# Nonlinear programming algorithms using trust regions and augmented Lagrangians with nonmonotone penalty parameters

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**Abstract**

A model algorithm based on the successive quadratic programming method for solving the general nonlinear programming problem is presented. The objective function and the constraints of the problem are only required to be differentiable and their gradients to satisfy a Lipschitz condition.

The strategy for obtaining global convergence is based on the trust region approach. The merit function is a type of augmented Lagrangian. A new updating scheme is introduced for the penalty parameter, by means of which monotone increase is not necessary.

Global convergence results are proved and numerical experiments are presented.

**Key words:** Nonlinear programming, successive quadratic programming, trust regions, augmented Lagrangians, Lipschitz conditions.

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## 1 Introduction

Every minimization problem with nonlinear equality and inequality constraints can be reduced, by means of the introduction of slack variables, to the standard form

$$\text{Minimize } f(x) \text{ subject to } \quad C(x) = 0, \;\; l \leq x \leq u. \tag{1}$$

In this work, it is assumed that $f : I\!R^n \to I\!R$, $C : I\!R^n \to I\!R^m$ have continuous first partial derivatives, and $-\infty \leq l_i < u_i \leq \infty$ for all $i = 1, \ldots, n$. All along the paper we denote

$$g(x) = \nabla f(x)$$

and

$$A(x) = (\nabla C_1(x), \ldots, \nabla C_m(x))^T.$$

Formulation (1) is used in many successful practical algorithms for nonlinear programming, like those based on the generalized reduced gradient (see [15, 16]) and on the augmented Lagrangian approach (see [6, 7]).

In this paper, a general algorithm is introduced, based on successive quadratic programming (SQP), for solving (1). The contribution of this work generalizes, in many aspects, the approach given in [8] for equality constrained optimization. As in [8], the main model algorithm considered here has the following characteristics:

(i) A trust region approach is used as globalization strategy.

(ii) The algorithm generates steps whose "normal" and "tangential" components satisfy mild conditions on adequate models.

(iii) The merit function is of the augmented Lagrangian type, where a large amount of freedom for the estimation of Lagrange multipliers is admitted.

The convergence proofs make it easier to work with a new formulation of the augmented Lagrangian, where the "penalty parameter" $\theta$ belongs to $[0, 1]$. As it is well known, the classical augmented Lagrangian is

$$\hat{\psi}(x, \lambda, \rho) = \ell(x, \lambda) + \rho\varphi(x),$$

where $\lambda \in I\!R^m$, $\rho > 0$, $\ell(x, \lambda)$ is the Lagrangian function given by

$$\ell(x, \lambda) = f(x) + C(x)^T \lambda$$

and

$$\varphi(x) = \frac{1}{2}\|C(x)\|_2^2.$$

In this work, the following merit function is defined:

$$\psi(x, \lambda, \theta) = \theta\ell(x, \lambda) + (1 - \theta)\varphi(x). \tag{2}$$

So, the obvious relation $\rho = (1 - \theta)/\theta$ can be established between $\theta$ and the classical penalty parameter $\rho$, and the monotone increase of $\rho$ corresponds to the monotone decrease of $\theta$. In the convergence theory presented, the monotone decrease of $\theta$ is not necessary. Instead, a nonmonotone strategy is defined that ensures convergence and allows one to test different "degrees of nonmonotonicity". In a recent paper, El-Alem [9] defined a different nonmonotone strategy for the penalty parameter of a specific algorithm for equality constrained minimization. Nonmonotone strategies for penalty parameters, associated to nonsmooth merit functions were proposed in [4] and [26].

An arbitrary norm is used in the definition of the trust regions. However, the $\infty-$norm is the natural choice for practical implementations, since it fits well with the bounds of the problem.

An important feature of this theory is that, unlike the one presented in [8], second differentiability of the objective function or of the constraints is not needed. Instead, a Lipschitz condition for the gradients of these functions is used. The optimality conditions of (1) can be viewed as a nonsmooth system of equations. Local convergence of Newton-like methods for this type of system has been established recently. See [13, 22, 24]. The global convergence theory presented in this work is complementary to those local analyses.

This paper is organized as follows: in Section 2 the definition of the main model algorithm is motivated, while its precise description is given in Section 3. In Section 4 it is proved that, under mild conditions, the algorithm is well defined. The feasibility of every limit point generated by the algorithm is established in Section 5. In Section 6 it is proved that there exists a limit point that is stationary for problem (1). In Section 7, numerical experiments are reported. Finally, in Section 8, some conclusions are stated and lines for future research are suggested.

**Notation**

The symbol $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^n$, $|\cdot|$ denotes the Euclidean norm of vectors or matrices and the corresponding scalar product is denoted $\langle x, y\rangle$.

If $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the segment that has $x$ and $y$ as extreme points.

The expressions $\mathcal{N}(B)$ and $\mathcal{R}(B)$ denote the null-space of a matrix $B$ and its column space, respectively.

In many places throughout this paper, if the simplification does not lead to confusion, some of the arguments of a function will be omitted. For example, $Q(s)$ will denote $Q(H, x, \lambda, s)$ when there is no ambiguity on the arguments $H, x$ and $\lambda$.

In the convergence proofs, the $O(.)$ notation will be frequently used. Namely, it is said that $a = O(b)$ (where $a, b \geq 0$) if a constant $c \geq 0$ (depending only on the data of the problem) exists such that $a \leq cb$. Sometimes, $a \leq O(b)$ will be written instead of $a = O(b)$ to stress the meaning of the underlying inequality. Many proofs can be more concisely written using this notation. In these cases, it will be easy to verify that, in fact, the implicit constants only depend on the problem data.

# 2 Motivation

Let $x \in \mathbb{R}^n$ $(l \leq x \leq u)$ and $\lambda \in \mathbb{R}^m$ be approximations of the solution of (1) and the optimal Lagrange multipliers respectively. Suppose that $Q(H, x, \lambda, s)$ is a quadratic (Taylor-like) approximation of $\ell(x + s, \lambda)$. So,

$$Q(H, x, \lambda, s) = \frac{1}{2} s^T H s + \nabla \ell(x, \lambda)^T s + \ell(x, \lambda), \tag{3}$$

where $H \in \mathbb{R}^{n \times n}$ is symmetric. A naive Newtonian procedure for finding a new, probably better, approximation $x_+ \equiv x + s_c$ of the solution of (1) involves finding the solution $s_c$ to the quadratic program

$$\text{Minimize } Q(H, x, \lambda, s) \tag{4}$$

$$\text{subject to} \quad A(x)s + C(x) = 0, \;\; l \leq x + s \leq u. \tag{5}$$

Most SQP methods for solving (1) are based on modifications of the scheme (4)–(5). Unfortunately, this quadratic program can be unbounded or infeasible. Unboundedness can be avoided by means of the introduction of a regularization (or trust-region) constraint

$$\|s\| \leq \delta > 0. \tag{6}$$

Observe that, if $\|\cdot\| = \|\cdot\|_\infty$, (4)–(6) continues being a quadratic programming problem. The problem of infeasibility is more serious, since the set of increments $s$ that satisfy (5) (or (5)–(6)) can be empty and, in this case, it is not clear how to carry out the algorithm. In this paper, the lines of [8] are followed for defining a relaxed problem that can be solved even when the constraints of (4)–(6) are incompatible. See [3, 5, 21, 23, 29] for related strategies. The idea consists on dividing the iteration into two phases. First, the problem

$$\text{Minimize } M(x,s) \text{ subject to } l \le x+s \le u, \quad \|s\| \le 0.8\delta \tag{7}$$

is considered, where

$$M(x,s) = \frac{1}{2}|A(x)s + C(x)|^2. \tag{8}$$

(Observe that $\nabla M(x,s) = A(x)^T(C(x)+A(x)s)$ and $\nabla M(x,0) = A(x)^T C(x) = \nabla\varphi(x)$.) Secondly, if $s_{nor}$ is a solution of (7), the increment $s_c$ is obtained solving the feasible quadratic program

$$\text{Minimize } Q(H,x,\lambda,s) \tag{9}$$

$$\text{subject to} \quad A(x)s = A(x)s_{nor}, \;\; l \le x+s \le u, \quad \|s\| \le \delta. \tag{10}$$

Clearly, (7)–(10) is consistent with (4)–(6), in the sense that the solution of both problems coincide when (4)–(6) is solvable, $\delta$ is large enough and $C(x)$ is in the column space of $A(x)$.

Now, the exact minimization of $M$ and $Q$ considered in (7) and (9) can be very costly in critical (especially large-scale) problems. Therefore, it is necessary to develop procedures for computing (7)–(10) only in an approximate sense. Most of the description of our main model algorithm consists in giving a precise meaning to (7) and (9) when "Minimize" is replaced by "Minimize approximately". The idea is that an increment $s_{nor}$ will be considered as an approximate solution of (7) if it produces a decrease of $M(x,s)$ of the same order of magnitude as the decrease produced by a "pseudo-Cauchy step" related to $M(x,s)$.

Analogously, an increment $s_c$ will be an approximate solution of (9)–(10) if the decrease $Q(H,x,\lambda,s_c) - Q(H,x,\lambda,s_{nor})$ is of the same order as the decrease produced by a "pseudo-Cauchy step" on the feasible region of this quadratic programming problem.

After the computation of the trial increment $s_c$ and the trial point $x+s_c$, it must be decided whether this point can be accepted as a new estimate of the solution of (1). For this purpose, the merit function defined by (2) is used. This function is a convex combination of $\ell$ and $\varphi$, so that its first term measures progress in optimality and the second one measures improvement of feasibility. The merit function at the trial point involves not only $x+s_c$ and the penalty parameter $\theta$ but also a new estimate $\lambda + \Delta\lambda$ of the Lagrange multipliers. In the theory presented here, as in [8] for equality constrained problems, the new estimate of the vector of multipliers can be (almost) completely arbitrary, subject only to a boundedness condition. As a result, all efficient techniques for estimating multipliers (see [12, 27, 28], and others) can be used.

To complete the definition of the merit function $\psi$, we need to choose a suitable penalty parameter related to $x$, $s_c$, $\lambda$ and $\Delta\lambda$. To make this decision, observe that, if $\|s\|$ is small enough,

$$C(x+s) \approx C(x) + A(x)s, \tag{11}$$

$$\varphi(x+s) \approx M(x,s) \tag{12}$$

and

$$\ell(x+s,\lambda) \approx Q(H,x,\lambda,s). \tag{13}$$

Therefore, by (11) and (13), we have that, for small $\|s\|$ and bounded $\|\Delta\lambda\|$,

$$\ell(x+s,\lambda+\Delta\lambda) = f(x+s) + C(x+s)^T(\lambda+\Delta\lambda) = \ell(x+s,\lambda) + C(x+s)^T\Delta\lambda$$

$$\approx Q(H,x,\lambda,s) + (C(x)+A(x)s)^T\Delta\lambda. \tag{14}$$

Let us define $Ared(x,\lambda,s,\Delta\lambda,\theta)$, the "actual reduction" of the merit function from $(x,\lambda)$ to $(x+s,\lambda+\Delta\lambda)$, as

$$Ared(x,\lambda,s,\Delta\lambda,\theta) = \psi(x,\lambda,\theta) - \psi(x+s,\lambda+\Delta\lambda,\theta). \tag{15}$$

By (2), (12) and (14) we have that

$$Ared(x,\lambda,s,\Delta\lambda,\theta) \approx \theta[Q(H,x,\lambda,0) - Q(H,x,\lambda,s) - (C(x)+A(x)s)^T\Delta\lambda]$$
$$+ (1-\theta)[M(x,0) - M(x,s)]. \tag{16}$$

Expression (16) justifies the definition of the "predicted reduction" of the merit function from $(x,\lambda)$ to $(x+s,\lambda+\Delta\lambda)$ as

$$Pred(H,x,\lambda,s,\Delta\lambda,\theta) = \theta[Q(H,x,\lambda,0) - Q(H,x,\lambda,s) - (C(x)+A(x)s)^T\Delta\lambda]$$
$$+ (1-\theta)[M(x,0) - M(x,s)]. \tag{17}$$

Consequently, as usual in trust-region methods,

$$Ared(x,\lambda,s,\Delta\lambda,\theta) \approx Pred(H,x,\lambda,s,\Delta\lambda,\theta). \tag{18}$$

Now, ideally it would be necessary for the merit function at the new point to be sufficiently smaller than its value at the current point, which means that $Ared(x,\lambda,s_c,\Delta\lambda,\theta)$ should be sufficiently larger than 0. So, by (18), it would be natural to choose the penalty parameter in such a way that $Pred(H,x,\lambda,s_c,\Delta\lambda,\theta)$ is sufficiently larger than 0. But, by the computation of $s_c$, only the second term on the definition of $Pred$ is guaranteed to be nonnegative, and the first one is nonnegative if the second is null. So, it is possible to choose a penalization parameter $\theta > 0$ small enough to ensure that

$$Pred(H,x,\lambda,s_c,\Delta\lambda,\theta) \geq \frac{1}{2}[M(x,0) - M(x,s)] \geq 0. \tag{19}$$

By (18) and (19), $Ared$ should be sufficiently positive if $\delta$ is small enough. In fact, the trial point $x+s_c$ is accepted when

$$Ared(x,\lambda,s,\Delta\lambda,\theta) \geq 0.1 Pred(H,x,\lambda,s_c,\Delta\lambda,\theta), \tag{20}$$

otherwise the trust region radius $\delta$ is reduced and the process (7)–(10) is restarted. It will be proved that, after a finite number of reductions of $\delta$, the test (20) is fulfilled, so that a new point $x+s_c$ can be accepted.

In the next section, the details are completed and the algorithm schematized in this section is rigorously formulated.

# 3 Description of the main model algorithm

In this section, the iteration that defines our successive quadratic programming algorithm is described. In order to improve the readibility of this description (avoiding an excessively large number of algorithmic parameters), many of the parameters that define the specific implementation are replaced by their recommended values, usually numbers between 0 and 1. The reader can recognize that, whenever a number in decimal notation appears in the description of the algorithm, it can be replaced by another number belonging to $(0,1)$.

We define $\Omega = \{x \in I\!R^n \mid l \leq x \leq u\}$ and we say that $z \in \Omega$ is $\varphi-$*stationary* if it satisfies the first order optimality conditions of

$$\text{Minimize } \varphi(x) \quad \text{subject to } x \in \Omega. \tag{21}$$

A point $z$ is said to be *feasible* if $z \in \Omega$ and $C(z) = 0$. As usual, a feasible point $z$ is said to be *regular* if the gradients of the active constraints at $z$ are linearly independent.

A lower bound $\delta_{min} > 0$ is used for the trust region radius at the beginning of each iteration. So, the $k$-th iteration begins with $\delta \geq \delta_{min}$. Positive constants $L_1$ and $L_2$ will be defined as upper bounds for the Lagrange multiplier estimates and the norms of the Hessian approximations, respectively.

Given the current estimate $x$ of the solution of (1), the estimate of the Lagrange multiplier vector $\lambda$ and a symmetric matrix $H$ (which, generally, represents an approximation of $\nabla^2 \ell(x, \lambda)$, at least in $C^2$ problems), the following algorithm explains how to obtain the "next iterate" $x_+$ and the next estimate of the vector of Lagrange multipliers $\lambda_+$.

**Algorithm 3.1**

Assume that $x \in \Omega$, $\lambda \in I\!R^m$, $\|\lambda\| \leq L_1$, $\delta \geq \delta_{min}$, $H = H^T \in I\!R^{n \times n}$, $\|H\| \leq L_2$, $\theta_{bound} = 1$. If $x$ is a stationary point of (1), $x$ is $\varphi-$stationary but not feasible, or if $x$ is feasible and non-regular, the execution of the algorithm terminates. Otherwise, the steps for obtaining the next iterate $x_+$ are the following.

**Step 0.** *Compute a feasible descent direction for the constraints or, if the point is feasible, for the Lagrangian.*

If $x$ is not $\varphi-$stationary, compute $d_{nor}(x) \in I\!R^n$ such that

$$l \leq x + d_{nor}(x) \leq u \tag{22}$$

and

$$d_{nor}(x)^T \nabla \varphi(x) = d_{nor}(x)^T \nabla M(x, 0) < 0. \tag{23}$$

If $x$ is feasible, regular, and non-stationary for the nonlinear programming problem (1), define $d_{nor}(x) = 0$, $s_{nor} = s_{nor}(x, \delta) = 0$ and $d_{tan}(x) \in I\!R^n$, such that

$$l \leq x + d_{tan}(x) \leq u,$$

$$A(x) d_{tan}(x) = 0,$$

and

$$d_{tan}(x)^T \nabla \ell(x, \lambda) = d_{tan}(x)^T g(x) < 0. \tag{24}$$

(Observe that $\nabla \ell(x, \lambda) - g(x) \in \mathcal{R}(A(x)^T)$.)

**Step 1.** *Compute the decrease step for the constraints.*

If $x$ is not feasible, compute $s_{nor}^{dec} = s_{nor}^{dec}(x, \delta) \in I\!R^n$, the solution of

$$\text{Minimize } M(x, s)$$

$$\text{subject to } l \leq x + s \leq u, \;\; \|s\| \leq 0.8\delta, \;\; s = t d_{nor}(x), \;\; t \geq 0.$$

**Step 2.** *Compute the "normal step".*

If $x$ is not feasible, compute $s_{nor} = s_{nor}(x, \delta) \in I\!R^n$ such that $l \leq x + s_{nor} \leq u$, $\|s_{nor}\| \leq 0.8\delta$ and

$$M(x, 0) - M(x, s_{nor}) \geq 0.9[M(x, 0) - M(x, s_{nor}^{dec})]. \tag{25}$$

**Step 3.** *Compute the decrease step for the Lagrangian function on the tangent space.*

If $x$ is feasible, set $d_{tan} = d_{tan}(H, x, \lambda, \delta) = d_{tan}(x)$ (defined at Step 0, in this case). Otherwise, compute $d_{tan} = d_{tan}(H, x, \lambda, \delta) \in I\!R^n$ such that

$$l \leq x + s_{nor} + d_{tan} \leq u,$$

$$A(x) d_{tan} = 0,$$

and

$$d_{tan}^T \nabla Q(H, x, \lambda, s_{nor}) < 0. \tag{26}$$

If a vector satisfying the three conditions above does not exist, define $d_{tan} = 0$. Otherwise (which includes the case where $s_{nor} = 0$, i.e. $x$ is feasible), define $s_{tan}^{dec} = s_{tan}^{dec}(H, x, \lambda, \delta)$ the solution of

$$\text{Minimize } Q(s_{nor} + s)$$

$$\text{subject to } l \leq x + s_{nor} + s \leq u, \;\; \|s_{nor} + s\| \leq \delta, \;\; s = t d_{tan}, \;\; t \geq 0.$$

**Step 4.** *Compute the "tangent step".*

Compute $s_{tan} = s_{tan}(H, x, \lambda, \delta) \in I\!R^n$ such that

$$A(x) s_{tan} = 0, \;\; l \leq x + s_{nor} + s_{tan} \leq u, \;\; \|s_{nor} + s_{tan}\| \leq \delta, \tag{27}$$

and

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}) \geq 0.9[Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec})]. \tag{28}$$

**Step 5.** *Compute the update of the Lagrange multipliers and the current step.*

Compute $\Delta\lambda \in I\!R^m$ such that $\|\lambda + \Delta\lambda\| \leq L_1$ and define

$$s_c = s_c(H, x, \lambda, \delta) = s_{nor} + s_{tan}.$$

(In practical implementations, $\Delta\lambda$ depends on $H, x, \lambda$, and $\delta$.)

**Step 6.** *Compute an adequate penalty parameter for the current step.*

Choose $\theta = \theta(H, x, \lambda, s_c, \Delta\lambda, \delta) \in [0, \theta_{bound}]$ such that

$$Pred(H, x, \lambda, s_c, \Delta\lambda, \theta) \geq 0.5[M(x, 0) - M(x, s_c)]. \tag{29}$$

Set $\theta_{bound} \leftarrow \theta$.

**Step 7.** *Accept or reject the current step $s_c$.*

If

$$Ared(x, \lambda, s_c, \Delta\lambda, \theta) \geq 0.1 Pred(H, x, \lambda, s_c, \Delta\lambda, \theta) \tag{30}$$

define

$$x_+ = x + s_c, \quad \lambda_+ = \lambda + \Delta\lambda.$$

In this case, if $k$ is the current iteration number, define

$$\delta_k = \delta, \quad \theta_k = \theta, \quad s_k = s_c, \quad \Delta\lambda_k = \Delta\lambda \tag{31}$$

and finish the iteration setting $x_{k+1} = x_+$, $\lambda_{k+1} = \lambda_+$.

If (30) does not hold, choose a new $\delta$ belonging to $[0.1\delta, 0.9\delta]$ and go to Step 1.

This completes the description of the algorithm.

**Remarks.**

If $x$ is not $\varphi$-stationary, then $M(x,0) - M(x,s_c) = M(x,0) - M(x,s_{nor}) > 0$. Thus, (29) is always satisfied for $\theta = 0$ and, in this case, $Pred$ does not depend on $\lambda, \Delta\lambda$ at all. It will be shown in Sections 4 and 5 that, even with this strange choice, the algorithm is well defined and converges to a stationary point of $\varphi$. However, should $\theta = 0$ be chosen, it would not be possible to prove convergence to stationary points of (1) because, in that case, the weight of the Lagrangian on the merit function would be null. In fact, the choice of $\theta$ by means of Algorithm 5.1 in Section 5 excludes the possibility of the null choice. It is necessary to stress that, throughout the paper, increasing $\theta$ across different iterations will be admitted. Nevertheless, by the choice (29), the sequence of penalty parameters used within a single iteration must be nonincreasing.

In Section 2, it was mentioned that the requirements on the "normal step" and the "tangent step" that are sufficient to guarantee convergence of the algorithm are that the decrease in the corresponding quadratics produced by those steps should be of the same order as the decrease produced by certain "Cauchy steps". Conditions (25) and (28) state those requirements rigorously. (Of course, 0.9 can be replaced by any other number between 0 and 1.) Clearly, $s_{nor}^{dec}$ and $s_{tan}^{dec}$ satisfy (25) and (28) respectively, but standard choices of $d_{nor}$ and $d_{tan}$ could produce algorithms with poor local convergence behavior when $s_{nor} = s_{nor}^{dec}$ and $s_{tan} = s_{tan}^{dec}$ are chosen.

At the beginning of each iteration, we choose $\delta \geq \delta_{min}$. This inequality is the only requirement that $\delta$ must satisfy for the convergence proofs. As usual in trust-region algorithms, $\delta$ can be chosen as a multiple of $\delta_k$ (safeguarded by $\delta_{min}$ in our case) according to the quotient $Ared/Pred$. Clever choices can help to improve practical performance of algorithms.

Algorithm 3.1 admits a large number of specific implementations. In particular, $\Delta\lambda$ can be computed, at Step 5, using quite sophisticated procedures for estimating Lagrange multipliers. However, even the null choice $(\lambda, \Delta\lambda \equiv 0)$ is admissible from the point of view of global convergence properties.

# 4 The algorithm is well defined

In this section, it is proved that Algorithm 3.1 is well defined. This means that, if it does not terminate at the current point $x$, a new point $x_+$ satisfying (30) can be found after repeating steps 1 to 7 a finite number of times. It is interesting to observe that the sufficient conditions that guarantee this property are very weak. In particular, $d_{nor}$ and $d_{tan}$ are merely directions that satisfy the angle conditions (23) and (24)-(26). Later, we will see that for obtaining convergence of the algorithm we need stronger conditions on $d_{nor}$ and $d_{tan}$.

For proving this property, two cases will be considered. Firstly, when the current point $x \in \Omega$ is not $\varphi-$stationary. Secondly, when $x$ is feasible and regular but it is not a stationary point of (1). Observe that in the remaining cases (when $x$ is $\varphi-$stationary but not feasible, $x$ feasible and non-regular and $x$ stationary for (1)), the algorithm terminates. The first case is considered in Lemma 4.1 and the second one in Lemma 4.2.

**Lemma 4.1.** *Assume that $x$ is not a stationary point of (21). Then, after a finite loop on Algorithm 3.1, the next iterate $x_+$ is obtained.*

*Proof.* Since $x$ is not $\varphi-$stationary, $d_{nor} \neq 0$ is computed at Step 0 of the algorithm. We define, for all $\delta > 0$,

$$v(\delta) = t(\delta) d_{nor},$$

where

$$t(\delta) = \max \{t > 0 \mid [x, x + t d_{nor}] \subset \Omega, \text{ and } \|t d_{nor}\| \leq 0.8\delta\}.$$

Since $x + d_{nor} \in \Omega$, we have that $\|t(\delta) d_{nor}\| = 0.8\delta$ if $\delta \leq \|d_{nor}\|$. Define $c = -\frac{1}{2} d_{nor}^T \nabla\varphi(x)/\|d_{nor}\| > 0$. By elementary properties of one-dimensional quadratics, there exists $\overline{\delta} \in (0, \|d_{nor}\|]$ such that, for all $\delta \in (0, \overline{\delta})$,

$$M(0) - M(v(\delta)) \geq -\frac{1}{2} d_{nor}^T \nabla\varphi(x) t(\delta) = c\|d_{nor}\| t(\delta) = 0.8c\delta.$$

Therefore, by the computations at steps 1 and 2 of Algorithm 3.1, we have that

$$M(0) - M(s_{nor}(\delta)) \geq 0.72c\delta.$$

Moreover, since $A s_{tan} = 0$, we also have that, for $\delta \in (0, \overline{\delta})$,

$$M(0) - M(s_c(\delta)) \geq 0.72c\delta.$$

Therefore, by the choice of $\theta$ at Step 6, for $\delta \in (0, \overline{\delta})$, we have that

$$Pred(H, x, \lambda, s_c(\delta), \Delta\lambda, \theta) \geq 0.36c\delta. \tag{32}$$

Now, by the differentiability of $f$ and $C$, the boundedness of $\|\Delta\lambda(\delta)\|$ and the definition of $Q, M$ and $Pred$, we have that

$$\lim_{\delta \to 0} \frac{|Ared(\delta) - Pred(\delta)|}{\delta} = 0. \tag{33}$$

So, by (32),

$$\lim_{\delta \to 0} \left| \frac{Ared(\delta)}{Pred(\delta)} - 1 \right| = 0. \tag{34}$$

This implies that, $\delta$ being small enough, the inequality (30) necessarily takes place. This proves the desired result. □

**Lemma 4.2**. *Assume that $x$ is a feasible and regular point for the nonlinear programming problem (1) where the Kuhn-Tucker conditions do not hold. Then, after a finite loop on Algorithm 3.1, the new iterate $x_+$ is obtained.*

*Proof.* In this case, we have that $s_{nor} = s_{nor}(\delta) = 0$ for all $\delta > 0$. Since $x$ is feasible, regular and non-stationary, the vector $d_{tan}(x) \neq 0$, computed at Step 0 of the algorithm, exists. As in Lemma 4.1, we define, for all $\delta > 0$,

$$p(\delta) = t(\delta) d_{tan},$$

where

$$t(\delta) = \max \{t > 0 \mid [x, x + t d_{tan}] \subset \Omega, \text{ and } \|t d_{tan}\| \leq \delta\}.$$

Since $x + d_{tan} \in \Omega$, we have that $\|t(\delta) d_{tan}\| = \delta$ whenever $\delta \leq \|d_{tan}\|$. Define, in this case,

$$c = -\frac{1}{2} d_{tan}^T \nabla\ell(x)/\|d_{tan}\| = -\frac{1}{2} d_{tan}^T g(x)/\|d_{tan}\| = -\frac{1}{2} d_{tan}^T \nabla Q(0)/\|d_{tan}\| > 0.$$

Analogously to Lemma 4.1, since $Q(s_{tan}^{dec}) \leq Q(p(\delta))$, there exists $\overline{\delta} \in (0, \|d_{tan}\|]$ such that, for all $\delta \in (0, \overline{\delta})$,

$$Q(0) - Q(s_{tan}^{dec}) \geq -\frac{1}{2} d_{tan}^T \nabla Q(0) t(\delta) = -\frac{1}{2} \frac{d_{tan}^T \nabla Q(0)}{\|d_{tan}\|} \delta = c\delta.$$

So, for all $\delta \in (0, \bar{\delta})$,

$$Q(0) - Q(s_c(\delta)) \geq Q(0) - Q(s_{tan}^{dec}) \geq 0.9c\delta. \tag{35}$$

Moreover, since, in this case $M(0) = M(s_c) = 0$, $C(x) = 0$ and $As_c = 0$, we have that

$$Pred(H, x, \lambda, s_c(\delta), \Delta\lambda, \theta) = \theta[Q(0) - Q(s_c(\delta))] \geq 0.5[M(0) - M(s_c)]$$

for all $\delta \in (0, \overline{\delta})$, $\theta \in [0, 1]$. This means that $\theta$ is not decreased through any sequence of possible decreases of $\delta$ in the algorithmic loop. Therefore, by (35),

$$Pred(H, x, \lambda, s_c(\delta), \Delta\lambda, \theta) \geq 0.9\theta' c\delta, \tag{36}$$

for all $\delta \in (0, \overline{\delta})$, where $\theta'$ is the value of $\theta$ that satisfies (29) at the beginning of the loop. The equality (33) follows as in Lemma 4.1 and, therefore, (34) follows from (33) and (36). As in Lemma 4.1, the desired result follows from (34). $\square$

# 5 Every limit point is $\varphi-$stationary

In Section 4, it was shown that Algorithm 3.1 terminates at the $k-$th iteration only in the following situations:

(a) $x_k$ is $\varphi-$stationary but not feasible;
(b) $x_k$ is feasible and non-regular;
(c) $x_k$ is feasible and stationary for the nonlinear programming problem (1).

In this section and in the next one, the behavior of infinite sequences defined by the algorithm is investigated. Some additional assumptions will be introduced as soon as they are necessary. The first one states that $d_{nor}(x)$, at Step 0 of Algorithm 3.1, must be chosen as a continuous function of the current point.

**A1. Continuity Assumption on $d_{nor}$.** *For all $x \in \Omega$, the mapping $d_{nor}$ is continuous at $x$.*

A possible choice that satisfies Assumption A1 is

$$d_{nor}(x) = P(x - \gamma \nabla \varphi(x)) - x, \tag{37}$$

where $P$ is the orthogonal projection on $\Omega$ and $\gamma > 0$ is a fixed scaling parameter. The continuity of $d_{nor}$ follows, in this case, from the continuity of $\nabla\varphi$ and $P$. The parameter $\gamma$ can be important to make the units of $x$ compatible with those of $\nabla\varphi$ in very badly scaled problems.

Assumption A1 is still very weak, and it is interesting to observe that, together with the angle condition (23) (and Assumptions A2 and A3 stated later), it suffices to guarantee convergence to $\varphi-$stationary points. In Section 6, we will see that, for obtaining convergence to critical points of the nonlinear programming problem, we need further assumptions on $d_{nor}$ and $d_{tan}$.

In Lemma 5.1, it is proved that, in the neighborhood of a point which is not $\varphi-$stationary, the predicted decrease of the merit function is proportional to the trust region radius $\delta$.

**Lemma 5.1.** *Assume that $x_* \in \Omega$ is not $\varphi-$stationary and that Assumption A1 holds. Then, there exist $\varepsilon_1, \delta_1', c_1 > 0$ such that for all $x \in \Omega$, $\|x - x_*\| \leq \varepsilon_1$, if Algorithm 3.1 is applied to $x$ and (29) is satisfied, the predicted reduction satisfies*

$$Pred(x, \theta, \delta) \geq c_1 \min \{\delta, \delta_1'\}.$$

*Proof.* Since $x_*$ is not $\varphi-$stationary, we have that

$$d_{nor}(x_*)^T \nabla \varphi(x_*) < 0.$$

So, by the continuity of $d_{nor}(x)$ and $\nabla \varphi(x)$, there exists $\varepsilon_1 > 0$ such that, whenever $x \in \Omega$ and $\|x - x_*\| \leq \varepsilon_1$,

$$2\|d_{nor}(x_*)\| \geq \|d_{nor}(x)\| \geq \frac{\|d_{nor}(x_*)\|}{2} > 0 \tag{38}$$

and

$$d_{nor}(x)^T \nabla M(x, 0) \equiv d_{nor}(x)^T \nabla \varphi(x) \leq 0.5 d_{nor}(x_*)^T \nabla \varphi(x_*) < 0. \tag{39}$$

By the continuity of $A(x)$, $|A(x)^T A(x)|$ is bounded for $\|x - x_*\| \leq \varepsilon_1$. So, by (39) and elementary properties of one-dimensional quadratics there exists $t_1 \in (0, 1)$ such that

$$M(x, 0) - M(x, t d_{nor}(x)) \geq -0.25 d_{nor}(x_*)^T \nabla \varphi(x_*) t > 0 \tag{40}$$

whenever $\|x - x_*\| \leq \varepsilon_1$, $t \in (0, t_1]$.

Define

$$\delta_1' = \frac{t_1 \|d_{nor}(x_*)\|}{2}.$$

If $0 < \delta \leq \delta_1'$ and $\|x - x_*\| \leq \varepsilon_1$ we have, from (38), that

$$\frac{\delta}{2\|d_{nor}(x)\|} \leq \frac{\delta_1'}{\|d_{nor}(x_*)\|} < t_1.$$

Therefore, by (38) and (40),

$$M(x, 0) - M(x, \frac{\delta}{2\|d_{nor}(x)\|} d_{nor}(x)) \geq \frac{-0.25 d_{nor}(x_*)^T \nabla \varphi(x_*) \delta}{2\|d_{nor}(x)\|}$$

$$\geq \frac{-0.25 d_{nor}(x_*)^T \nabla \varphi(x_*) \delta}{4\|d_{nor}(x_*)\|}.$$

Since $\|\frac{\delta}{2\|d_{nor}(x)\|} d_{nor}(x)\| \leq 0.8\delta$, it follows that

$$M(x, 0) - M(x, s_c(x, \delta)) \geq 0.9[M(x, 0) - M(x, s_{nor}^{dec}(x, \delta))]$$

$$\geq \frac{-0.05625 d_{nor}(x_*)^T \nabla \varphi(x_*)}{\|d_{nor}(x_*)\|} \delta. \tag{41}$$

If $\delta > \delta_1'$, the definition of $s_{nor}^{dec}$ implies that

$$[M(x, 0) - M(x, s_{nor}^{dec}(x, \delta))] \geq [M(x, 0) - M(x, s_{nor}^{dec}(x, \delta_1'))].$$

Since (41) holds for $\delta = \delta_1'$, it follows that

$$M(x, 0) - M(x, s_c(x, \delta)) \geq 0.9[M(x, 0) - M(x, s_{nor}^{dec}(x, \delta_1'))]$$

$$\geq \frac{-0.05625 d_{nor}(x_*)^T \nabla\varphi(x_*)}{\|d_{nor}(x_*)\|}\delta_1'. \tag{42}$$

The desired result follows from (41), (42) and (29) by taking

$$c_1 = \frac{-0.028125 d_{nor}(x_*)^T \nabla\varphi(x_*)}{\|d_{nor}(x_*)\|}.$$

□

The following assumption replaces second derivative assumptions generally used in trust-region-SQP algorithms. It will be used to ensure that the predicted reduction of the merit function is a second-order approximation of the actual reduction.

**A2. Bounded variation assumption.** *For all $x, y \in \Omega$,*

$$\|\nabla f(x) - \nabla f(y)\| \leq O(\|x-y\|), \tag{43}$$

and

$$\|A(x) - A(y)\| \leq O(\|x-y\|). \tag{44}$$

The Lipschitz conditions (43) and (44) imply that, for all $x, y \in \Omega$,

$$|f(y) - f(x) - \nabla f(x)^T(y-x)| \leq O(\|y-x\|^2) \tag{45}$$

and

$$\|C(y) - C(x) - A(x)(y-x)\| \leq O(\|y-x\|^2). \tag{46}$$

The following Lemma states that, in a neighborhood of a non-$\varphi-$stationary point, the criterion of acceptance of a trial step will be satisfied if the trust region radius is smaller than a fixed quantity $\delta_2'$.

**Lemma 5.2.** *Assume that $x_*$ is not $\varphi-$stationary and that Assumptions A1 and A2 hold. Then, there exist $\varepsilon_2, \delta_2' > 0$ such that for all $x \in \Omega$, $\|x - x_*\| \leq \varepsilon_2$, $\delta \leq \delta_2'$, if Algorithm 3.1 is applied to $x$ and (29) is satisfied, the following inequality holds:*

$$Ared(x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta) \geq 0.1 Pred(H, x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta).$$

*Proof.* By (45), (46) and the boundedness of $\Delta\lambda$,

$$|Ared(x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta) - Pred(H, x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta)| \leq O(\|s_c(\delta)\|^2) \leq O(\delta^2).$$

Therefore, by Lemma 5.1,

$$\left| \frac{Ared(x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta)}{Pred(H, x, \lambda, s_c(\delta), \Delta\lambda(\delta), \theta)} - 1 \right| \leq O(\delta)$$

and the required result follows from this inequality. □

In the following Lemma, it will be proved that, if the sequence generated by Algorithm 3.1 admits a limit point which is not $\varphi-$stationary, the actual reduction of the merit function is bounded away from zero.

**Lemma 5.3.** *Suppose that $x_*$ is not $\varphi-$stationary and that Assumptions A1 and A2 hold. Let $K_1$ be an infinite set of indices such that*

$$\lim_{k\in K_1} x_k = x_*.$$

*Then $\{\delta_k, k \in K_1\}$ is bounded away from 0. Moreover, there exists $c_2 > 0$ such that for large enough $k \in K_1$,*

$$Ared(x_k, \lambda_k, s_k, \Delta\lambda_k, \theta_k) \geq c_2.$$

*Proof.* If $k \in K_1$ is large enough, we have that $\|x_k - x_*\| \leq \varepsilon_2$, where $\varepsilon_2$ is defined in Lemma 5.2. By the criterion of acceptance of the trust region radius $\delta$ given in Algorithm 3.1, and Lemma 5.2, it is easy to see that, when $\|x_k - x_*\| \leq \varepsilon_2$, rejected steps smaller than $\delta_2'$ cannot occur. Therefore, $\delta_k$ is bounded away from zero. Now, by Lemmas 5.1 and 5.2,

$$Ared(x_k, \lambda_k, s_k, \Delta\lambda_k, \theta_k) \geq 0.1 Pred(H_k, x_k, \lambda_k, s_k, \Delta\lambda_k, \theta_k) \geq 0.1 c_1 \min\{\delta_k, \delta_1'\},$$

and the required result follows from this inequality since $\delta_k$ is bounded away from zero. □

The fact proved in Lemma 5.3 is related to unboundedness of the merit function as far as, assymptotically, the merit function used at all the iterations does not change. In the following algorithm, a specific choice of the penalization parameter is given with the aim of assuring that, essentially, the merit function will not change if $k$ is large enough.

**Algorithm 5.1**

This algorithm has exactly the same steps as Algorithm 3.1, with a specific choice of $\theta$ at Step 6 of the $k-$th iteration.

The parameter $\theta$ that satisfies (29) is chosen according to

$$\theta_k^{min} = \min\{1, \theta_0, \ldots, \theta_{k-1}\},$$

$$\theta_k^{large} = (1 + \frac{N}{(k+1)^{1.1}})\theta_k^{min}, \tag{47}$$

where $N \geq 0$ is a number that reflects the "degree of non-monotonicity" desired for the penalty parameter. Let us define

$$\theta^{sup} = \theta^{sup}(x, \delta) = \sup\{\theta \in [0,1] \mid Pred(H, x, \lambda, s_c, \Delta\lambda, \theta) \geq 0.5[M(x,0) - M(x, s_c)]\}.$$

If $\delta$ is the first trust region radius tested at the current iteration, define

$$\theta' = \theta'(x, \delta) = \theta_k^{large}. \tag{48}$$

Otherwise, set

$$\theta' = \theta'(x, \delta) = \theta(x, \delta'), \tag{49}$$

where $\delta'$ is the trust region radius tested immediately before $\delta$ at the current iteration.

Finally, the value of $\theta$ that satisfies (29) is given by

$$\theta = \theta(x, \delta) = \min\{\theta^{sup}, \theta'\}. \tag{50}$$

The assymptotic stability of $\theta_k$ is proved in the following lemma.

**Lemma 5.4.** *Assume that Algorithm 5.1 generates an infinite sequence $\{x_k\}$. Then, the sequence $\{\theta_k\}$ is convergent.*

*Proof.* The sequence $\{\theta_k^{min}\}$ is nonincreasing and bounded below, so it is convergent. Moreover, $\{\theta_k^{large}\}$ converges to the same limit, since $|\theta_k^{min} - \theta_k^{large}| \to 0$. But, by the choice of $\theta$, we have that $\theta_k \leq \theta_k^{large}$, and, by the definition of $\theta_k^{min}$, we have that $\theta_{k+1}^{min} \leq \theta_k$. Therefore, $\{\theta_k\}$ is enclosed between two sequences ($\{\theta_{k+1}^{min}\}$ and $\{\theta_k^{large}\}$) that have the same limit. So, $\{\theta_k\}$ is convergent. □

The following assumption makes the unboundedness of the merit function used in Algorithm 5.1 impossible.

**A3. Compactness assumption.** *The sequence $\{x_k\}$ generated by Algorithm 5.1 is bounded.*

The compactness assumption is quite natural when dealing with bound constrained problems, because the sequence $\{x_k\}$ is obviously bounded if the bounds $l$ and $u$ are finite ($\Omega$ is compact). By the boundedness of $\|H_k\|$ and $\lambda_k$, it follows that $\ell(x_k, \lambda)$, $\varphi(x_k)$ and $\psi(x_k, \lambda, \theta)$ are bounded.

**Theorem 5.5.** *Let $\{x_k\}$ be the (infinite) sequence generated by Algorithm 5.1 and assume that Assumptions A1, A2 and A3 hold. Then, every limit point of $\{x_k\}$ is $\varphi-$stationary.*

*Proof.* Suppose that $x_* \in \Omega$ is a limit point of $\{x_k\}$ and that it is not $\varphi-$stationary. Let us write, for simplicity,

$$\ell_k = \ell(x_k, \lambda_k), \quad \varphi_k = \varphi(x_k), \quad \psi_k = \psi(x_k, \lambda_k, \theta_k)$$

for all $k \in I\!N$. Then, for all $k \in I\!N$ we have that

$$\begin{aligned}
\psi_{k+1} &= \theta_{k+1}\ell_{k+1} + (1-\theta_{k+1})\varphi_{k+1} \\
&= \theta_{k+1}\ell_{k+1} + (1-\theta_{k+1})\varphi_{k+1} - [\theta_k\ell_{k+1} + (1-\theta_k)\varphi_{k+1}] + [\theta_k\ell_{k+1} + (1-\theta_k)\varphi_{k+1}] \\
&= (\theta_{k+1} - \theta_k)\ell_{k+1} + (\theta_k - \theta_{k+1})\varphi_{k+1} + [\theta_k\ell_{k+1} + (1-\theta_k)\varphi_{k+1}] \\
&= (\theta_k - \theta_{k+1})(\varphi_{k+1} - \ell_{k+1}) + [\theta_k\ell_k + (1-\theta_k)\varphi_k] - \beta_k \\
&= (\theta_k - \theta_{k+1})(\varphi_{k+1} - \ell_{k+1}) + \psi_k - \beta_k,
\end{aligned} \tag{51}$$

where $\beta_k \equiv Ared(x_k\lambda_k, s_k, \Delta\lambda_k, \theta_k) \geq 0$ for all $k \in I\!N$ and, by Lemma 5.3, $\beta_k \geq c_2 > 0$ for an infinite set of indices. Now, by the choice of $\theta$ at Algorithm 5.1, we have that

$$\theta_k - \theta_{k+1} + \frac{\theta_k N}{(k+1)^{1.1}} \geq 0 \tag{52}$$

for all $k \in I\!N$. By the compactness assumption, there exists an upper bound $c > 0$ such that

$$|\varphi_k - \ell_k| \leq c$$

for all $k \in I\!N$. Therefore, by (51) and (52),

$$\begin{aligned}
\psi_{k+1} &= (\theta_k - \theta_{k+1} + \frac{\theta_k N}{(k+1)^{1.1}})(\varphi_{k+1} - \ell_{k+1}) + \psi_k - \beta_k - \frac{\theta_k N}{(k+1)^{1.1}}(\varphi_{k+1} - \ell_{k+1}) \\
&\leq (\theta_k - \theta_{k+1} + \frac{\theta_k N}{(k+1)^{1.1}})c + \psi_k - \beta_k + \frac{\theta_k N}{(k+1)^{1.1}}c
\end{aligned}$$

$$= (\theta_k - \theta_{k+1})c + \psi_k - \beta_k + 2\frac{\theta_k N}{(k+1)^{1.1}} c.$$

Writing the inequality above for $k = 0, 1, 2, \ldots$ and adding the inequations, we obtain

$$\psi_k \le (\theta_0 - \theta_k)c + \sum_{j=0}^{k-1} \frac{2\theta_j c N}{(j+1)^{1.1}} - \sum_{j=0}^{k-1} \beta_j + \psi_0 \le 2c + \sum_{j=0}^{k-1} \frac{2cN}{(j+1)^{1.1}} - \sum_{j=0}^{k-1} \beta_j + \psi_0 \tag{53}$$

for all $k \in I\!N$. Since the series $\sum_{j=0}^{\infty} \frac{2cN}{(j+1)^{1.1}}$ is convergent and $\beta_k > 0$ for all $k$, (53) implies that $\psi_k$ is unbounded below, thus contradicting the compactness assumption. $\square$

# 6 The algorithm finds a critical point

In this section, it will be proved that, under suitable assumptions, there exists a limit point of the model algorithm that is a stationary point of (1).

In Algorithm 6.1, a specific choice for the tangent direction $d_{tan}$ will be made. The idea is that this step should also be a continuous function of the current point, as it is in the case of $d_{nor}$. Roughly speaking, if $d_{tan}$ is continuous and $s_{nor}$ is small, the decreasing properties of the quadratic model of the Lagrangian on the tangent space will hold for points that are close to feasibility (see Lemmas 6.1 and 6.2). Consequently, the freedom of the choice of $s_{nor}$ must be restricted by imposing that the size of this vector must be of the same order as $\|C(x)\|$.

**A4. First algorithmic assumption.** *The choice of $s_{nor}$ at Step 2 of Algorithm 3.1 is such that*

$$\|s_{nor}(x, \delta)\| \le O(\|C(x)\|). \tag{54}$$

It will be seen later that (54) holds for natural choices of $s_{nor}$.

**Algorithm 6.1**

This is a particular case of Algorithm 5.1, where the choice of $d_{tan}$ at steps 0 and 3 of the algorithm is the following:

$$d_{tan} = P_x(-\eta \nabla Q(s_{nor})), \tag{55}$$

where $P_x(z)$ is the orthogonal projection of $z$ on the set

$$\mathcal{T} \equiv \{y \in \mathcal{N}(A(x)) \mid l \le x + s_{nor} + y \le u\}$$

and $\eta > 0$ is a fixed scaling parameter.

The positive parameter $\eta$ makes the scaling of $x$ compatible with that of $\nabla Q$. As in the definition (37) of $d_{nor}$, $d_{tan}$ can be interpreted as the minimizer of a simple quadratic model of the Lagrangian on the tangent set $\mathcal{T}$. So, the angle conditions (24) and (26) are satisfied by this choice of $d_{tan}$. The problem of finding the projection $d_{tan}$ tends to be easier if $\eta$ is small.

In the proof of Lemma 6.1 two technical propositions are used. The proof of the first one is straightforward.

**Proposition 1.** *Define, for all $z \in I\!R^n$, $P(z)$ the orthogonal projection of $z$ on the polytope given by $Ax = b$, $a_1 \le x \le a_2$. Assume that $z_1 - z_2 \in \mathcal{R}(A^T)$. Then $P(z_1) = P(z_2)$.*

**Proposition 2.** *For all $A \in \mathbb{R}^{(m+\nu)\times n}, b \in \mathbb{R}^{m+\nu}, y \in \mathbb{R}^n$, let us define $\Phi(A,b,y)$ the orthogonal projection of $y$ on the polyhedron defined by*

$$\langle a_i, z\rangle \leq b_i, \quad i = 1,\ldots,m,$$

$$\langle a_i, z\rangle = b_i, \quad i = m+1,\ldots,m+\nu,$$

*where $a_1^T,\ldots,a_{m+\nu}^T$ are the rows of $A$, $b = (b_1,\ldots,b_{m+\nu})^T$. Let $\bar{A} = (\bar{a}_1^T,\ldots,\bar{a}_{m+\nu}^T)^T$, $\bar{b} = (\bar{b}_1,\ldots,\bar{b}_{m+\nu})^T$ , $\bar{y} \in \mathbb{R}^n$ be such that $\bar{z} = \Phi(\bar{A},\bar{b},\bar{y})$,*

$$\langle \bar{a}_i, \bar{z}\rangle = \bar{b}_i \quad \textit{for} \quad i \in \{1,\ldots,p\} \cup \{m+1,\ldots,m+\nu\},$$

$$\langle \bar{a}_i, \bar{z}\rangle < \bar{b}_i \quad \textit{for} \quad i \in \{p+1,\ldots,m\}$$

*and suppose that the vectors $\{\bar{a}_i, i \in \{1,\ldots,p\} \cup \{m+1,\ldots,m+\nu\}\}$ are linearly independent. Then, $\Phi$ is continuous at $(\bar{A},\bar{b},\bar{y})$.*

*Proof.* To shorten the proof, let us consider the case $\nu = 0$. The extension to $\nu > 0$ is straightforward. By the Kuhn-Tucker conditions there exist $\lambda_1,\ldots,\lambda_p \geq 0$ such that

$$\bar{y} - \bar{z} = \bar{\lambda}_1 \bar{a}_1 + \ldots + \bar{\lambda}_p \bar{a}_p,$$

$$\langle \bar{a}_i, \bar{z}\rangle = \bar{b}_i, \quad i = 1,\ldots,p,$$

$$\langle \bar{a}_i, \bar{z}\rangle < \bar{b}_i, \quad i = p+1,\ldots,m. \tag{56}$$

It follows that $\bar{z}$ is also the projection of $\bar{y}$ on the polyhedron defined by

$$P \equiv \{z \in \mathbb{R}^n \mid \langle \bar{a}_i, z\rangle \leq \bar{b}_i, \quad i = 1,\ldots,p\}.$$

Assume that $(A^k, b^k, y^k) \to (\bar{A},\bar{b},\bar{y})$, and define

$$\bar{P}_k \equiv \{z \in \mathbb{R}^n \mid \langle a_i^k, z\rangle \leq b_i^k, \quad i = 1,\ldots,m\},$$

$$P_k \equiv \{z \in \mathbb{R}^n \mid \langle a_i^k, z\rangle \leq b_i^k, \quad i = 1,\ldots,p\}.$$

Let us call $\bar{z}^k$ the projection of $y^k$ on $\bar{P}_k$. Our objective is to prove that $\bar{z}^k \to \bar{z}$.

Let $z^k$ be the projection of $y^k$ on $P_k$. So, for all $k = 0,1,2,\ldots$ there exist $I_k \subset \{1,\ldots,p\}$, $\lambda_1^k,\ldots,\lambda_p^k \geq 0$ such that

$$y^k - z^k = \sum_{i=1}^{p} \lambda_i^k a_i^k,$$

$$\langle a_i^k, z^k\rangle \leq b_i^k, \quad i = 1,\ldots,p,$$

$$\langle a_i^k, z^k\rangle = b_i^k \text{ for } i \in I_k, \quad \lambda_i^k = 0 \text{ if } i \notin I_k.$$

Clearly, there exist only a finite number of different subsets $I_k$ (say $\{I^1,\ldots,I^q\}$) that are repeated for infinitely many indices $k$. Let us consider the subsequences that correspond to each of them, say $\{y^k, A^k, b^k, z^k\}_{k\in K_j}, j = 1,\ldots,q$. If $I^j \in \{I^1,\ldots,I^q\}$, $K_j \in \{K_1,\ldots,K_q\}$, we have, for all $k \in K_j$,

$$y^k - z^k = \sum_{i\in I^j} \lambda_i^k a_i^k, \quad \langle a_i^k, z^k\rangle = b_i^k \text{ for } i \in I^j, \tag{57}$$

$$\lambda_i^k \geq 0 \text{ for all } i \in I^j, \tag{58}$$

$$\langle a_i^k, z^k\rangle \leq b_i^k \text{ for } i = 1,\ldots,p. \tag{59}$$

Without loss of generality, assume that $I^j = \{1, \ldots, r\}$. Define $A_r^k$ as the matrix whose rows are the first $r$ rows of $A^k$, and $\tilde{b}_r^k = (b_1^k, \ldots, b_r^k)^T$. By (57), $(z^k, \lambda^k)$ solves the $(n+r) \times (n+r)$ linear system

$$\begin{bmatrix} I & (A_r^k)^T \\ A_r^k & 0 \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} y^k \\ \tilde{b}_r^k \end{bmatrix}$$

By the linear independence of the first $p$ rows of $\bar{A}$ and the convergence of $A^k$, we have that, for $k$ large enough, the matrix of this linear system is nonsingular and converges to an invertible matrix. Therefore, there exists $(z, \lambda)$ such that

$$\lim_{k \in K_j} z^k = z \text{ and } \lim_{k \in K_j} \lambda^k = \lambda.$$

Taking limits in (57), (58) and (59), we see that $z$ satisfies the Kuhn-Tucker conditions for the projection of $\bar{y}$ on $P$, so $z = \bar{z}$. Since this reasoning holds for each of the subsequences defined by $K_1, \ldots, K_q$, we have that

$$\lim_{k \to \infty} z^k = \bar{z}. \tag{60}$$

So, by (56), for large enough $k$ we have that

$$\langle a_i^k, z^k \rangle < b_i^k \quad \text{if} \quad i > p. \tag{61}$$

By (57), (58), (59) and (61), $z^k$ is the projection of $y^k$ on $\bar{P}_k$, that is $z^k = \bar{z}^k$. So, by (60), $\bar{z}^k \to \bar{z}$ and the proof is complete. □

In Lemma 6.1 it is proved that, in a neighborhood of a feasible, regular and non-stationary point $x_*$, the directional derivative of the quadratic model of the Lagrangian along $d_{tan}$ is bounded away from zero.

**Lemma 6.1.** *Assume that the infinite sequence $\{x_k\}$ is generated by Algorithm 6.1 and that Assumption A4 holds. Suppose that the subsequence $\{x_k\}_{k \in K_1}$ converges to the feasible and regular point $x_* \in \Omega$, which is not a stationary point of (1). Then, there exist $k_1, c_1 > 0$, such that, for all $x \in \{x_k \mid k \in K_1,\ k \geq k_1\}$,*

$$-\nabla Q(s_{nor}(x, \delta))^T d_{tan}(H, x, \lambda, \delta) \geq c_1. \tag{62}$$

*Moreover, $\|d_{tan}(H, x, \lambda, \delta)\|$ is bounded and bounded away from 0 for $x \in \{x_k \mid k \in K_1,\ k \geq k_1\}$.*

*Proof.* For all $x \in \{x_k\}$ we have that

$$d_{tan}(H, x, \lambda, \delta) = P_x(-\eta \nabla Q(s_{nor}(x, \delta))) = P_x(-\eta[H s_{nor}(x, \delta) + \nabla \ell(x)]).$$

By the contractive property of the orthogonal projection (see, for example, [30]),

$$|P_x(-\eta[H s_{nor}(x, \delta) + \nabla \ell(x)]) - P_x(-\eta \nabla \ell(x))| \leq \eta |H| |s_{nor}(x, \delta)|.$$

So, by Assumption A4, and the boundedness of $|H|$, we have that

$$|d_{tan}(H, x, \lambda, \delta) - P_x(-\eta \nabla \ell(x))| \leq O(\|C(x)\|).$$

Now, by Proposition 1, $P_x(-\eta \nabla \ell(x)) = P_x(-\eta g(x))$. Therefore,

$$|d_{tan}(H, x, \lambda, \delta) - P_x(-\eta g(x))| \leq O(\|C(x)\|). \tag{63}$$

So, by the continuity of $g(x)$ and the convergence of $\{x_k\}_{k\in K_1}$,

$$|g(x)^T P_x(-\eta g(x)) - g(x)^T d_{tan}(H, x, \lambda, \delta)| \le O(\|C(x)\|) \quad \text{for} \quad k \in K_1. \tag{64}$$

Observe that, for all $k \in K_1$, $P_x(-\eta g(x))$ is the solution of

$$\text{Minimize } | - \eta g(x) - z|^2 \text{ subject to } A(x)z = 0, \quad l \le x + s_{nor} + z \le u.$$

Let us define $P_{x_*}(-\eta g(x_*))$ the solution of

$$\text{Minimize } | - \eta g(x_*) - z|^2 \text{ subject to } A(x_*)z = 0, \quad l \le x_* + z \le u. \tag{65}$$

Since $x_*$ is not a stationary point of (1) and $x_*$ is regular, it follows that 0 is not a solution of (65). So, $P_{x_*}(-\eta g(x_*)) \ne 0$. Moreover, since 0 is a feasible point of (65), we have that

$$| - \eta g(x_*) - P_{x_*}(-\eta g(x_*))|^2 < | - \eta g(x_*)|^2,$$

which implies that $g(x_*)^T P_{x_*}(-\eta g(x_*)) < 0$.

By Proposition 2, $P_x(-\eta g(x))$ is a continuous function of $x$ and $s_{nor}$ for all regular $x$. So, there exist $c_1', c_2', c_3' > 0$ and $k' \in I\!N$ such that, for all $x \in \{x_k \mid k \in K_1, k \ge k'\}$,

$$c_1' \le \|P_x(-\eta g(x))\| \le c_2' \quad \text{and} \quad g(x)^T P_x(-\eta g(x)) \le -c_3'. \tag{66}$$

Now, by (63), (64), (66), the continuity of $C(x)$ and the feasibility of $x_*$, there exists $k'' \ge k'$ such that, whenever $x \in \{x_k \mid k \in K_1,\ k \ge k''\}$,

$$\frac{c_1'}{2} \le \|d_{tan}(H, x, \lambda, \delta)\| \le 2c_2' \quad \text{and} \quad g(x)^T d_{tan}(H, x, \lambda, \delta) \le -\frac{c_3'}{2}.$$

So, $\|d_{tan}(H, x, \lambda, \delta)\|$ is bounded and bounded away from zero for $x \in \{x_k \mid k \in K_1,\ k \ge k''\}$. Finally, since $d_{tan} \in \mathcal{N}(A(x))$, by Assumption A4 and the boundedness of $\|d_{tan}\|$ and $|H|$, we have, for $x \in \{x_k \mid k \in K_1,\ k \ge k''\}$,

$$\nabla Q(s_{nor})^T d_{tan} = (H s_{nor} + \nabla \ell(x))^T d_{tan}$$

$$= g(x)^T d_{tan} + d_{tan}^T H s_{nor} \le -\frac{c_3'}{2} + O(\|C(x)\|).$$

So, (62) follows defining $c_1 = c_3'/4$ and $k_1 \ge k''$ such that $O(\|C(x_k)\|) \le c_3'/4$ whenever $k \in K_1,\ k \ge k_1$. $\square$

The following Lemma 6.2 complements Lemma 6.1, showing that, in a neighborhood of a feasible, regular and nonstationary point, the decrease of the quadratic model of the Lagrangian (from $s_{nor}$ to $s_c$) is proportional to the trust region radius $\delta$.

**Lemma 6.2.** *Assume, as in Lemma 6.1, that the infinite sequence $\{x_k\}$ is generated by Algorithm 6.1, Assumption A4 holds and the subsequence $\{x_k\}_{k\in K_1}$ converges to the feasible and regular point $x_* \in \Omega$, which is not a stationary point of (1). Then, there exist $c_2 > 0$, $k_2 \ge k_1$, $\delta' \in (0, \delta_{min})$ such that, when $x \in \{x_k \mid k \in K_1,\ k \ge k_2\}$, the quadratic model of the Lagrangian satisfies*

$$Q(x, s_{nor}(x, \delta)) - Q(x, s_c(x, \delta)) \ge c_2 \min\{\delta, \delta'\}. \tag{67}$$

*Proof.* By Step 4 of Algorithm 6.1, we have:

$$Q(s_{nor}) - Q(s_c) = Q(s_{nor}) - Q(s_{nor} + s_{tan}) \ge 0.9[Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec})]. \tag{68}$$

Now, writing $s_{tan}^{dec} = td_{tan}$, we obtain:

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) = -\frac{1}{2}(s_{tan}^{dec})^T H s_{tan}^{dec} - \nabla Q(s_{nor})^T s_{tan}^{dec}$$
$$= -\frac{1}{2} t^2 d_{tan}^T H d_{tan} - t \nabla Q(s_{nor})^T d_{tan}.$$

Suppose that $d_{tan}^T H d_{tan} \leq 0$. By Lemma 6.1, there exists $k_1 \in I\!N$ such that when $k \in K_1, k \geq k_1$,

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) \geq -t_{sup} \nabla Q(s_{nor})^T d_{tan} \geq t_{sup} c_1 \tag{69}$$

where

$$t_{sup} = \sup \{\tau \geq 0 \mid l \leq x + s_{nor} + \tau d_{tan} \leq u, \text{ and } \|s_{nor} + \tau d_{tan}\| \leq \delta\}. \tag{70}$$

By the definition of $d_{tan}$ in Algorithm 6.1, we have that

$$l \leq x + s_{nor} + d_{tan} \leq u.$$

So, we have that $t_{sup} \geq t_1$, where

$$t_1 = \sup \{\tau \in [0,1] \mid \|s_{nor} + \tau d_{tan}\| \leq \delta\}.$$

Therefore, either $t_1 = 1$ or $\|s_{nor} + t_1 d_{tan}\| = \delta$. In the second case, since $\|s_{nor}\| \leq 0.8\delta$, we have that $t_1 \|d_{tan}\| \geq 0.2\delta$. Therefore, either $t_{sup} \geq 1$ or $t_{sup} \geq 0.2\delta / \|d_{tan}\|$. By Lemma 6.1, there exists $c > 0$ such that $\|d_{tan}\| \leq c$ when $x \in \{x_k \mid k \in K_1, k \geq k_1\}$. Consequently,

$$t_{sup} \geq \min \{1, 0.2\delta/c\} = (0.2/c) \min \{c/0.2, \delta\}.$$

So, by (69), we have that

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) \geq 0.2(c_1/c) \min \{c/0.2, \delta\} \tag{71}$$

when $x \in \{x_k \mid k \in K_1, k \geq k_1\}$.

If $d_{tan}^T H d_{tan} > 0$, and $s_{tan}^{dec} = t_{sup} d_{tan}$, where $t_{sup}$ is defined as in (70), an elementary property of one-dimensional quadratics shows that

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) \geq -\frac{t_{sup}}{2} \nabla Q(s_{nor})^T d_{tan}.$$

Thus, following the same arguments as in the case $d_{tan}^T H d_{tan} \leq 0$, we prove that

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) \geq 0.1(c_1/c) \min \{c/0.2, \delta\} \tag{72}$$

when $x \in \{x_k \mid k \in K_1, k \geq k_1\}$.

Finally, we only need to consider the case where

$$d_{tan}^T H d_{tan} > 0, \quad s_{tan}^{dec} = td_{tan} \text{ and } t < t_{sup}.$$

In this case, $t$ is the unconstrained minimizer of $Q(s_{nor} + td_{tan})$. So,

$$t = -\frac{\nabla Q(s_{nor})^T d_{tan}}{d_{tan}^T H d_{tan}},$$

and

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) = -\frac{1}{2} t \nabla Q(s_{nor})^T d_{tan}.$$

Now, by Lemma 6.1, $\|d_{tan}\|$ and $\|H\|$ are bounded for $x \in \{x_k \mid k \in K_1, k \geq k_1\}$. So,

$$t \geq c_1/c',$$

where $c'$ is a uniform bound of $d_{tan}^T H d_{tan}$. Therefore, in this case, we have that

$$Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) \geq c_1^2/(2c'). \tag{73}$$

By (71), (72) and (73), we obtain that, when $x \in \{x_k \mid k \in K_1, k \geq k_1\}$,

$$\begin{aligned} Q(s_{nor}) - Q(s_{nor} + s_{tan}^{dec}) &\geq 0.1(c_1/c) \min\{c/0.2,\ 5cc_1/c',\ \delta\} \\ &= 0.1c_1/c \min\{\min\{c/0.2,\ 5cc_1/c'\}, \delta\}. \end{aligned}$$

Therefore, by (68), the desired result follows taking $k_2 \geq k_1$,

$$c_2 = 0.09c_1/c \quad \text{and} \quad \delta' = \min\{\delta_{min}, 5c, 5cc_1/c'\}.$$

□

Lemma 6.3 concerns the decrease of the quadratic model of the Lagrangian, not from $s_{nor}$ to $s_c$ (as in Lemma 6.2) but from 0 to $s_c$. In fact, in Lemma 6.2 it was proved that the decrease from $s_{nor}$ to $s_c$ is proportional to $\delta$. Now, it must be taken into account that an increase from 0 to $s_{nor}$ can occur. The idea is to prove that, when $\|C(x)\|$ is less than some multiple of $\delta$, that possible increase is negligible and, so, the decrease on the quadratic model of the Lagrangian is proportional to $\delta$, even considering the terms that depend on $\Delta\lambda$. As a consequence, since the first term of the convex combination that defines $Pred$ is sufficiently positive, it will not be necessary to decrease the penalty parameter within this region.

**Lemma 6.3.** *Assume that the hypotheses of Lemmas 6.1 and 6.2 hold. Then, there exist $\alpha, c_3, k_3 > 0$ such that, when $x \in \{x_k \mid k \in K_1,\ k \geq k_3\}$, and $\|C(x)\| \leq \alpha\delta$,*

$$[Q(x,0) - Q(x, s_c(x,\delta))] - (A(x)s_c(x,\delta) + C(x))^T \Delta\lambda \geq c_3 \min\{\delta, \delta'\}$$

*and*

$$\theta^{sup}(x,\delta) = 1,$$

*where $\theta^{sup}(x,\delta)$ is defined in Algorithm 5.1 and $\delta'$ is defined in Lemma 6.2.*

*Proof.* By Lemma 6.2, Assumption A4, the continuity of $C(x)$, the convergence of $\{x_k\}_{k\in K_1}$ and the boundedness of $\|\lambda_k\|$ and $\|\Delta\lambda_k\|$, we have that

$$\begin{aligned} &[Q(0) - Q(s_c)] - (As_c + C(x))^T \Delta\lambda \\ &\geq Q(s_{nor}) - Q(s_c) - |Q(0) - Q(s_{nor})| - |As_c + C(x)||\Delta\lambda| \\ &\geq Q(s_{nor}) - Q(s_c) - |Q(0) - Q(s_{nor})| - |C(x)||\Delta\lambda| \\ &\geq c_2 \min\{\delta, \delta'\} - O(\|C(x)\|), \end{aligned} \tag{74}$$

if $x \in \{x_k \mid k \in K_1,\ k \geq k_2\}$, where $c_2$ and $\delta'$ are defined in Lemma 6.2. Clearly, the first part of the thesis follows from this inequality by means of an adequate choice of $\alpha$. But, by Assumption A4, $M(0) - M(s_c) = M(0) - M(s_{nor}) \leq O(\|C(x)\|)$, so

$$Pred(x,1,\delta) - 0.5(M(x,0) - M(x,s_c)) \geq c_3 \min\{\delta, \delta'\} - O(\|C(x)\|).$$

Therefore, choosing $\alpha$ in a proper way, we prove the second part of the thesis. $\square$

In Lemma 6.4, an additional consequence of the existence of a feasible, regular and non-stationary limit point of $\{x_k\}$ is derived. Roughly speaking, if $\theta_k$ is bounded away from zero, it is possible to construct a sequence of actual reductions that is also bounded away from zero. This causes a violation of the compactness assumption. Therefore, under the assumptions of Lemmas 6.1–6.3, it can be concluded that $\theta_k \to 0$.

**Lemma 6.4.** *Assume that the hypotheses of Lemmas 6.1–6.3 and, in addition, that Assumptions A1, A2 and A3 hold. Then*

$$\lim_{k\to\infty} \theta_k = 0.$$

*Proof.* Suppose, by contradiction, that the sequence $\{\theta_k\}$ does not converge to 0. Since, by Lemma 5.4, $\{\theta_k\}$ is convergent, there exists $k_4 \geq k_3, \hat{\theta} > 0$ such that

$$\theta_k \geq \hat{\theta}$$

for $k \geq k_4$. Assume now that $x \in \{x_k \mid k \in K_1,\ k \geq k_4\}$. Since $M(x,0) - M(x,s_c) \geq 0$, we have that

$$Pred(x,\theta,\delta) \geq \theta[Q(x,0) - Q(x,s_c(x,\delta)) - (A(x)s_c(x,\delta) + C(x))^T \Delta\lambda)].$$

So, by Lemma 6.2, Assumption A4 and the continuity of $C(x)$ and $A(x)$, if $x \in \{x_k \mid k \in K_1,\ k \geq k_4\}$ we have, as in (74), that

$$Pred(x,\theta,\delta) \geq \theta c_2 \min\{\delta,\delta'\} - O(\|C(x)\|).$$

Now, by (48)–(50), the sequence of penalty parameters $\theta$ effectively used to test the acceptance criterion (30) within a single iteration is nonincreasing. So, for each $(\theta,\delta)$ effectively used to test (30) at the iteration which corresponds to $x$, we have that

$$Pred(x,\theta,\delta) \geq \hat{\theta} c_2 \min\{\delta,\delta'\} - O(\|C(x)\|). \tag{75}$$

On the other hand, by the bounded variation Assumption A2, and the boundedness of $H$,

$$|Ared(x,\theta,\delta) - Pred(x,\theta,\delta)| \leq O(\delta^2).$$

Therefore, there exists $\tilde{\delta} \in (0,\delta') \subset (0,\delta_{min})$ such that, whenever $\delta \in (0,\tilde{\delta})$ and $x \in \{x_k \mid k \in K_1,\ k \geq k_4\}$,

$$|Ared(x,\theta,\delta) - Pred(x,\theta,\delta)| \leq \hat{\theta} c_2 \tilde{\delta}/40.$$

Let $k_5 \geq k_4$ be such that for all $x \in \{x_k \mid k \in K_1, k \geq k_5\}$,

$$O(\|C(x)\|) \leq c_2 \hat{\theta} \tilde{\delta}/20,$$

where $O(\|C(x)\|)$ is the function considered in (75). If $x \in \{x_k \mid k \in K_1,\ k \geq k_5\}$ we have that, for all $(\theta,\delta)$ effectively used to test (30) at the iteration which corresponds to $x$,

$$Pred(x,\theta,\delta) \geq \hat{\theta} c_2 \min\{\delta,\delta'\} - \hat{\theta} c_2 \tilde{\delta}/20.$$

If, in addition, $\delta \in [\tilde{\delta}/10, \tilde{\delta})$,

$$Pred(x,\theta,\delta) \geq \hat{\theta} c_2 \tilde{\delta}/10 - \hat{\theta} c_2 \tilde{\delta}/20 = \hat{\theta} c_2 \tilde{\delta}/20.$$

So, for all $\delta \in [\tilde{\delta}/10, \tilde{\delta})$, $x \in \{x_k \mid k \in K_1,\ k \geq k_5\}$,

$$\frac{|Ared(x,\theta,\delta) - Pred(x,\theta,\delta)|}{Pred(x,\theta,\delta)} \leq 0.5.$$

This means that for some $\delta \in [\tilde{\delta}/10, \tilde{\delta})$ the stepsize must be accepted at Step 7 of Algorithm 3.1. Therefore $\delta_k$ is bounded away from 0 for $k \in K_1, k \geq k_5$. So,

$$Ared(x_k, \theta_k, \delta_k) \geq 0.1 Pred(x_k, \theta_k, \delta_k)$$

and, by (75), the right-hand side of the former inequality is bounded away from 0 for $k \in K_1, k \geq k_5$. Therefore, by the argument used in Theorem 5.5 and the convergence of $\{\theta_k\}$, it follows that the merit function $\psi(x_k, \lambda_k, \theta_k)$ is unbounded. This contradicts the compactness Assumption A3. Thus, the lemma is proved. □

At this point, it is useful to visualize the plane $(\|C(x)\|, \delta)$ as divided by the line $\|C(x)\| = \alpha\delta$. The pairs $(\|C(x)\|, \delta)$ such that $\|C(x)\| \leq \alpha\delta$, can be thought as belonging to the "good zone" of the plane. In Lemma 6.3 it was shown that, within this zone, the first term of the convex combination $Pred$ is positive and proportional to $\delta$. Moreover, it was proved that when $\|C(x)\| \leq \alpha\delta$, the corresponding $\theta^{sup}$ is equal to 1, therefore there is no need for the penalty parameter to be decreased. In Lemma 6.5 it will be proved that, although it is possible that the penalty parameter needs to be decreased "in the bad zone", it will always remain larger than a multiple of $\delta$. To prove that property, an additional assumption on the decrease $M(x,0) - M(x, s_{nor})$ is necessary.

**A5. Second algorithmic assumption.** *For each feasible and regular $x_* \in \Omega$, there exist $c_4, \varepsilon_0 > 0$ such that, whenever $\|C(x)\| \geq \alpha\delta$,*

$$[M(x,0) - M(x, s_{nor}(x,\delta))] \geq c_4 \delta \|C(x)\|, \tag{76}$$

*if Algorithm 6.1 is applied to $x$, and $\|x - x_*\| \leq \varepsilon_0$.*

**Remark on the choice of the normal step.**

Before continuing the convergence proof, let us show that a choice of $s_{nor}$ that satisfies Assumptions A4 and A5 is possible. In a neighborhood of a feasible and regular $x_* \in \Omega$, there exists $v_{nor}(x) \in I\!R^n$ such that

$$\|v_{nor}(x)\| \leq O(\|C(x)\|), \quad l \leq x + v_{nor}(x) \leq u$$

and

$$A(x) v_{nor}(x) + C(x) = 0. \tag{77}$$

(A possible choice of $v_{nor}(x)$ comes from selecting $m$ "basic" variables $[x]_i$ such that $l_i < [x]_i < u_i$ and the corresponding columns of $A(x)$ form a nonsingular $m \times m$ matrix $B(x)$. This is possible at $x_*$ due to the regularity assumption, and in a neighborhood of $x_*$ by continuity. The vector $v_{nor}(x)$ is formed by the components of $-B(x)^{-1}C(x)$ at the proper $m$ positions with zeroes at the remaining $n - m$ positions. Of course, other choices are possible.) Then, $s_{nor}(x,\delta)$ is chosen according to the following rules:

(a) If $\|v_{nor}(x)\| \leq 0.8\delta$, take $s_{nor}(x,\delta) = v_{nor}(x)$. In this case, Assumption A5 follows from (77), since $M(x,0) - M(x, v_{nor}(x)) = (1/2)|C(x)|^2$.

(b) If $\|v_{nor}(x)\| > 0.8\delta$, define

$$\tilde{v}_{nor}(x,\delta) = \frac{0.8\delta}{\|v_{nor}(x)\|} v_{nor}(x)$$

and choose

$$s_{nor}(x,\delta)) = \tilde{v}_{nor}(x,\delta)) \quad \text{if} \quad M(x,\tilde{v}_{nor}(x,\delta)) \le M(x, s_{nor}^{dec}(x,\delta)), \tag{78}$$

and

$$s_{nor}(x,\delta)) = s_{nor}^{dec}(x,\delta)) \quad \text{otherwise}. \tag{79}$$

If we compute $d_{nor}(x)$ by (37), Assumption A4 is satisfied by this choice. Observe that

$$M(x,0) - M(x,\tilde{v}_{nor}(x,\delta)) \ge \frac{0.8\delta}{\|v_{nor}(x)\|} M(x,0) = \frac{0.4\delta}{\|v_{nor}(x)\|} |C(x)|^2.$$

But $\|v_{nor}(x)\| \le O(\|C(x)\|)$, so

$$M(x,0) - M(x,\tilde{v}_{nor}(x,\delta)) \ge 0.4\delta O(|C(x)|).$$

Therefore, Assumption A5 follows from (78)-(79).

**Lemma 6.5.** *Assume that the hypotheses of Lemmas 6.1-6.3 and Assumption A5 hold. If $x \in \{x_k \mid k \in K_1, k \ge k_3\}$ and $\|C(x)\| \ge \alpha\delta$, then $\delta/\theta^{sup}(x,\delta)$ is uniformly bounded.*

*Proof.* When $\theta^{sup}(x,\delta) \ne 1$, a trivial calculation shows that

$$\theta^{sup}(x,\delta) = \frac{M(0) - M(s_{nor})}{2[M(0) - M(s_{nor}) - Q(0) + Q(s_c) + (As_c + C(x))^T \Delta\lambda]}.$$

So, by Assumptions A4 and A5, if $x \in \{x_k \mid k \ge k_3, k \in K_1\}$ and $\|x - x_*\| \le \varepsilon_0$, we have that

$$\frac{1}{2\theta^{sup}(x,\delta)} = 1 + \frac{Q(s_c) - Q(s_{nor})}{M(0) - M(s_{nor})} + \frac{Q(s_{nor}) - Q(0) + [As_c + C(x)]^T \Delta\lambda}{M(0) - M(s_{nor})}$$

$$\le 1 + \frac{|Q(0) - Q(s_{nor}) - (As_c + C(x))^T \Delta\lambda|}{M(0) - M(s_{nor})} \le 1 + \frac{O(\|C(x)\|)}{c_4 \|C(x)\| \delta} \le 1 + O(1/\delta).$$

Therefore, $\delta/\theta^{sup}(x,\delta)$ is bounded. □

The following lemma is crucial. In Lemma 6.3, we proved that $\theta$ does not need to be decreased in the "good zone". However, it could be possible that many iterations enter into the bad zone, through successive decreases of $\delta$. In Lemma 6.6 we prove that, at most we enter into the bad zone only once at each iteration. That is, for $k$ large enough, the first trust region radius $\delta$ tried within the bad zone will necessarily be accepted. For this reason, $\theta$ will not be excessively decreased. The proof of the lemma requires a careful analysis of the first-order approximations of $C$ and $f$.

**Lemma 6.6.** *Assume that the hypotheses of Lemmas 6.1–6.3 and Assumptions A2 and A5 hold. Then, there exist $k_6 \ge k_3$, $\tilde{\theta} \in (0,1]$ such that, when $x \in \{x_k \mid k \in K_1,\ k \ge k_6\}$, $\|C(x)\| \ge \alpha\delta$ and $\theta \le \tilde{\theta}$ satisfies (29), then*

$$Ared(x,\theta,\delta) \ge 0.1 Pred(x,\theta,\delta).$$

*Proof.* By the bounded variation Assumption A2, we have that

$$Ared(x,\theta,\delta) = \theta[\ell(x,\lambda) - \ell(x + s_c(\delta), \lambda + \Delta\lambda)] + (1-\theta)[\varphi(x) - \varphi(x + s_c(\delta))]$$

$$= \theta[Q(x,0) - Q(x, s_c(\delta) - (C(x) + A(x)s_c(\delta))^T \Delta\lambda + O(\delta^2)]$$

$$+(1-\theta)[|C(x)|^2 - |C(x+s_c(\delta))|^2]/2.$$

But, by (46),

$$|C(x)|^2 - |C(x+s_c)|^2 = -s_c^T A(x)^T A(x) s_c - 2 s_c^T A(x)^T C(x) + \|C(x)\| O(\delta^2) + O(\delta^3)$$
$$= 2(M(x,0) - M(x,s_c)) + \|C(x)\| O(\delta^2) + O(\delta^3).$$

So,

$$\begin{aligned} &Ared(x,\theta,\delta) \\ &= \theta[Q(x,0) - Q(x,s_c(\delta)) - (C(x) + A s_c(\delta))^T \Delta\lambda + O(\delta^2)] \\ &+(1-\theta)[M(x,0) - M(x,s_c(\delta)) + \|C(x)\| O(\delta^2) + O(\delta^3)] \\ &= Pred(x,\theta,\delta) + \theta O(\delta^2) + (1-\theta)[\|C(x)\| O(\delta^2) + O(\delta^3)]. \end{aligned} \tag{80}$$

Therefore, when $\|C(x)\| \geq \alpha\delta$,

$$|Ared(x,\theta,\delta) - Pred(x,\theta,\delta)| \leq \theta \|C(x)\| O(\delta) + \|C(x)\| O(\delta^2). \tag{81}$$

But, since $\theta$ satisfies (29), Assumption A5 implies that, for large enough $k \in K_1$,

$$Pred(x,\theta,\delta) \geq 0.5[M(x,0) - M(x,s_c)] \geq \frac{c_4}{2} \|C(x)\| \delta.$$

So, $\delta \|C(x)\| / Pred(x,\theta,\delta)$ is uniformly bounded. Therefore, dividing both terms of (81) by $Pred(x,\theta,\delta)$, we have that

$$\left| \frac{Ared(x,\theta,\delta)}{Pred(x,\theta,\delta)} - 1 \right| \leq O(\theta) + O(\delta) \leq O(\theta) + O(\|C(x)\|/\alpha).$$

The desired result follows from this inequality. □

In the following Lemma it is proved that, if all the limit points of the sequence generated by the algorithm are feasible and regular, one of them is necessarily stationary. Observe that the previous results on this section were obtained under the assumption that there exists a subsequence that converges to a feasible, regular and non-stationary point. The technique used in Lemma 6.7 consists of showing that, when those hypotheses hold, an additional sequence that necessarily converge to a stationary point can be constructed.

**Lemma 6.7.** *Let $\{x_k\}$ be an infinite sequence generated by Algorithm 6.1. Assume that all the limit points of $\{x_k\}$ are feasible and regular and that Assumptions A1–A5 hold. Then there exists a limit point of the sequence $\{x_k\}$ that is a stationary point of (1).*

*Proof.* By the Compactness Assumption A3, there exists a convergent subsequence $\{x_k\}_{k \in K_0}$. If the limit of this subsequence is a stationary point of (1), we are done. Otherwise, the hypotheses of Lemma 6.4 hold (with $K_1 = K_0$), so $\lim_{k\to\infty} \theta_k = 0$. Therefore, again by Assumption A3, there exists $k_7 \in I\!N$ such that $1 + N/(k_7+1)^{1.1} \leq 2$ ($N$ is defined in (47)) and $K_1$, an infinite subset of $\{k_7, k_7+1, k_7+2, \ldots\}$ such that $\{x_k\}_{k \in K_1}$ is convergent (say, $\lim_{k \in K_1} x_k = x_*$),

$$\lim_{k \in K_1} \theta^{sup}(x_k, \underline{\delta}_k) = 0 \tag{82}$$

and, by (48)–(50),

$$\theta^{sup}(x_k, \underline{\delta}_k) \leq \theta'(x_k, \underline{\delta}_k) \leq \theta_k^{large} \tag{83}$$

for all $k \in K_1$, where $\underline{\delta}_k$ is one of the trust region radii tested at iteration $k$. In fact, the second inequality of (83) always holds by (48)–(49), while the first must hold for an infinite set of indices (otherwise, by (50), $\theta^{sup}$ could not tend to 0).

By the hypothesis of the lemma, $x_*$ is regular. Again, if $x_*$ is a stationary point of (1) the result is proved. Otherwise, the hypotheses of Lemmas 6.1–6.6 hold for $\{x_k\}_{k \in K_1}$. Without loss of generality, we can asume that $k \geq k_6 \geq k_3$,

$$\theta^{sup}(x_k, \underline{\delta}_k) \leq \tilde{\theta}/2 < 1 \quad \text{and} \quad \theta_k \leq \tilde{\theta}/2 \tag{84}$$

for all $k \in K_1$, where $k_3$ is defined in Lemma 6.3 and $k_6$ is defined in Lemma 6.6. By Lemma 6.3, $\theta^{sup}(x_k, \delta) = 1$ whenever $k \in K_1$ and $\|C(x_k)\| \leq \alpha\delta$. So, by (82) and (84),

$$\|C(x_k)\| > \alpha\underline{\delta}_k \tag{85}$$

for all $k \in K_1$. Therefore, since $\|C(x_k)\| \to 0$,

$$\lim_{k \in K_1} \underline{\delta}_k = 0. \tag{86}$$

Assume, without loss of generality, that

$$\underline{\delta}_k \leq 0.1\delta' < 0.1\delta_{min} \tag{87}$$

for all $k \in K_1$, where $\delta'$ is defined in Lemma 6.2 and used in Lemma 6.3. By (87), $\underline{\delta}_k$ cannot be the first trust region radius tried at iteration $k$. Let us call $\hat{\delta}_k$ the trust region radius tried at iteration $k$ immediately before $\underline{\delta}_k$. Let us call $\hat{\theta}_k$ the penalty parameter associated to that rejected step. Since $k_7 \geq N$, by (84) and the choice of the penalty parameter at Algorithm 5.1, we have that $\hat{\theta}_k \leq \tilde{\theta}$ for all $k \in K_1$. Therefore, Lemma 6.6 can be applied, giving

$$\|C(x_k)\| < \alpha\hat{\delta}_k \text{ for all } k \in K_1. \tag{88}$$

Moreover, since $\underline{\delta}_k \in [0.1\hat{\delta}_k, 0.9\hat{\delta}_k]$, (87) implies that

$$\hat{\delta}_k \leq 10\underline{\delta}_k \leq \delta' < \delta_{min} \text{ for all } k \in K_1. \tag{89}$$

Now, by (48)–(50), (82) and Lemma 6.3, we have

$$\begin{aligned} \hat{\theta}_k =& \ \min\,\{\theta'(x_k, \hat{\delta}_k), \theta^{sup}(x_k, \hat{\delta}_k)\} = \theta'(x_k, \hat{\delta}_k) \\ \geq& \ \min\,\{\theta'(x_k\hat{\delta}_k), \theta^{sup}(x_k, \underline{\delta}_k)\} = \theta^{sup}(x_k, \underline{\delta}_k) \end{aligned} \tag{90}$$

for all $k \in K_1$. Now, by Lemma 6.3, (89) and the definition of $Pred$,

$$Pred(x_k, \hat{\theta}_k, \hat{\delta}_k) \geq \hat{\theta}_k c_3 \hat{\delta}_k. \tag{91}$$

Moreover, repeating the arguments that lead to (80), we obtain

$$|Ared(x_k, \hat{\theta}_k, \hat{\delta}_k) - Pred(x_k, \hat{\theta}_k, \hat{\delta}_k)| \leq \hat{\theta}_k O(\hat{\delta}_k^2) + (1 - \hat{\theta}_k)[\|C(x_k)\| O(\hat{\delta}_k^2) + O(\hat{\delta}_k^3)] \tag{92}$$

for all $k \in K_1$. So, by (91) and (92),

$$\frac{|Ared(x_k, \hat{\theta}_k, \hat{\delta}_k) - Pred(x_k, \hat{\theta}_k, \hat{\delta}_k)|}{Pred(x_k, \hat{\theta}_k, \hat{\delta}_k)} \leq \frac{|Ared(x_k, \hat{\theta}_k, \hat{\delta}_k) - Pred(x_k, \hat{\theta}_k, \hat{\delta}_k)|/\hat{\theta}_k}{c_3 \hat{\delta}_k}$$

$$\leq \frac{O(\hat{\delta}_k^2) + \frac{1-\hat{\theta}_k}{\hat{\theta}_k}[\|C(x_k)\|O(\hat{\delta}_k^2) + O(\hat{\delta}_k^3)]}{c_3\hat{\delta}_k}$$

$$= O(\hat{\delta}_k) + (1-\hat{\theta}_k)[\|C(x_k)\|O(\hat{\delta}_k) + O(\hat{\delta}_k^2)]/\hat{\theta}_k \tag{93}$$

for all $k \in K_1$.

Now, by Lemma 6.5 and (85), $\underline{\delta}_k/\theta^{sup}(x_k, \underline{\delta}_k)$ is bounded for $k \in K_1$. So, by (89) and (90), $O(\hat{\delta}_k)/\hat{\theta}_k$ is also bounded for $k \in K_1$. Finally, by (86), (89) and the feasibility of $x_*$, the right-hand side of (93) tends to 0 for $k \in K_1$. This implies that for $k$ large enough, the acceptance criterion (30) is satisfied with the trust region radius $\hat{\delta}_k$. This is a contradiction, since we had assumed that the trust region radius $\hat{\delta}_k$ was rejected. □

Theorem 6.8 condenses the results proved in Sections 5 and 6.

**Theorem 6.8.** *Let $\{x_k\}$ be an infinite sequence generated by Algorithm 6.1. Suppose that Assumptions A1–A5 hold. Then, all the limit points of $\{x_k\}$ are $\varphi-$stationary. Moreover, if all the limit points of $\{x_k\}$ are feasible and regular, there exists a limit point $x_*$ that is a stationary point of the nonlinear programming problem (1). In particular, if all the $\varphi-$stationary points are feasible and regular, there exists a subsequence of $\{x_k\}$ that converges to a feasible, regular and stationary point of (1).*

*Proof.* This result follows directly from Theorem 5.5 and Lemma 6.7. □

# 7 Numerical experiments

In this section, a specific implementation of Algorithm 6.1 is described. A FORTRAN (double precision arithmetic) code was written corresponding to this implementation. No claim whatsoever is made here with regards taking the best decisions with respect to all the degrees of freedom that characterize the main model algorithm. However, it will be interesting to show that even a preliminary numerical implementation produces reliable results in a number of test problems. Some parameters ($\delta_{min}$, $L_1$, $\gamma$, $\eta$) of the main model algorithm are dimensional and should be chosen, in practical cases, according to the scaling of the problem. However, the same values of these parameters were used in all of the tests, in order to establish a common basis for comparison. The parameters and specific procedures used are given below.

(i) If some Lagrange multiplier estimate $\lambda$ is such that $\|\lambda\|_\infty > L_1$, it is projected on the box $\|\lambda\|_\infty \leq L_1$, where $L_1 = 10^4$.

(ii) The lower bound for the trust region radius is chosen as $\delta_{min} = 10^{-4}$. The initial $\delta$ at the first iteration of the algorithm is $10^3$.

(iii) The vector $d_{nor}$ is chosen as in (37) with $\gamma = 10^{-3}$. The "normal step" $s_{nor}$ is computed as an approximate minimizer of $M(x, s)$ on the box $l \leq x + s \leq u$, $\|s\|_\infty \leq 0.8\delta$. For this computation the algorithm described in [11], with modifications introduced in [1], is used. In addition to (25), a convergence criterion for this procedure is that the projected gradient at the final point should be less than a fraction of $10^{-4}$ of the projected gradient at $s = 0$. Preconditioners are not used as auxiliary tools for the conjugate gradient iterations used in this algorithm. However, systems with less than 100 variables are solved using (sparse) Cholesky factorizations.

(iv) The well-known software MINOS 5.4 [18] is used as an active-set quadratic programming solver to find the tangent step $s_{tan}$ by means of the approximate minimization of $Q(s_{nor} + s)$ subject to the constraints (27). $\Delta\lambda$ is chosen as the vector of estimates of multipliers given by

MINOS for this quadratic program. The solution found (allowing a maximum of 200 MINOS-iterations) is accepted as tangent step when the inequality (67) holds with $c_2 = 10^{-6}, \delta' = 1$. Only in the very improbable case that this inequality is not satisfied is it necessary to proceed to the computation of $d_{tan}$ as in (55) with $\eta = 10^{-3}$. The computation of $d_{tan}$ involves an easy strictly convex quadratic programming problem which can also be solved using MINOS 5.4. Obviously, this algorithmic trick does not affect the convergence proofs.

(v) It can be assumed that, for all $x \in \mathbb{R}^n$ there exists a sequence $x^j \to x$ such that $\{\nabla^2 f(x^j)\}$ and $\{\nabla^2 C_i(x^j)\}$ exist and are convergent for all $i = 1, \ldots, m$. Consequently, given the current point $x$, the Hessian Lagrangian approximations are chosen as

$$H = \lim_{j \to \infty} [\nabla^2 f(x^j) + \sum_{i=1}^{m} \lambda_i \nabla^2 C_i(x^j)]. \tag{94}$$

Of course, when $x$ is a point at which second derivatives exist, $H$ is the true Hessian of the Lagrangian.

(vi) The trust region radius is updated according to rules similar than those described in [6], with the safeguards $\delta \geq \delta_{min}$ at the beginning of the iteration and "new $\delta$"$\in [0.1\delta, 0.9\delta]$ when (30) is not satisfied.

(vii) After the computation of $s_{nor}$ for a given $\delta$, (54) and (76) are controlled in the following way. If $\|C(x_k)\|_\infty \leq 0.1$, the inequalities

$$\|s_{nor}(x,\delta)\|_\infty \leq 10^4 \|C(x_k)\|_\infty \quad \text{and} \quad M(x,0) - M(x, s_{nor}) \geq 10^{-6} \min\{\delta, 1\} \, \|C(x_k)\|_\infty \tag{95}$$

are tested. If the inequalities (95) do not hold, $s_{nor}$ is computed as in the Remark that follows Assumption A5. In all of the performed tests this change was never necessary.

(viii) The intermediate penalty parameter $\theta'$ is chosen using (48) when $\delta \geq \delta_{min}$, and using (49) when $\delta < \delta_{min}$. This apparent modification of Algorithm 5.1 was motivated by preliminary numerical experiments. However, it is easy to observe that it does not represent a real alteration of the model algorithm, since it can always be interpreted that the first trust-region radius tried at iteration $k$ is the last one which is greater than $\delta_{min}$. (The only constraint for the initial trust-region radius at each iteration is that it must be greater than $\delta_{min}$.) So, only when the trust-region radius is less than $\delta_{min}$, it is necessary to decrease the penalty parameter, as required by the convergence theory.

(viii) The algorithm terminates when

a) $\|x_{k+1} - x_k\|_\infty \leq \max \{10^{-6}, 10^{-6}\|x_k\|_\infty\}$ or

b) $\|C(x_k)\|_\infty \leq 10^{-6}$ and $|Q_k(0) - Q_k(s_c) - \Delta\lambda_k^T(A(x_k)s_c + C(x_k))| \leq 10^{-8}$.

## 7.1 Experiments with $C^2$-problems

The first set of test problems (solved using a SUN SPARC-station 2+) was taken from the data base CUTE [2]. A total of 45 constrained optimization problems were selected, where the analytic second derivatives of the functions are available. The selection of problems was made trying to cover from small to relatively large problems, with a wide scope of non-linearity. The initial points and initial estimates of Lagrange multipliers were the ones given by CUTE.

For comparison, the test problems were also solved by the well-known code LANCELOT [6], with the option BANDSOLVER-PRECONDITIONER-CG-SOLVER (semibandwidth=5).

The numerical results are given in Table 1. The following abbreviations are used

$m$ : number of constraints.

$n$ : number of variables.

$NF$ : number of evaluations of $f$ (also, number of iterations of LANCELOT).

$NG$ : number of gradient evaluations (also, number of iterations of the new algorithm).

$ET$ : execution time in seconds.

$N$ : degree of nonmonotonicity of the penalty parameter $\theta$, according to (47).

In preliminary experiments, the new algorithm was tested using the following values for $N$ (see Algorithm 5.1): $0, 10^2, 10^4, 10^6$. When $N = 0$ the strategy for $\theta$ is monotone, which corresponds to a monotone increase of the classical penalty parameter $\rho$. When $N$ grows, larger values of $\theta$ have the chance of being accepted in the test (29). In the table, only the results for $N = 0$ and $N = 10^6$ are shown.

| Problem | | | LANCELOT | | New ($N = 0$) | | New ($N = 10^6$) | |
|---|---|---|---|---|---|---|---|---|
| name | m | n | NF , NG | ET | NF , NG | ET | NF , NG | ET |
| AUG2D | 100 | 220 | 18 , 19 | 4.64 | 3 , 3 | 6.15 | 3 , 3 | 6.02 |
| CATENARY | 32 | 99 | 103 , 92 | 10.07 | 153 , 136 | 218.16 | 125 , 102 | 150.86 |
| CLNLBEAM | 200 | 303 | 129 , 125 | 133.05 | 6 , 6 | 34.47 | 6 , 6 | 34.50 |
| CORKSCRW | 350 | 506 | 185 , 183 | 2067.50 | 30 , 26 | 437.70 | 32 , 26 | 426.33 |
| COSHFUN | 20 | 81 | 292 , 242 | 21.72 | 51 , 27 | 14.63 | 39 , 27 | 10.01 |
| DISC2 | 23 | 35 | 97 , 80 | 2.92 | (**) 74 , 56 | 4.92 | (**) 74 , 56 | 4.75 |
| DIXCHLNV | 50 | 100 | 192 , 193 | 82.75 | 15 , 14 | 168.69 | 15 , 14 | 142.38 |
| DTOC1NA | 196 | 298 | 11 , 12 | 5.14 | 12 , 12 | 249.72 | 12 , 12 | 157.70 |
| DTOC2 | 396 | 598 | 36 , 37 | 28.26 | 7 , 6 | 322.76 | 7 , 6 | 199.31 |
| DTOC4 | 998 | 1499 | 18 , 19 | 24.34 | 5 , 5 | 891.43 | 5 , 5 | 706.78 |
| EIGENACO | 55 | 110 | 26 , 26 | 5.11 | 2 , 1 | 0.38 | 2 , 1 | 0.39 |
| EIGMAXA | 101 | 101 | 7 , 8 | 1.08 | 5 , 4 | 1.03 | 5 , 4 | 1.03 |
| EXPFITC | 502 | 507 | 85 , 86 | 75.05 | 14 , 14 | 23.54 | 14 , 14 | 23.67 |
| HAGER1 | 100 | 201 | 6 , 7 | 0.96 | 2 , 1 | 4.52 | 2 , 1 | 4.44 |
| HAGER2 | 100 | 201 | 8 , 9 | 1.38 | 2 , 1 | 4.12 | 2 , 1 | 4.12 |
| HALDMADS | 42 | 48 | 67 , 63 | 4.61 | 19 , 14 | 2.91 | 12 , 10 | 1.42 |
| HIMMELBI | 12 | 112 | 42 , 43 | 7.69 | 10 , 9 | 4.96 | 10 , 9 | 4.96 |
| HIMMELBK | 14 | 24 | 200 , 168 | 12.42 | 6 , 6 | 0.97 | 6 , 6 | 0.98 |
| HS108 | 13 | 22 | 26 , 23 | 0.58 | 20 , 14 | 0.94 | 18 , 14 | 0.83 |
| HS114 | 11 | 18 | 893 , 773 | 9.41 | 12 , 10 | 0.49 | 12 , 10 | 0.48 |
| HS116 | 14 | 27 | 2606 , 2327 | 67.57 | 245 , 177 | 11.60 | 52 , 44 | 3.32 |
| HVYCRASH | 150 | 204 | 256 , 223 | 48.03 | 53 , 39 | 217.89 | 52 , 39 | 230.20 |
| HYDROELL | 1008 | 2017 | 136 , 137 | 1262.44 | 11 , 10 | 96.37 | 11 , 10 | 96.14 |
| LAUNCH | 28 | 44 | (*) 3000 , 2971 | 682.99 | 3 , 1 | 0.45 | 3 , 1 | 0.48 |
| LEAKNET | 153 | 156 | 1097 , 1088 | 1538.48 | 164 , 164 | 97.42 | 164 , 164 | 97.42 |
| MADSSCHJ | 98 | 149 | 1221 , 1052 | 1081.00 | 10 , 8 | 15.59 | 10 , 8 | 15.48 |
| MANNE | 200 | 500 | 10 , 12 | 3.72 | 4 , 4 | 11.37 | 4 , 4 | 11.59 |
| MINMAXBD | 20 | 25 | 609 , 504 | 17.33 | 209 , 169 | 10.71 | 165 , 136 | 8.72 |
| MOSARQP1 | 30 | 930 | 10 , 11 | 19.28 | 4 , 3 | 59.32 | 4 , 3 | 59.25 |
| NGONE | 323 | 373 | (*) 3000 , 2700 | 2119.43 | 41 , 33 | 106.56 | 23 , 20 | 71.54 |
| OET2 | 1002 | 1005 | 69 , 63 | 139.12 | 8 , 6 | 34.05 | 7 , 7 | 20.51 |
| OPTCTRL3 | 80 | 122 | 46 , 47 | 2.96 | 5 , 5 | 3.74 | 5 , 5 | 3.75 |
| ORTHREGA | 256 | 517 | 177 , 165 | 73.47 | 141 , 129 | 2414.87 | 33 , 30 | 511.51 |
| ORTHREGD | 50 | 103 | 450 , 380 | 32.79 | 11 , 9 | 9.46 | 11 , 9 | 9.45 |
| PENTAGON | 15 | 21 | 44 , 45 | 0.86 | 6 , 5 | 0.27 | 6 , 5 | 0.25 |
| PRODPL0 | 29 | 69 | 42 , 39 | 2.48 | 12 , 9 | 2.18 | 15 , 13 | 2.49 |
| PT | 501 | 503 | 43 , 44 | 16.20 | 3 , 2 | 1.89 | 3 , 2 | 2.00 |
| QPCBLEND | 74 | 114 | 304 , 306 | 237.93 | 2 , 1 | 1.46 | 2 , 1 | 1.59 |
| READING1 | 100 | 202 | 691 , 629 | 160.29 | 9 , 7 | 8.86 | 9 , 7 | 8.87 |
| SSEBNLN | 96 | 218 | 36 , 37 | 9.22 | (**) 5 , 3 | 4.48 | 28 , 26 | 29.02 |
| SVANBERG | 100 | 200 | 76 , 66 | 34.03 | 10 , 9 | 21.44 | 10 , 9 | 21.33 |
| TFI1 | 1001 | 1004 | 197 , 168 | 148.53 | 13 , 11 | 22.00 | 14 , 11 | 22.57 |
| TFI3 | 5001 | 5004 | 44 , 44 | 491.16 | 4 , 3 | 34.15 | 4 , 3 | 34.03 |
| TRAINF | 202 | 408 | 76 , 77 | 59.68 | 5 , 5 | 36.21 | 5 , 5 | 36.27 |
| UBH1 | 60 | 99 | 49 , 50 | 2.32 | 4 , 3 | 1.78 | 4 , 3 | 1.78 |

Table 1: First set of numerical results.

The symbol (*) indicates that Lancelot failed to achieve the solution in 3000 iterations. In the cases marked (**), the new algorithm converged to a nonfeasible point (stationary point of $\varphi(x)$). In all the other cases both algorithms arrived at the same solution.

From Table 1, it can be observed that, when the execution time is considered, the new algorithm with $N = 0$ outperforms Lancelot for 28 problems out of 45 (including the two cases where Lancelot failed to attain the solution). A comparison on the number of function evaluations taken by both algorithms shows that Lancelot was beaten in 41 cases.

For $N = 10^6$, the new algorithm has shown a slightly better performance. Although being faster than Lancelot in only 27 cases, it failed to find a feasible solution for just one problem. Moreover, it spent less time and took less iterations than the algorithm with $N = 0$ in many occasions.

Table 2 summarizes the performance of the new algorithm. The values that appear in columns 2 through 5 indicate how many times each algorithm has outperformed the other. The first number on each column corresponds to the execution time, while the number in parenthesis refers to the comparision on the number of function evaluations.

| compared performance | Lancelot | New ($N = 0$) | Lancelot | New ($N = 10^6$) |
|---|---|---|---|---|
| more than 10 times better | 5 (0) | 9 (16) | 3 (0) | 10 (17) |
| from 2 to 10 times better | 6 (0) | 9 (20) | 8 (0) | 11 (20) |
| from 1.1 to 2 times better | 3 (1) | 8 (3) | 4 (1) | 4 (3) |
| almost indistiguishable | 1 (1) | 1 (1) | 2 (1) | 2 (1) |
| failures in obtaining the solution | 2 | 2 | 2 | 1 |

Table 2: A summary of the results.

## 7.2 Experiments without second derivatives

For the experiments presented in this section, the objective function is not twice differentiable everywhere. Consider the problem of estimating a solution of the differential equation

$$-y'' + e^y = F(t), \qquad t \in [0, 2\pi], \tag{96}$$

that fits a set of noisy observations $(t_1, y_1), \ldots, (t_m, y_m)$. (In the experiments, $F(t) = \sin(t) + e^{\sin(t)}$, so $\sin(t)$ is a particular solution of (96).) Problems of this type, although much more complicated, arise, for example, in numerical weather forecasting [19]. Dividing $[0, 2\pi]$ into $ndiv$ equally spaced intervals, defining $h = 1/ndiv$, and discretizing (96) using central differences, the unknown variables of the problem turn out to be the values of $y$ at the grid points $ih$, $i = 0, 1, 2, \ldots, ndiv$, denoted by $x_{i+1} \equiv y(ih)$. So, using the a priori knowledge $-1 \le y(t) \le 1$ for all $t$, the constraints of the problem are

$$\frac{-x_{i-1} + 2x_i + x_{i+1}}{h^2} + e^{x_i} = F((i-1)h), \quad i = 2, \ldots, ndiv, \tag{97}$$

and

$$-1 \le x_i \le 1, \quad i = 1, ndiv + 1. \tag{98}$$

Assume that the observed $t_i$ are grid points, say

$$t_1 = j_1 h, \ldots, t_m = j_m h.$$

The objective function of the problem will be

$$f(x) = f(y(0h), y(1h), \ldots, y(ndiv\ h)) = \sum_{\nu=1}^{m} \phi(|y(j_\nu h) - y_\nu|), \tag{99}$$

where $\phi$ is a measure of the deviation of the theoretical $y$ with respect to the observed one. Usually, $\phi(t) = t^2$ is used. However, if some observations are subject to large errors (outliers), a deviation function less subject to extreme values is needed. The deviation function used in the experiments of this section is defined by $\phi(z) = z^2$ if $z \leq 0.25$, and $\phi(z) = 0.5(z - 0.25) + 0.0625$ otherwise. So, the objective function $f$ has no second derivatives at points where the argument of $\phi$ is 0.25.

The experiments with this problem were designed in the following way:

(a) First, $m$ and $p$ ($m < p$) were chosen (the number of observed data and the number of outliers).

(b) The values of $m$ equally spaced grid points $t_i$ and $p$ equally space outliers were chosen.

(c) For each "normal" $t_i$, the observed $y_i$ was chosen randomly in $[\sin(t_i) - 0.1, \sin(t_i) + 0.1]$.

(d) For each outlier, the observed $y_i$ was chosen randomly in $[\sin(t_i) - 0.4, \sin(t_i) + 0.4]$.

A total of 14 problems were tested. The results are given in the following tables, for different values of the monotonicity index $N$. The number of iterations and the number of functional evaluations used to attain convergence are reported, as well as the *sup*−norm of the difference between the final computed $y$ and the "true solution" $\sin(t)$. In the experiments reported in Table 2, true Hessian-Lagrangian approximations in the sense of (94) are used. In Table 3, $H_k$ is the null $n \times n$ matrix for all $k$. So, the experiments reported in this table correspond to the behavior of an approximate sequential linear programming method.

| $m$ | $p$ | $N = 0$ | $N = 10^2$ | $N = 10^6$ |
|---|---|---|---|---|
| 100 | 0 | 6, 6, 4.E-4 | 6, 7, 4.E-4 | 6, 7, 4.E-4 |
| | 5 | 6, 6, 4.E-4 | 6, 7, 4.E-4 | 6, 7, 4.E-4 |
| | 10 | 6, 6, 3.E-2 | 5, 6, 3.E-2 | 5, 6, 3.E-2 |
| 60 | 0 | 6, 6, 4.E-4 | 6, 6, 4.E-4 | 6, 6, 4.E-4 |
| | 3 | 6, 6, 1.E-2 | 6, 6, 1.E-2 | 6, 6, 1.E-2 |
| | 6 | 5, 6, 3.E-2 | 5, 6, 3.E-2 | 5, 6, 3.E-2 |
| 40 | 0 | 7, 7, 1.E-2 | 7, 7, 1.E-2 | 7, 7, 1.E-2 |
| | 2 | 7, 7, 3.E-2 | 7, 7, 3.E-2 | 7, 7, 3.E-2 |
| | 4 | 4, 5, 2.E-2 | 4, 5, 2.E-2 | 4, 5, 2.E-2 |
| 20 | 0 | 5, 5, 4.E-4 | 5, 5, 4.E-4 | 5, 5, 4.E-4 |
| | 1 | 5, 5, 4.E-4 | 5, 5, 4.E-4 | 5, 5, 4.E-4 |
| | 2 | 6, 6, 3.E-2 | 6, 6, 3.E-2 | 6, 6, 3.E-2 |
| 10 | 0 | 5, 5, 4.E-4 | 5, 5, 4.E-4 | 5, 5, 4.E-4 |
| | 1 | 6, 8, 4.E-4 | 7, 7, 4.E-4 | 7, 7, 4.E-4 |

Table 2: Second set of numerical results. True Hessian-Lagrangian.

| $m$ | $p$ | $N=0$ | $N=10^2$ | $N=10^6$ |
|---|---|---|---|---|
| 100 | 0 | 7, 7, 4.E-4 | 7, 7, 4.E-4 | 7, 7, 4.E-4 |
| | 5 | 7, 7, 4.E-4 | 7, 7, 4.E-4 | 7, 7, 4.E-4 |
| | 10 | 14, 37, 3.E-2 | 10, 29, 3.E-2 | 9, 28, 3.E-2 |
| 60 | 0 | 6, 7, 4.E-4 | 6, 7, 4.E-4 | 6, 7, 4.E-4 |
| | 3 | 12, 33, 1.E-2 | 12, 33, 1.E-2 | 12, 33, 1.E-2 |
| | 6 | 10, 29, 3.E-2 | 10, 29, 3.E-2 | 10, 29, 3.E-2 |
| 40 | 0 | 11, 30, 1.E-2 | 11, 30, 1.E-2 | 11, 30, 1.E-2 |
| | 2 | 10, 29, 3.E-2 | 11, 32, 3.E-2 | 11, 32, 3.E-2 |
| | 4 | 10, 29, 3.E-2 | 10, 29, 3.E-2 | 10, 29, 3.E-2 |
| 20 | 0 | 6, 6, 4.E-4 | 6, 6, 4.E-4 | 6, 6, 4.E-4 |
| | 1 | 7, 7, 4.E-4 | 7, 7, 4.E-4 | 7, 7, 4.E-4 |
| | 2 | 8, 26, 3.E-2 | 8, 26, 3.E-2 | 8, 26, 3.E-2 |
| 10 | 0 | 6, 7, 4.E-4 | 6, 7, 4.E-4 | 6, 7, 4.E-4 |
| | 1 | 6, 7, 4.E-4 | 6, 7, 4.E-4 | 6, 7, 4.E-4 |

Table 3: Second set of numerical results. Null Hessian approximation.

The aim of these experiments is only to show that the algorithm works well even for problems without second derivatives. Unfortunately, the number of problems is not sufficient to permit an analysis of the influence of the monotonicity parameter $N$.

## 8 Final remarks

To the best of our knowledge, the method introduced in this paper is the first algorithm for general nonlinear programming (in the form (1)) with the following characteristics:

(a) the method is a globalized sequential quadratic programming algorithm;
(b) it uses trust regions as a globalization strategy;
(c) the merit function is an augmented Lagrangian (thus, it is differentiable);
(d) the estimates of the Lagrange multipliers are arbitrary;
(e) the behavior of the penalty parameter is not monotone;
(f) the steps on the normal and on the tangent spaces obey mild conditions;
(g) second derivatives of the objective function and constraints are not necessary.

All these features are valuable for the development of practical algorithms. Sequential quadratic programming is the most natural extension of Newton's method to constrained optimization and the trust-region approach allows one to deal consistently with infeasibility of quadratic subproblems. The possibility of using arbitrary estimates of Lagrange multipliers permits one to take advantage of well-known procedures to compute clever estimates that fit well with the augmented Lagrangian as a merit function. The nonmonotonicity feature tends to avoid the inheritance of unnecessary extreme values of penalty parameters from the first few iterations. Finally, mild conditions for the normal and the tangent step open the possibility of many useful implementations for large-scale problems.

It was proved that the method is globally convergent under suitable conditions on the problem. In fact, the compactness Assumption A3 holds if $\Omega$ is bounded, the feasibility of limit points is guaranteed if all the stationary points of (21) are feasible and the optimality of a limit point depends only on the regularity of feasible points.

A particular version of the model algorithm introduced in this paper was implemented. Among the many decisions that are necessary for a practical implementation, some reasonable ones were taken according to common sense and algorithmic availability. No claim is made with respect to have taken the best choices concerning parameters and internal procedures. In fact, the numerical experiments presented here have two main objectives: the first is to proceed to a preliminary evaluation of the nonmonotone strategy for the penalty parameter against the monotone one. It seemed that in most cases both strategies are essentially equivalent, but in some critical cases the nonmonotone strategy is clearly the best. For example, in the problem SSEBNLN the algorithm with monotone penalty parameter converges to a stationary nonfeasible point of (21), while the nonmonotone algorithm converges to a solution. This is due to a premature decrease of the penalty parameter at the first iterations which produces a merit function with excessive weight of the feasibility term. In other problems (for example, ORTHREGA) this phenomenon only caused a clear loss of efficiency of the monotone strategy. The second objective of the numerical experiments was to test the reliability of the new algorithm, by means of its comparison with a well established software for nonlinear programming. There was no intention here to draw strong conclusions regarding this comparison, but only to test if the set of ideas introduced in this paper deserve future development and further implementations. In these experiments it was corroborated that the new algorithm obtains the same solutions as the particular version of LANCELOT tested here does, and that the computer time used for finding these solutions is, in general, quite affordable.

Finally, since the existence of second derivatives is not necessary for the definition or the convergence of the algorithm, some tests were performed concerning problems where the objective function is not twice differentiable.

Among the possible lines for continuation of this work, the development of a local convergence analysis can be cited. It can be conjectured that, when the Lagrange multipliers are properly estimated, under suitable local conditions, the first $\delta$ tried at each iteration must be accepted. So, if second derivatives exist and true Hessians are used, quadratic convergence should be obtained and, if quasi-Newton approximations of the Hessians are used, superlinear convergence should take place. See, for example, [20]. As it was mentioned in the introduction, local convergence studies of SQP methods for problems without second derivatives are available ([13, 24, 25]). So, the objective of forthcoming local proofs will be to show that, with proper choices of the Lagrange multiplier estimates, the initial trial step is accepted in a neighborhood of the solution (that is, the method does not suffer from the so called "Maratos effect" [10]).

It should also be interesting to extend the method in order to consider constraints of type $h_i(x) \leq 0$ directly (without the introduction of slack variables). The implementation of a particular case of the general method, using alternative iterative linear algebra procedures for solving the subproblems, should also be very relevant in order to accomplish the solution of very large-scale problems.

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