second equation. Of course, these examples could appear arbitrary, but this is also the case of the examples described in [4] which seem to favor $p-$step methods.

2. If the pure local method does not work, the present research shows how to correct it in order to achieve global convergence. It is not clear what should be done if we used a "several-iterations" strategy. In the best case, we could be wasting a lot of time for global unsuccessful trials.

3. From the considerations above it seems that, in difficult problems, a "several-iterations" strategy should be associated to internal globalization schemes. In other words, when dealing with block $i$, the "several-iterations" strategy should not consist on performing "several" local Inexact-Newton (or Newton) iterations but on performing the same number of *global* Inexact-Newton iterations. Of course, such procedure can work very well but, rigorously speaking, an additional

external globalization procedure should be necessary to guarantee convergence. So, the globalization introduced in this paper, that uses only local internal methods, is appealing.

As a result, we think that both "one-iteration" and "several-iterations" strategies deserve attention and can be effective for practical large-scale block systems. The correct and effective way of implementing "several-iterations" strategies is, however, still unclear and deserves future research.

Let us finish with some remarks regarding potential applications. Many engineering problems are described by systems of differential-algebraic equations (DAEs). Parametric sensitivity of the (DAE) model may yield useful information. See [8, 10]. The general DAE system with parameters is of the form

$$F\left(t, y, y', p\right) = 0, \quad y\left(0\right) = y_0, \quad y \in I\!R^{n_y},\ p \in I\!R^{n_p},$$

with sensitivity equations

$$\frac{\partial F}{\partial y} s_i + \frac{\partial F}{\partial y'} s_i + \frac{\partial F}{\partial p} = 0, \quad i = 1, \ldots, n_p,$$

where $s_i = dy/dp_i$. Defining

$$Y = \left[y, s_1, \ldots, s_{n_p}\right]^\top$$

and

$$I\!F = \left[F(t, y, y', p), \frac{\partial F}{\partial y} s_1 + \frac{\partial F}{\partial y'} s_1 + \frac{\partial F}{\partial p_1}, \ldots, \frac{\partial F}{\partial y} s_{n_p} + \frac{\partial F}{\partial y'} s_{n_p} + \frac{\partial F}{\partial p_{n_p}}\right],$$

the combined system can be written as

$$I\!F\left(t, Y, Y', p\right) = 0, \quad Y_0 = \left[y_0, \frac{dy_0}{dp_1}, \ldots, \frac{dy_0}{dp_1}\right]^\top.$$

Approximating the solution to the combined system by an implicit numerical method leads to a block-lower triangular nonlinear algebraic system. When $y$ (or $p$) is of large dimension, an iterative method for solving the Newtonian equation is necessary. Up to our present knowledge, several codes for solving this problem exist. Some of them, like DASSLSO, DASPKSO (see [10]) and DSL48S (see [8]), exploit the lower-triangular structure of the Jacobian but not in the way suggested here. The implementation of the algorithm described in this paper for sensitivity analysis problems can be the subject of research in the near future.

**Acknowledgements**

The authors are indebted to two anonymous referees whose criticism helped a lot to improve this paper. In particular, “Referee 1” suggested the present form of Lemma 3, from which $q$-linear convergence follows. “Referee 2” made insightful remarks concerning a first version of the global algorithm and drew our attention to possible applications of the BIN algorithm.

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